DE GRUYTER Open Mathematics 2020; 18: 1531–1539

Research Article

Zahid Raza, Mohammed M. M. Jaradat*, Mohammed S. Bataineh, and Faiz Ullah

On the sandpile model of modified wheels II

https://doi.org/10.1515/math-2020-0094
received May 10, 2019; accepted October 1, 2020

Abstract: We investigate the abelian sandpile group on modified wheels $\hat{W}_n$ by using a variant of the dollar game as described in [N. L. Biggs, *Chip-Firing and the critical group of a graph*, J. Algebr. Comb. **9** (1999), 25–45]. The complete structure of the sandpile group on a class of graphs is given in this paper. In particular, it is shown that the sandpile group on $\hat{W}_n$ is a direct product of two cyclic subgroups generated by some special configurations. More precisely, the sandpile group on $\hat{W}_n$ is the direct product of two cyclic subgroups of order a_n and $3a_n$ for n even and of order a_n and $2a_n$ for n odd, respectively.

Keywords: abelian sandpile group, critical configuration, stable configuration, dollar game, modified wheels

MSC 2020: 05C05, 05C25, 15A18, 05C50

1 Introduction

The abelian sandpile model (ASM) was defined in [2] for any graph. This model is a prime example of self-organized criticality [3], which has transformed the understanding of how complexity arises in nature. The ASM is related to the chip-firing game introduced by Spencer [4].

The algebraic structure of the sandpile group has been completely determined for the following family of graphs: wheel graphs, trees, regular trees, thick trees, wired regular trees, cycles, complete graphs, multipartite complete graphs, wheels, some thick wheels, the square of the cycle $(C_n)^2$, dihedral group graph, Möbius ladder, $3n$ twisted bracelets, generic threshold graphs, the Cartesian products $(P_4) \times (C_n)$, $(K_3) \times (C_n)$, $(K_m) \times (P_n)$, $(C_4) \times (C_n)$, and $(K_m) \times (C_n)$. The abstract structure of the sandpile group has been partially described for the hypercube and the Cartesian product of complete graphs.

However, the combinatorial structure of the recurrent configurations has only been described for a thick tree (a graph is a thick tree if it is chordal and every two separators are either equal or disjoint). In [1], it has been shown that the set of critical configurations can be used to give the structure of an abelian group by giving critical configurations for a wheel graph W_n. For general references on the ASM (denoted by SP (G,q)), see, e.g., [5–14] and on the critical group (denoted by K(G)), see for instance [15–24]. The definitions and statements of theorems presented in this section have been taken from [1]. We only consider undirected graphs.

The sandpile model is a discrete model (cellular automaton) originally defined on a rectangular grid of the standard square grid in the plan which was extended on the graphs as well.

*** Corresponding author: Mohammed M. M. Jaradat,** Department of Mathematics, Statistics and Physics, Qatar University, Doha, Qatar, e-mail: mmjst4@qu.edu.qa
Zahid Raza: Department of Mathematics, College of Sciences, University of Sharjah, Sharjah, UAE, e-mail: zraza@sharjah.ac.ae
Mohammed S. Bataineh: Department of Mathematics, College of Sciences, University of Sharjah, Sharjah, UAE; Department of Mathematics, Yarmouk University, Irbid, Jordan, e-mail: bataineh71@hotmail.com
Faiz Ullah: Department of Mathematics, National University of Computer and Emerging Sciences, Lahore Campus, Pakistan, e-mail: faizullah57@yahoo.com

An (undirected) ASM begins with any connected undirected graph G with vertex set $V(G)$ and edge set $E(G)$, together with a *distinguished vertex* $q \in V(G)$ called the central or hub vertex. To each vertex of the graph G, we associate a value (grains of sand, chips) referred to as the (initial) configuration of the sandpile. The dynamics of the model at iteration $i \in \mathbb{N}$ are then defined as follows:

- Choose a random vertex according to some probability distributions (usually uniform).
- Add one grain of sand to this vertex while letting the grain numbers for all other vertices unchanged.
- If all vertices are stable. In this case, continue with the next iteration.
- If at least one vertex is unstable, then whole configuration is said to be unstable. In this case, choose any unstable vertex at random. Topple this vertex by reducing its grain number by number of the neighbors and by increasing the grain numbers of each of its direct neighbors by one.
- Due to the redistribution of grains, the toppling of one vertex can render other vertices unstable. Thus, repeat the toppling procedure until all vertices of eventually become stable and continue with the next iteration.

A dollar game on a graph G starts with dollars at each vertex of a graph. At each step of the game a vertex is “fired,” that is, $\deg(v)$ dollars move from the fired vertex to the adjacent vertices, one dollar going along each edge incident with the fired vertex. A vertex v can be fired if and only if the number of dollars currently held at v is at least $\deg(v)$ (degree of v).

Let s be a configuration of the game on the graph G. By a *configuration* we mean that s is a function defined on the vertices $V(G)$ such that $s(v)$ is the number of dollars at each vertex v.

Suppose that S is a non-empty finite sequence of (not necessarily distinct) vertices of G, such that, starting from s, the vertices can be fired in the order of S. If v occurs $x(v)$ times in S, x shall be referred to as the representative vector for s. After “firing” a change in the configuration occurs. The configuration s' obtained after the sequence of firings S is given by

$$s'(v) = s(v) - x(v)\deg(v) + \sum_{w \neq v} x(w)v(v, w),$$

where $v(v, w)$ is the number of edges joining v and w. Each time a vertex v is fired it loses $\deg(v)$ dollars, and each time a vertex $w \neq v$ is fired v gains $v(v, w)$ dollars. It should be noted that s' always exists and is unique.

The Laplacian matrix L is given by

$$(L)_{v_i v_j} = \begin{cases} \deg(v_i), & i = j, \\ -1, & i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

The relationship between s and s' can be written concisely if we define the *Laplacian* matrix Q as follows:

$$(Q)_{vw} = \begin{cases} -v(v, w), & v \neq w, \\ \deg(v), & v = w. \end{cases}$$

In terms of Q the relationship between s and s' is given as

$$s' = s - Qx.$$

Definition 1.1. The family of graph denoted by $\hat{W}_n$ is constructed from an n-wheel W_n by adding one vertex on each of the cycle edge of the wheel. In particular, let W_n be the wheel with vertices $q, v_1, v_2, \ldots, v_n$, where we fixed the central vertex as q. Then adding one extra vertex in between vertices v_i for all $i = 1, 2, \ldots, n$, we obtained the family $\hat{W}_n$. This family has $2n + 1$ vertices and $3n$ edges. The set of vertices are given by $V(\hat{W}_n) = \{q, v_1, v_2, \ldots, v_{2n}\}$.

Definition 1.2. A configuration s is said to be stable if it satisfies the following inequality:

$$0 \leq s(v) \leq \deg(v)\ (v \neq q).$$

Note that a configuration with 2 dollars at each vertex is stable for W_n, see [1], but not for $\hat{W}_n$.

A sequence of firing is *q-legal* if and only if each occurrence of a vertex ($v \neq q$) follows a configuration t with $t(v) \geq \deg(v)$ and each occurrence of q follows a stable configuration.

A configuration r of the dollar game on a graph is said to be *recurrent* if there is a q-legal sequence for r which leads to the same configuration.

A configuration which is both recurrent and stable is referred to as the *critical* configuration.

Note that not all stable configurations are critical (e.g., the configuration with zero dollars at every vertex is stable but not recurrent).

The theory of dollar game is based on a *confluence property*, i.e., if we start with a given configuration s, then there may be many different sequences which are possible starting from s, but all of these sequences will lead to the same outcome. For the proof of this property, we direct the reader to [1].

We shall study a variant of the *dollar game* in which the fixed central vertex q is allowed to go into debt. The central vertex q is fired if and only if no other firing is possible. Thus, in this variant, a configuration s is an integer-valued function satisfying:

$$s(v) \geq 0\ \ (v \neq q)\ \text{ and }\ s(q) = -\sum s(v) \leq 0.$$

Equivalently, any dollars going into the center are ignored. We shall refer to a sequence as a *proper sequence* if it does not contain the central vertex q.

Given a configuration s and $v \in V(G)\backslash\{q\}$, toppling unstable non-central vertices on a finite connected graph until no unstable non-central vertex remains leads to a unique stable configuration s', which is called the stabilization of s. Given two stable configurations s and w, we can define the operation $s\star w \to (z + w)'$, corresponding to the vertex-wise addition of dollars followed by the stabilization of the resulting sandpile. Under the operation $\star$, the set of critical configurations forms an abelian group isomorphic to the cokernel of the reduced graph Laplacian Q', i.e., $SP(G, q) \cong \mathbf{Z}^{n-1}/\mathbf{Z}^{n-1}Q'$, whereby n denotes the number of vertices (including the center). More generally, the set of critical configurations (stable and recurrent) forms a commutative monoid under the operation $\star$. The minimal ideal of this monoid is then isomorphic to the group of recurrent configurations.

The group formed by the recurrent configurations, as well as the group $\mathbf{Z}^{n-1}/\mathbf{Z}^{n-1}Q'$ to which the former is isomorphic, is most commonly referred to as the sandpile group. Other common names for the same group are critical group, Jacobian group, or (less often) Picard group. Note, however, that some authors only denote the group formed by the recurrent configurations as the sandpile group, while reserving the name Jacobian group or critical group for the (isomorphic) group defined by $SP(G, q) = \mathbf{Z}^{n-1}/\mathbf{Z}^{n-1}Q'$. Given the isomorphisms stated above, the order of the sandpile group is the determinant of Q', which by the matrix tree theorem is the number of spanning trees of the graph.

Let $l > 1$ be any positive integer. Then $l \cdot a$ denotes the $\star$-sum of l copies of a configuration a in $SP(G, q)$. Equivalently, $l \cdot a$ is the unique critical configuration $s' = \gamma(a + a + \cdots + a)$, where the + sign represents vector addition.

For example, let $a = (2, 0, 2, 1, 2, 1, 2, 1)$ be a configuration on $\hat{W}_4$ and $l = 3$, then $3 \cdot a = a + a + a$ is equal to the following critical configuration $3 \cdot a = (1, 0, 1, 1, 2, 1, 2, 1)$.

Example 1.3. Consider the graph $\hat{W}_2$. A configuration on this graph is determined by a vector $(s(v_1), s(v_2), s(v_3), s(v_4))$ denoting the number of dollars at the vertices v_1, v_2, v_3, v_4. Instead of referring to the vertices as v_1, v_2, v_3, and v_4, we number the vertices as $0, 1, 2, 3$ with the number 0 assigned to a vertex with degree 3. Also note that the vertices $0, 1, 2, 3$ are non-central vertices. A configuration is stable if and only if $0 \leq s(v) \leq 2$ for $v = 0, 2$ and $0 \leq s(v) \leq 1$ for $v = 1, 3$. There are 12 critical configurations. The zero element is $(2, 0, 2, 0)$ and the sandpile group $SP(\hat{W}_2, q)$ is the direct sum of two cyclic groups of orders 2 and 6 generated by configurations a and b, respectively.

2 Results

In this section, we give our main theorem by providing the complete structure of the sandpile group on a class of graphs. In particular, we show that the sandpile group is a direct product of two cyclic groups generated by some special configurations. That is, if n is even, these two subgroups are of order a_n and $3a_n$, and if n is odd they are of order a_n and $2a_n$, respectively.

The vertices of the graph $\hat{W}_n$ are numbered as $0, 1, 2, 3, \dots, 2n-1$, where the vertex numbered 0 has degree 3. Let a_n be a sequence of positive integers with initial conditions $a_1 = 0$, $a_2 = 1$, $a_3 = 2$, $a_4 = 5$,

$$a_i = 4a_{i-2} - a_{i-4}.$$

Case I: If $n = 2k$ is an even number and define the following four configurations on the graph $\hat{W}_n$ as:

$$a = \begin{cases} 2, & v \equiv 0 \bmod 2, \\ 0, & v = 1, \\ 1, & \text{otherwise}, \end{cases} \qquad b = \begin{cases} 2, & v = 2, \\ 0, & v = 3, \\ 1, & \text{otherwise}, \end{cases} \qquad c = \begin{cases} 2, & v \equiv 0 \bmod 4, \\ 1, & \text{otherwise}, \end{cases} \qquad c' = \begin{cases} 2, & v \equiv 2 \bmod 4, \\ 1, & \text{otherwise}. \end{cases}$$

Lemma 2.1. *If n is even number, i.e., $n = 2k$, then b is a critical configuration and*

$$a_{n+1} \cdot b = \begin{cases} c', & \text{if} \quad k \equiv 0 \bmod 2, \\ c, & \text{if} \quad k \equiv 1 \bmod 2. \end{cases}$$

Proof. Clearly b is stable. It is easy to check that the sequence of vertices $q, 2, 1, 0, 2n-1, 2n-2, 2n-3, \dots, 4, 3$ is q-legal for b, and since each vertex is fired once the resulting configuration is $b - Qu = b$, which shows that b is critical. The proof of second statement is obtained by taking induction on n. The statement is true for $n = 4$ and $n = 6$. The inductive hypothesis states that for all integers l less than or equal to n, where $l = 2k'$. We have $a_{l+1} \cdot b = c'$ when $k' \equiv 0 \bmod 2$, and $a_{l+3} \cdot b = c$ when $k' \equiv 1 \bmod 2$. We shall show that $a_{l+5} \cdot b = c'$. Since $a_{l+5} = 4a_{l+3} - a_{l+1}$ and by inductive hypothesis, we have $a_{l+5} \cdot b = \gamma(4c - c') = \gamma(s)$, where

$$s = \begin{cases} 2, & v \equiv 0 \bmod 4, \\ 7, & v \equiv 2 \bmod 4, \\ 3, & \text{otherwise}. \end{cases}$$

It can be easily verified that the following sequence is a q-legal sequence for this configuration and results in c'

$$1, 2, 3, \dots, 2n-1, 0, 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, \dots, 2n-1, 0, 1, 2, 3, \dots, 2n-1,$$
$$1, 2, 3, 4, \dots, 2n-1, 0, 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, \dots, 2n-1, 0, 1, 2, \dots, 2n-1.$$

Now, we shall prove that $a_{l+7} \cdot b = c$. Since $a_{l+7} = 4a_{l+5} - a_{l+3}$ and by induction hypothesis, we have $a_{l+7} \cdot b = \gamma(4c' - c) = \gamma(s')$, where

$$s' = \begin{cases} 7, & v \equiv 0 \bmod 4, \\ 2, & v \equiv 2 \bmod 4, \\ 3, & \text{otherwise}. \end{cases}$$

It can be easily verified that the following sequence is a q-legal sequence for this configuration and results in c

$$0, 1, 2, 3, \dots, 2n-1, 0, 1, 3, 4, 5, 7, 8, 9, 11, 12, 13, \dots, 2n-1, 0, 1, 2, 3, \dots, 2n-1,$$
$$0, 1, 2, 3, \dots, 2n-1, 0, 1, 2, 4, 5, 7, 8, 9, 11, 12, 13, \dots, 2n-1, 2, 3, 4, 5, \dots, 2n-1, 0, 1. \qquad \square$$

Lemma 2.2. *The configurations c and c' are critical configurations of order 3 in the abelian sandpile group $SP(\hat{W}_n, q)$.*

Proof. Clearly c' is stable and the following sequence of vertices is q-legal for c' $q, 2, 3, 4, \ldots, 2n-1, 0, 1$, since each vertex is fired once, c' is a recurrent configuration and hence critical. The following sequence of firings is a q-legal sequence and leads to identity for $3 \cdot c'$.

$$0, 1, 2, 3, \ldots, 2n-1, 0, 1, 2, 3, \ldots, 2n-1, 0, 1, 3, 5, \ldots, 2n-1, 0, 1, 2, \ldots, 2n-1,$$
$$1, 2, \ldots, 2n-1, 0, 1, 3, 5, 7, \ldots, 2n-1, 0, 2, 4, 6, 8, \ldots, 2n-2, 1, 3, 5, 7, \ldots, 2n-1.$$

By definition c is stable and the given sequence of vertices is q-legal for c $0, 1, 2, 3, \ldots, 2n-1$, which shows that c is a recurrent configuration and hence critical. Similarly, it can be checked that the following sequence is a q-legal sequence and leads to identity for $3 \cdot c$.

$$0, 1, 2, \ldots, 2n-1, 1, 2, 3, \ldots, 2n-1, 0, 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, \ldots,$$
$$2n-1, 0, 1, 2, \ldots, 2n-1, 1, 2, 3, \ldots, 2n-1, 0, 1, 3, 5, 7, \ldots, 2n-1, 0, 2, 4, 6, \ldots, 2n-2, 1, 3, 5, \ldots, 2n-1. \quad \square$$

Lemma 2.3. *Let t be a critical configuration on $\hat{W}_n$ such that*
- *$\alpha \cdot t = t + t + \cdots + t$ makes all the vertices unstable (except for those which have zero dollars) for any positive integer α,*
- *all the even numbered vertices have the same number of dollars,*
- *there is only one odd numbered vertex with zero dollars.*

Then the only odd numbered vertex with zero dollars will always have zero dollars at the same position after stabilizing.

Proof. Let us consider a configuration t that satisfies the aforementioned conditions and let an odd numbered vertex say v_i has zero dollars where index i is odd. In the very first round of firings, the vertex v_i does not fire but receives 2 dollars from its neighboring vertices (as they were destabilized initially). In other rounds of firings, the number of dollars at odd numbered vertices remains constant because the odd numbered vertices are not directly connected to the central vertex so whatever they loose they gain it back through their neighbors. $\square$

Since all the even numbered vertices had equal dollars initially, they stabilize at the same time. Once this happens, start firing the odd numbered vertices. The vertex v_i then loses each of its dollar to its neighbors v_{i-1} and v_{i+1} which are even numbered vertices. If the neighbors of v_i are destabilized after receiving a dollar each from v_i, then fire all the destabilized even numbered vertices. As a result v_i gains 2 dollars and is destabilized. Repeat the procedure of firing odd and even numbered vertices alternately until firing of v_i does not destabilize its neighbors.

Example 2.4. Let $p = (2, 0, 2, 1, 2, 1, 2, 1)$ be a configuration on $\hat{W}_4$ that satisfies the conditions stated in the aforementioned lemma and $\alpha = 3, 22, 105$, then

$$3 \cdot p = (1, 0, 1, 1, 2, 1, 2, 1), \quad 22 \cdot p = (1, 0, 1, 1, 2, 0, 2, 1), \quad 105 \cdot p = (2, 0, 2, 1, 2, 1, 2, 1).$$

Example 2.5. It is easy to see that the following configuration $m = (2, 0, 1, 1, 1, 1, 1, 1)$ does not satisfy the conditions stated in the aforementioned lemma and $\alpha = 13, 16, 18$ positive integer then,

$$13 \cdot m = (2, 1, 1, 1, 2, 0, 2, 1), \quad 16 \cdot m = (1, 1, 2, 1, 1, 1, 2, 1), \quad 18 \cdot m = (2, 1, 1, 1, 1, 1, 2, 1).$$

Example 2.6. Let $r = (2, 1, 2, 0, 2, 1, 2, 0, 2, 1)$ be a configuration on $\hat{W}_5$, which satisfies the conditions of the aforementioned lemma, but has zero dollars at more than one vertex and $\alpha = 3, 5$ is the positive integer then, we obtained the following critical configuration after the stabilization of vertices, $3 \cdot r = (2, 1, 0, 1, 2, 1, 2, 1, 0, 1)$, $5 \cdot r = (1, 1, 1, 1, 2, 0, 2, 1, 1, 1)$, which shows that the conditions in the lemma are not sufficient.

Note. For defining permutations of the rim vertices, we denote the vertices on the rim of $\hat{W}_n$ by the residue classes $-(n-1), -(n-2), \ldots, -1, 0, +1, +2, \ldots, +(n-1) \pmod n$ as shown in Figure 1.

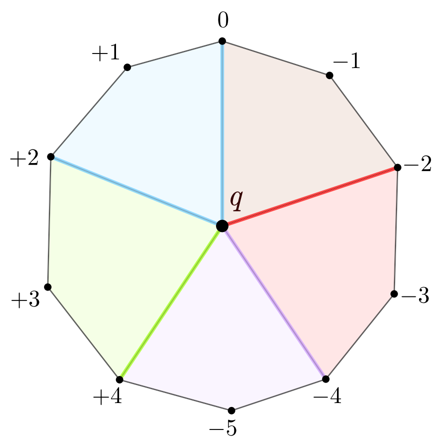

Figure 1: The vertices on the rim of $\hat{W}_n$.

Theorem 2.7. *If n is even, the abelian sandpile group $SP(\hat{W}_n, q)$ is isomorphic to the direct product of two cyclic groups of orders a_n and $3a_n$. In particular,*

$$SP(\hat{W}_n, q) \approx \frac{\mathbb{Z}}{a_n\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3a_n\mathbb{Z}},$$

where the cyclic groups are generated by configurations a and b, respectively.

Proof. Let $\pi_{-1,n-1}$ and $\pi_{-2,-3}$ denote the permutations of the rim vertices defined as follows:

$$\pi_{-1,n-1}(i) = \begin{cases} -1, & \text{if } i = -1, \\ n-1, & \text{if } i = n-1, \\ -(i+2), & \text{if } i = 1, 2, 3, \ldots, n-3, \\ n, & \text{if } i = n-2, \\ n-2, & \text{if } i = n, \\ i-2, & \text{otherwise,} \end{cases} \qquad \pi_{-2,-3}(i) = \begin{cases} -2, & \text{if } i = -2, \\ -3, & \text{if } i = -3, \\ -i+4, & \text{if } i = -5, -4, \\ -(n-1), & \text{if } i = +(n-1), \\ -(n-2), & \text{if } i = n, \\ i+2, & \text{otherwise.} \end{cases}$$

Every multiple of a is symmetrical with respect to $\pi_{-1,n-1}$, that is, $(\alpha \cdot a)(v) = (\alpha \cdot a)(\pi_{-1,n-1}(v))$. On the other hand, $(\beta \cdot b)(v) = (\beta \cdot b)(\pi_{-2,-3}(v))$. Since both the defined permutations generate a group which acts transitively on the rim, the only configurations that are symmetrical with respect to both permutations are those in which every rim vertex has the same number of dollars or all the even numbered vertices have the same number of dollars and all the odd numbered vertices have the same number of dollars. So, if $(\alpha \cdot a) = (\beta \cdot b)$, both configurations are in fact the critical configurations in which all the even numbered vertices have the same number of dollars and all the odd numbered vertices have the same number of dollars.

The required configuration cannot be a configuration in which every rim vertex has the same number of dollars because such a configuration is not critical for this particular family of graphs. Therefore, after deleting the non-critical configurations we are left with the configurations $(2, 0, 2, 0, \ldots, 2, 0)$ and $(2, 1, 2, 1, \ldots, 2, 1)$. By using Lemma 2.3, the choice $(2, 1, 2, 1, \ldots, 2, 1)$ is ruled out and hence $(2, 0, 2, 0, \ldots, 2, 0)$ is the required critical configuration, which is also the zero element of the sandpile group $SP(\hat{W}_n, q)$. It follows that the subgroup generated by a and b is the direct sum of the cyclic groups, which completes the proof.

Case II: If $n = 2k + 1$ is an odd number and define some configurations on $\hat{W}_n$.

$$d = \begin{cases} 2, & v \equiv 0 \bmod 2, \\ 1, & v = 1, 2n-1, \\ 0, & \text{otherwise,} \end{cases} \qquad g = \begin{cases} 2, & v \equiv 0 \bmod 4, \\ 0, & v \equiv 2 \bmod 4, \\ 1, & \text{otherwise,} \end{cases} \qquad f = \begin{cases} 2, & v \equiv 0 \bmod 2, \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 2.8. *If n is an odd number, then g is a critical configuration and $a_{n+1} \cdot g = f$.*

Proof. Clearly g is stable. It easy to check that the following sequence of vertices

$$q, 0, 1, 4, 5, \ldots, 2n-6, 2n-5, 2n-2, 2n-1, 3, 7, \ldots, 2n-3, 2, 6, \ldots, 2n-4$$

is q-legal for $g \Rightarrow g - Qu = g$. So g is recurrent, and therefore critical. The proof of the second statement is obtained by taking induction on n. It is easy to see that $n = 5$, i.e., $a_6 \cdot b = 19 \cdot b = \gamma(38, 19, 38, 0, 38, 0, 38, 19, 38, 19) = (2, 1, 2, 1, 2, 1, 2, 1, 2, 1) = f$. Suppose that the statement is true $\forall\, k \leq n$, where n, k are both odd, i.e., $a_{k+1} \cdot g = f$. We shall prove that $a_{k+3} \cdot g = f$. Since $a_{k+3} = 4a_{k+1} - a_{k-1}$ and by induction hypothesis we get, $a_{k+3} \cdot g = \gamma(4f - f) = \gamma(3f)$, where

$$3f = \begin{cases} 6, & v \neq q \text{ and } v \equiv 0 \bmod 2, \\ 3, & \text{otherwise.} \end{cases}$$

It can be verified that the following sequence is a q-legal sequence for this configuration and results in f.

$$0, 1, 2, \ldots, 2n-1, 0, 1, 2, \ldots, 2n-1, 0, 1, 2, \ldots, 2n-1, 0, 1, 2, \ldots, 2n-1,$$
$$1, 2, 3, \ldots, 2n-1, 0, 1, 3, 5, 7, \ldots, 2n-1, 0, 1, 2, 3, \ldots, 2n-1.$$ □

Lemma 2.9. *Prove that f is a critical configuration and its order is 2 in the abelian sandpile group $SP(\hat{W}_n, q)$.*

Proof. By definition, f is a stable configuration and since the following sequence of vertices $0, 1, 2, 3, \ldots, 2n-1$ is q-legal for f, therefore, f is a recurrent configuration and hence critical. The following sequence is a q-legal sequence for $2 \cdot f = f + f$ $0, 1, 2, \ldots, 2n-1, 0, 1, 2, \ldots, 2n-1, 1, 2, 3, \ldots, 2n-1, 0, 1, 3, 5, 7, \ldots, 2n-1, 0, 2, 4, 6, 8, \ldots, 2n-2, 2n-1, 1, 3, 5, 7, \ldots, 2n-3$ and stabilization results in the identity element of the group $SP(\hat{W}_n, q)$. □

Lemma 2.10. *Let t be a critical configuration of the abelian sandpile group $SP(\hat{W}_n, q)$ such that*
- *every even numbered vertex has the same number of dollars,*
- *the number of dollars at v_1 and v_{2n-1} is the same,*
- *the number of dollars at remaining odd vertices is zero,*
- *multiplying by α destabilizes all the vertices,*

then after stabilizing v_n will always have zero dollars.

Proof. It is easy to identify that the only configurations that satisfy the aforementioned conditions are d and e given as:

$$d = \begin{cases} 2, & v \equiv 0 \bmod 2, \\ 1, & v = 1, 2n-1, \\ 0, & \text{otherwise,} \end{cases} \qquad e = \begin{cases} 2, & v \equiv 0 \bmod 2, \\ 0, & \text{otherwise.} \end{cases}$$

The argument given below is based on the fact that if we start with a given configuration p, then there may be many different sequences of firings which are possible from p, but they all lead to the same outcome. So we define certain concrete configurations that can be reached starting from $\alpha \cdot d$ and $\alpha \cdot e$ which will lead to the desired result. First, we shall give the proof for configuration d.

In the initial round of firings, all vertices except the vertices with zero dollars fire and loose a dollar each. While the vertices with zero dollars gain 2 dollars each. Fire the vertices in the sequence $1, 2, \ldots, 2n-1$ a number of times till the following configuration is obtained:

$$m = \begin{cases} 4, & v = 0, \\ \alpha, & v = 1, 2n-1, \\ \alpha - 1, & v = 2, 2n-2, \\ 2, & \text{otherwise.} \end{cases}$$

The number of dollars at vertex n is 2. Also note that the number of dollars at the vertices 1 and $2n - 1$ is same. Similarly, the number of dollars at 2 and $2n - 2$ is the same. In fact, the number of dollars for the vertices occurring in the following pairs is same.

$$\{1, 2n - 1\}, \{2, 2n - 2\}, \{3, 2n - 3\}, \ldots, \{n - 1, n + 1\}.$$

This implies that a sequence of firings that destabilizes vertex 1 also destabilizes vertex $2n - 1$. Similarly, when vertex 2 destabilizes, vertex $2n - 2$ also destabilizes and so on. Also note that the configuration m is an unstable configuration. No matter what the sequence of firings is, n always gains a multiple of 2 dollars when its neighbors $n - 1$ and $n + 1$ fire. So when n stabilizes it will always give off 2 dollars to its neighbors when fired once. Hence, when n finally stabilizes it will be left with zero dollars, which leads to the desired result.

Now we give the proof for configuration e. Initially, the even numbered vertices fire only and contribute 2 dollars to the odd numbered vertices. After the first round of firings, the following configuration is obtained:

$$m = \begin{cases} \alpha - 3, & v \equiv 0 \bmod 2, \\ 2, & \text{otherwise.} \end{cases}$$

It is quite obvious that the above configuration is an unstable configuration. Therefore, the sequence of vertices $1, 2, \ldots, 2n - 1$ is fired till every vertex has two dollars. Following the sequence given as follows:

$$1, 2, 3, \ldots, 2n - 1, 0, 1, 2, 3, \ldots, 2n - 1, 0, 1, 2, 3, \ldots, 2n - 1,$$

we get, $\alpha \cdot e = e$, which surely has zero dollars at the vertex v_n. □

Theorem 2.11. *If n is an odd number, then the abelian sandpile group $SP(\hat{W}_n, q)$ is isomorphic to the direct product of two cyclic groups of order a_n and $2a_n$. In particular,*

$$SP(\hat{W}_n, q) \approx \frac{\mathbb{Z}}{a_n\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2a_n\mathbb{Z}},$$

where the cyclic groups are generated by the configurations d and g, respectively.

Proof. Let $\pi_{0,n}$ and π' denote the permutations of the rim vertices defined as follows:

$$\pi_{0,n}(v) = \begin{cases} 0, & \text{if } v = 0, \\ n, & \text{if } v = n, \\ +i, & \text{if } v = -i, \\ -i, & \text{if } v = 1, 2, \ldots, n - 1, \\ -(n - 2), & \text{if } v = n, \\ -(n - 1), & \text{if } v = +(n - 3), \end{cases}$$

$$\pi'(i) = \begin{cases} -(n - 2), & \text{if } i = n, \\ -(n - 1), & \text{if } i = +(n - 3), \\ i + 2, & i = \ldots, -5, -3, -1, 0, 1, 3, 5, \ldots, \\ i + 4, & i = -(n - 1), -(n - 3), \ldots, -4, +2, +4, \ldots, +(n - 5), \\ -(i - 2), & \text{otherwise.} \end{cases}$$

Every multiple of d is symmetrical with respect to $\pi_{0,n}$, that is, $(\alpha \cdot d)(v) = (\alpha \cdot d)(\pi_{0,n}(v))$. On the other hand, every multiple of g is symmetrical with respect to π', that is, $(\beta \cdot g)(v) = (\beta \cdot g)(\pi'(v))$. Since $\pi_{0,n}$ and π' generate a group which acts transitively on the rim, the only configurations which are symmetrical with respect to both permutations are those in which all the even numbered vertices have the same degree and all the odd numbered vertices have the same degree or every rim vertex has the same degree. So, if $\alpha \cdot d = \beta \cdot g$, then both configurations are in fact equal to the critical configuration $e = (2, 0, 2, 0, \ldots, 2, 0)$ because Lemma 13 ensures that the required critical configuration cannot be $(2, 1, 2, 1, \ldots, 2, 1) = f$, which completes the proof. □

Acknowledgments: Open Access funding was provided by the Qatar National Library. The authors are indebted to the anonymous referees for their valuable comments to improve the original version of this paper. Zahid Raza was supported by University of Sharjah under the grant number 1802144068 and MASEP research group.

References

[1] N. L. Biggs, *Chip-Firing and the critical group of a graph*, J. Algebraic Combin. **9** (1999), 25–45.
[2] D. Dhar, *Self-organized critical state of sandpile automaton models*, Phys. Rev. Lett. **64** (1990), no. 14, 1613–1616.
[3] P. Bak, C. Tang, and K. Wiesenfeld, *Self-organized criticality: an explanation for the 1/f noise*, Phys. Rev. Lett. **59** (1987), 381–384.
[4] J. Spencer, *Balancing vectors in the max norm*, Combinatorica **6** (1986), 55–65.
[5] W. Chen and T. Schedler, *Concrete and abstract structure of the sandpile group for thick trees with loops*, preprint, 2009. ArXiv:math/0701381.
[6] R. Cori and D. Rossin, *On the sandpile group of dual graphs*, European J. Combin. **21** (2000), no. 4, 447–459.
[7] A. Dartois, F. Fiorenzi, and P. Francini, *Sandpile group on the graph $\mathcal{D}_n$ of the dihedral group*, European J. Combin. **24** (2003), no. 7, 815–824.
[8] Y. P. Hou, C. W. Woo, and P. Chen, *On the sandpile group of the square cycle C_n^2*, Linear Algebra Appl. **418** (2006), no. 2–3, 457–467.
[9] Y. P. Hou, T. Lei, C. W. Woo, *On the sandpile group of the square cycle $K_3 \times C_n$*, Linear Algebra Appl. **428** (2008), no. 8–9, 1886–1898.
[10] L. Levine, *The sandpile group of a tree*, European J. Combin. **30** (2009), no. 4, 1026–1035.
[11] Z. Raza, S. Tariq, and M. T. Rahim, *Sandpile model on subdivided wheels $W_{n,l}$*, Util. Math. **105** (2017), 291–302.
[12] Z. Raza and S. Iqbal, *On the sandpile group of a family of graphs*, JP J. Alg. Num. Theo. Appl. **40** (2018), no. 3, 229–240.
[13] Z. Raza and S. A. Waheed, *On the sandpile model of modified wheels I*, Int. J. Nonlinear Sci. Numer. Simul. **15** (2014), no. 3–4, 207–213.
[14] J. Shen and Y. Hou, *On the sandpile group of $3 \times n$ twisted bracelets*, Linear Algebra Appl. **429** (2008), no. 8–9, 1894–1904.
[15] N. L. Biggs, *The critical group from a cryptographic perspective*, Bull. London Math. Soc. **29** (2007), 829–836.
[16] P. Chen and Y. Hou, *On the sandpile group of $P_4 \times C_n$*, European J. Combin. **29** (2008), no. 2, 532–534.
[17] P. G. Chen, Y. P. Hou, and C. W. Woo, *On the critical group of the Möbius ladder graph*, Australas. J. Combin. **36** (2006), 133–142.
[18] H. Christianson and V. Reiner, *The critical group of a threshold graph*, Linear Algebra Appl. **349** (2002), no. 1–3, 233–244.
[19] B. Jacobson, A. Niedermaier, and V. Reiner, *Critical groups for complete multipartite graphs and Cartesian products of complete graphs*, J. Graph Theory **44** (2003), 231–250.
[20] H. Liang, Y. L. Pan, and J. Wang, *The critical group of $K_m \times P_n$*, Linear Algebra Appl. **428** (2008), no. 11–12, 2723–2729.
[21] Z. Raza, *On the critical group of a family of graphs*, Novi Sad J. Math. **43** (2013), no. 1, 113–120.
[22] Z. Raza and N. Saleem, *The critical group of $C_m \vee P_2$*, Sci. Int. (Lahore) **24** (2012), no. 4, 333–336.
[23] Z. Raza and S. A. Waheed, *On the critical group of W_{4n}*, J. Appl. Math. Inform. **30** (2012), no. 5–6, 993–1003.
[24] E. Toumpakari, *On the sandpile group of regular trees*, European J. Combin. **28** (2007), 822–842.