KYUNGPOOK Math. J. 54(2014), 23-30
http://dx.doi.org/10.5666/KMJ.2014.54.1.23

Edge-Maximal θ_{2k+1}-Edge Disjoint Free Graphs

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Abstract. For two positive integers r and s, $\mathcal{G}(n;r;\theta_s)$ denotes to the class of graphs on n vertices containing no r of edge disjoint θ_s-graphs and $f(n;r;\theta_s) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;r;\theta_s)\}$. In this paper, for integers $r,\ k \geq 2$, we determine $f(n;r;\theta_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n;r;\theta_{2k+1})$.

1. Introduction

The graphs considered in this paper are finite, undirected and have no loops or multiple edges. Most of the notations that follow can be found in [6]. For a given graph G, we denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. The cardinalities of these sets are denoted by $\nu(G)$ and $\mathcal{E}(G)$, respectively. The cycle on n vertices is denoted by C_n. A theta graph θ_n is defined to be a cycle C_n to which we add a new edge that joins two non-adjacent vertices. We would like to mention that the method used in this paper follows the same lines used in [2] for the same authors.

Let G_1 and G_2 be graphs. The union of G_1 and G_2 is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. G_1 and G_2 are disjoint if and only if $V(G_1) \cap V(G_2) = \varnothing$; G_1 and G_2 are edge disjoint if $E(G_1) \cap E(G_2) = \varnothing$. If G_1 and G_2 are disjoint, we denote their union by $G_1 + G_2$. The intersection $G_1 \cap G_2$ of graphs G_1 and G_2 is defined similarly, but in this case we need to assume that $V(G_1) \cap V(G_2) \neq \varnothing$. The join $G \vee H$ of two disjoint graphs G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H. For two vertex disjoint subgraphs H_1 and H_2 of G, we let $E_G(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\mathcal{E}_G(H_1, H_2) = |E_G(H_1, H_2)|$.

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* Corresponding Author.
Received September 25, 2010; accepted November 7, 2012.
2010 Mathematics Subject Classification: Primary 05C35 and secondary 05C38.
Key words and phrases: Extermal graphs, Edge-disjoint, Theta graph.

In this paper we consider the Turán-type extermal problem with the odd edge-disjoint theta graphs being the forbidden subgraph. Since a bipartite graph contains no odd theta graph, the non-bipartite graphs have been considered by some authors. First, we recall some notations and terminologies. For a positive integer n and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n;\mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$-free graphs on n vertices, and

$$f(n;\mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;\mathcal{F})\}.$$

An important problem in extermal graph theory is determine the values of the function $f(n;\mathcal{F})$. Further, an additional goal is to characterize the extermal graphs $\mathcal{G}(n;\mathcal{F})$ where $f(n;\mathcal{F})$ is attained. This problem has been studied extensively by a number of authors [4, 5, 7, 8, 9]. In 1998, Jia proved that $\mathcal{E}(G) \leq \lfloor (n-2)^2/4 \rfloor + 3$ for $G \in \mathcal{G}(n;C_5)$ and $n \geq 10$. Furthermore, equality holds if and only if $G \in \mathcal{G}^*(n)$ where $\mathcal{G}^*(n)$ is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor (n-2)/2 \rfloor, \lceil (n-2)/2 \rceil}$. In 2007, Bataineh established the following result: Let $k \geq 3$ be a positive integer and $G \in \mathcal{G}(n;C_{2k+1})$. Then for large value of n, $\mathcal{E}(G) \leq \lfloor (n-2)^2/4 \rfloor + 3$. Furthermore, equality holds if and only if $G \in \mathcal{G}^*(n)$ where $\mathcal{G}^*(n)$ is as above.

Let $\mathcal{G}(n;r;\theta_s)$ denote to class of graphs on n vertices containing no r edge-disjoint θ_s-graphs and

$$f(n;r;\theta_s) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;r;\theta_s)\}.$$

Note that

$$\mathcal{G}(n;1;\theta_s) \subseteq \mathcal{G}(n;2;\theta_s) \subseteq \mathcal{G}(n;3;\theta_s) \subseteq \cdots \subseteq \mathcal{G}(n;r;\theta_s).$$

Let $\Omega(n,r)$ denote the class of graphs obtained by adding $r-1$ edges to the complete bipartite graphs $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. Figure 1 displays a member of $\Omega(n,2)$.

The Turán-type extermal problem with r odd edge-disjoint cycles being the forbidden subgraph, was studied by Bataineh and Jaradat [2], In fact, they proved that for $G \in \mathcal{G}(n;r;C_{2k+1}), k \geq 2$ and large value of n, $f(n;r;C_{2k+1}) \leq \lfloor n^2/4 \rfloor + r - 1$. Furthermore, equality holds if and only if $G \in \Omega(n,r)$. Recently, Bataineh et al [3] and Jaradat et al [10], proved the following results:

Theorem 1.1(Bataineh et al)**.** *For $n \geq 9$,*

$$f(n;\theta_5) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Furthermore, the bound is best possible.

Theorem 1.2(Jaradat et al)**.** *Let $k \geq 3$ be a positive integer and $G \in \mathcal{G}(n;\theta_{2k+1})$. Then for large n,*

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

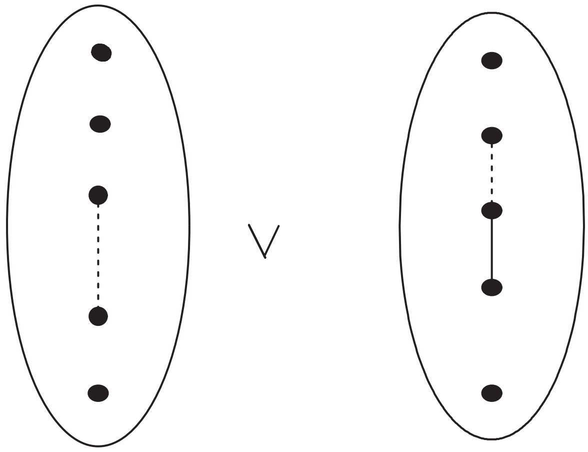

Figure 1: The figure represents a member of $\Omega(n, 2)$.

Furthermore, the bound is best possible.

Theorem 1.3(Jaradat et al)**.** *Let $k \geq 3$ be a positive integer and G be a graph on n vertices that contains no θ_{2k+1} graph as a subgraph. Then for large value of n,*

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Furthermore, equality holds if and only if G is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

We continue the work initiated in [3] and [10] by generalizing and extending the above theorem. In fact, we determine $f(n; r; \theta_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n; r; \theta_{2k+1})$ for $k, r \geq 2$.

In the rest of this paper, $N_G(u)$ stands for the set of neighbors of u in the graph G. Moreover, $G[X]$ denotes the induced subgraph of X in G.

2. Edge-Maximal θ_{2k+1} - Disjoint Free Graphs

In this section, we determine $f(n; r; \theta_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n; r; \theta_{2k+1})$ for $k, r \geq 2$. Observe that $\Omega(n, r) \subseteq \mathcal{G}(n; r; \theta_{2k+1})$ and every graph in $\Omega(n, r)$ contains $\lfloor n^2/4 \rfloor + r - 1$ edges. Thus, we have established that

$$f(n; r, \theta_{2k+1}) \geq \lfloor n^2/4 \rfloor + r - 1.$$

In the following theorem, we establish that equality holds. Further, we characterize the edge maximal members in $\mathcal{G}(n; r; \theta_{2k+1})$.

Theorem 2.1. *Let $k, r \geq 2$ be two positive integers and $G \in \mathcal{G}(n; r; \theta_{2k+1})$. For large value of n,*

$$f(n; r; \theta_{2k+1}) \leq \lfloor n^2/4 \rfloor + r - 1.$$

Furthermore, equality holds if and only if $G \in \Omega(n, r)$.

Proof. We prove this theorem using induction on r.

Step 1: We show the result for $r = 2$ and $k \geq 2$. Let $G \in \mathcal{G}(n, 2; \theta_{2k+1})$. If G contains no θ_{2k+1} as a subgraph, then by Theorem 1.3, $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$. Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1$. So, we need to consider the case when G has θ_{2k+1} as a subgraph. Assume $x_1 x_2 \ldots x_{2k+1} x_1 x_t$ be a θ_{2k+1} in G for some $3 \leq t \leq 2k$. Consider $H = G - \{e_1 = x_1x_2, e_2 = x_2x_3, \ldots, e_{2k+1} = x_{2k+1}x_1, e_{2k+2} = x_1x_t\}$. Observe that H cannot have θ_{2k+1} as otherwise G would have two edge-disjoint θ_{2k+1} as a subgraph. We now consider two cases:

Case 1: H is not a bipartite graph. Then we split this case into two subcases:

Subcase 1.1. $k = 2$. Then by Theorem 1.1

$$\mathcal{E}(H) \leq \lfloor (n-1)^2/4 \rfloor + 1.$$

Now,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(H) + 2k + 2 \\ &\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 2k + 3 \\ &< \left\lfloor \frac{n^2}{4} \right\rfloor + 1 \end{aligned}$$

for $n \geq 4k + 7$.

Subcase 1.2. $k \geq 3$. Then by Theorem 1.2

$$\mathcal{E}(H) \leq \lfloor (n-2)^2/4 \rfloor + 3.$$

Now,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(H) + 2k + 2 \\ &\leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 2k + 5 \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2k + 6, \end{aligned}$$

for $n \geq 2k + 6$, we have

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

Case 2: H is a bipartite graph. Let X and Y be the partition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is obtained when $|X| = \lfloor \frac{n}{2} \rfloor$ and $|Y| = \lceil \frac{n}{2} \rceil$. Thus, $\mathcal{E}(H) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. Restore the edges of the θ_{2k+1}-graph. We now consider the following subcases:

Subcase 2.1: One of X and Y contains two edges of the θ_{2k+1}-graph, say e_i and e_j in X. Let $y_1, y_2, \ldots, y_{k-1}$ be a set of vertices in $X - \{x_i, x_{i+1}, x_j, x_{j+1}\}$. We split this subcase into two subsubcases:

Subsubcase 2.1.1: i and j are not consecutive. Then $|N_Y(x_i) \cap N_Y(x_{i+1}) \cap N_Y(x_j) \cap N_Y(x_{j+1}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \ldots \cap N_Y(y_{k-1})| \leq k+2$, as otherwise G contains two edge-disjoint θ_{2k+1}-graph. Thus,

$$\mathcal{E}_G(\{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k+2)|Y| + k + 2.$$

So,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}_G(X - \{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) + \\
&\quad \mathcal{E}_G(\{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) + \mathcal{E}(G[X]) + \mathcal{E}(G[Y]) \\
&\leq (|X| - k - 3)|Y| + (k+2)|Y| + k + 2 + 2k + 2 \\
&\leq |X||Y| - |Y| + 3k + 4 \\
&\leq (|X| - 1)|Y| + 3k + 4
\end{aligned}$$

Observe that $|X| + |Y| = n$. The maximum of the above equation is when $|Y| = \lceil \frac{n-1}{2} \rceil$ and $|X| - 1 = \lfloor \frac{n-1}{2} \rfloor$. Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 3k + 4.$$

Hence, for $n \geq 6k + 9$,

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

Subsubcase 2.1.2: i and j are consecutive, say $j = i + 1$. Then by following the same arguments as in subsubcase 2.1.1 and by taking into the account that $|N_Y(x_i) \cap N_Y(x_{i+1}) \cap N_Y(x_{j+2}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \ldots \cap N_Y(y_{k-1})| \leq k+1$ and so $\mathcal{E}(\{x_i, x_{i+1}, x_{i+2}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k+1)|Y| + k + 1$, we get the same inequality.

Subcase 2.2: $\mathcal{E}(G[X]) = 1$ and $\mathcal{E}(G[Y]) = 0$ or $\mathcal{E}(G[X]) = 0$ and $\mathcal{E}(G[Y]) = 1$. Then

$$\begin{aligned}
\mathcal{E}(G) &\leq \mathcal{E}(H) + 1 \\
&\leq \left\lfloor \frac{n^2}{4} \right\rfloor + 1
\end{aligned}$$

One can observe from the above arguments that for $r = 2$ only time when we have equality is when G is obtained by adding an edge to the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. This leads to the class $\Omega(n, 2)$.

Step 2: Assume that the result is true for $r - 1$.

Step 3: We show the result is true for $r \geq 3$. To accomplish that we use similar arguments to those in Step 1. Let $G \in \mathcal{G}(n; r; \theta_{2k+1})$. If G contains no $r - 1$ edge-disjoint of θ_{2k+1}-graphs, then by the inductive step $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor + r - 2$. Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + r - 1$. So, we need to consider the case when G has $r - 1$ edge-disjoint of θ_{2k+1}-graphs. Assume that $\{\theta^{(i)} = x_{i1}x_{i2} \ldots x_{i(2k+1)}x_{i1}x_{it}\}_{i=1}^{r-1}$ be the set of $r - 1$ θ_{2k+1}-graphs. Consider $H = G - \cup_{i=1}^{r-1} E(\theta^{(i)})$. Observe that H cannot have θ_{2k+1}-graphs as otherwise G would have r edge-disjoint θ_{2k+1}-graphs. As in Step 1, we consider two cases:

Case I: H is not a bipartite graph. Then we consider two subcases

Subcase 1.1. $k = 2$. Then by Theorem 1.1

$$\mathcal{E}(H) \leq \lfloor (n-1)^2/4 \rfloor + 1.$$

Now,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(H) + (r-1)(2k+2) \\ &\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + (r-1)(2k+2) \\ &< \left\lfloor \frac{n^2}{4} \right\rfloor + 1 \end{aligned}$$

for $n \geq 2(2k+2)(r-1) + 1$.

Subcase 1.2. $k \geq 3$. Then by Theorem 1.2

$$\mathcal{E}(H) \leq \lfloor (n-2)^2/4 \rfloor + 3.$$

Thus,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(H) + (r-1)(2k+2) \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor + (r-1) - n + 4 + (2k+1)(r-1), \end{aligned}$$

for $n > 4 + (2k+1)(r-1)$,

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1.$$

Case II: H is a bipartite graph. Let X and Y be the partition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe that $|X| + |Y| = n$. The maximum of the above is obtained when $|X| = \lfloor \frac{n}{2} \rfloor$ and $|Y| = \lceil \frac{n}{2} \rceil$. Thus, $\mathcal{E}(H) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. Now, we consider the following two subcases:

Subcase II.I: There is $1 \leq m \leq r-1$ such that $\theta^{(m)}$ contains at least two edges, say $e_i = x_{mi}x_{m(i+1)}$ and $e_j = x_{mj}x_{m(j+1)}$, joining vertices of one of X and Y , say X. Let $y_1, y_2, \ldots, y_{k-1}$ be a set of vertices in $X - \{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}\}$. To this end we have two susubcases:

Subsubcase II.I.I: i and j are not consecutive. Then $|N_Y(x_{mi}) \cap N_Y(x_{m(i+1)}) \cap N_Y(x_{mj}) \cap N_Y(x_{m(j+1)}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \ldots \cap N_Y(y_{k-1})| \leq k+2$, as otherwise $H \cup \{e_i, e_j\}$ contains two edge-disjoint θ_{2k+1}-graphs and so G contains r edge-disjoint θ_{2k+1}-graphs. Thus, as in Subsubcase 2.1.1 of Step 1,

$$\mathcal{E}_H(\{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k+2)|Y| + k + 2.$$

And so,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(H) + |\cup_{i=1}^{r-1} E(\theta^i)| \\
&= \mathcal{E}_H(X - \{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) + \\
&\quad \mathcal{E}_H(\{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) + |\cup_{i=1}^{r-1} E(\theta^{(i)})| \\
&\leq (|X| - k - 3)|Y| + (k+2)|Y| + k + 2 + (r-1)(2k+2) \\
&= (|X| - 1)|Y| + k + 2 + (r-1)(2k+2)
\end{aligned}$$

Moreover, the maximum of the above inequality is obtained when $|Y| = \lceil \frac{n-1}{2} \rceil$ and $|X| - 1 = \lfloor \frac{n-1}{2} \rfloor$. Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + k + 2 + (r-1)(2k+2)$$

For $n \geq (6k+2)(r-1) + 7$, we have

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + (r-1).$$

Subsubcase II.I.II: i and j are consecutive, say $j = i+1$. Then by following the same arguments as in Subsubcase 2.1.2 of Step 1 and Subsubcase II.I.II of step 2, we get the same inequality.

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + (r-1).$$

Subcase II.II: Each $1 \leq m \leq r-1$, $\theta^{(m)}$ has exactly one edge belonging to one of X and Y. Let e be the edge of $\theta^{(1)}$ that belongs to one of X and Y. Then

$G - e \in \Omega(n, r-1) \subseteq \mathcal{G}(n; r-1; \theta_{2k+1})$ and so by the inductive step,

$$\begin{aligned}\mathcal{E}(G) &= \mathcal{E}(G-e)+1\\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor + r - 2 + 1\\ &= \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1.\end{aligned}$$

This completes the proof of the theorem. □

We can now characterize the extermal graphs. Throughout the proof, we noticed that the only time when we had equality was in the case when G was obtained by adding $r-1$ edges to the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. This leads rise to the class $\Omega(n, r)$.

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