What is Graph Theory ?

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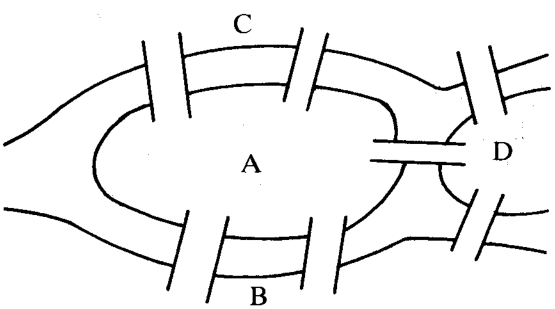
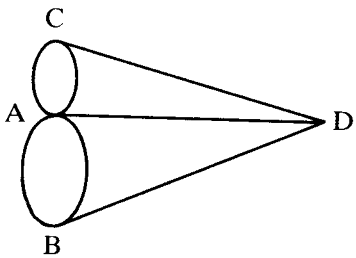
1. Historical Note

Graph theory has its roots in applied mathematics. The earliest recorded mention of the subject occurs in the works of Euler (1707 - 1782). He became the father of graph theory when in 1736 he settled a famous unsolved problem of his day called the Konigsberg Bridge problem. There were two islands linked to each other and to the banks of the Pragel River by seven bridges (Fig. 1). The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the starting point.



(Fig. 1)



(Fig. 2)

In proving that the problem is unsolvable Euler replaced each area by a point and each bridge by a line joining the corresponding points, thereby producing a "graph". Showing that the problem is unsolvable is equivalent to showing that the graph cannot be traversed in a certain way (Fig. 2).

In 1847 Kirchhoff developed the theory of trees in order to solve the system of simultaneous linear equation which gave the current in each branch and around each circuit of an electric network. He replaced an electric network by its corresponding combinatorial structure consisting only of points and lines without any indication of the type of electrical element represented by individual lines. In effect he replaced each electrical network by its underlying graph and showed that it is not necessary to consider every cycle in the graph of an electric network separately in order to solve the system of equations.

In 1857, Cayley discovered the important class of graphs called trees in the very natural setting of organic chemistry. He was engaged in enumerating the isomers of the saturated hydrocarbons C_nH_{2n+2} with a given number n of carbon atoms. He stated the problem abstractly as follows: find the number of trees with p points in which every point has degree 1 or 4. He did not immediately succeed in solving this and so he altered the problem until he was able to enumerate rooted tree (in which one point is distinguished from the others), trees with points of degree at most 4, and finally the chemical problem of trees in which every point has degree 1 or 4.

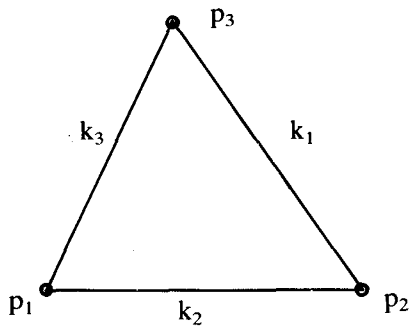
The four colour problem: The most famous problem in graph theory, which was solved only a few years ago, is the four colour problem. The problem was communicated to DeMorgan by Guthrie about 1850. This remarkable problem can be explained in five minutes by any mathematician to the so-called man in the street. At the end of the explanation, both will understand the problem but neither will be able to solve it. The problem states that any map on a plane or the surface of a sphere can be coloured with only four colours so that any two adjacent countries have the same colour. This problem is a problem in graph theory because every map yields a graph in which the countries (including the exterior region) are the points and two points are joined by a line whenever the corresponding countries are adjacent. Such a graph can be drawn in the plane without intersecting lines. Thus if it is possible to colour the points of every planar graph with four or fewer colours so that adjacent points have different colours, then the four colour problem will have been proved.

2. Basic Concepts and Definitions

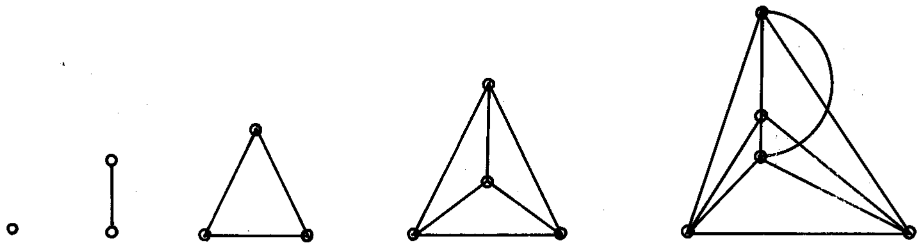
Let E and K be two disjoint sets, with an incidence relation such that each element of K is incident with either one or two elements of E. Then E and K with this incidence relation is called **graph** G = EUK. E is called the set of vertices and K the set of **edges** of G. Let $k \in K$ and $p, q \in E$. The following statements have the same meaning: i) an edge k is incident with p or p and q, ii) k joins p and q, iii) p and q are in k, iv) p and q are the end points of k, v) k meets p or p and q.

An edge, that is incident with one vertex (i.e. it joins the same two vertices) is called a **loop.** If two vertices are joined by more than one edge, then these are called **multiple edges.** A graph is said to be without loops and multiple edges if it possesses neither loops nor multiple edges. All graphs considered here shall be, if not specifically pointed out, without loops and multiple edges. An edge k is denoted by (p,q) if k joins the vertices p and q. Since the graph considered are without multiple edges it can be said that the edge k = (p,q). A graph is called **finite** if it has only finitely many verticles and edges. A graph, that has neither vertices nor edges is called the **empty graph.** Let G be a graph and p, q two vertices of G. Consider $k_v = (P_{v-1}, P_v)$, $v = 1, \ldots\ldots, n$ where $p_o = p$ and $p_n = q$. Then the graph which is made of the vertices $p_o, p_1, \ldots\ldots, p_n$ and the edges $k_1, \ldots., k_n$ is called a **path** from p to q in G with **length n,** where n is the number of its edges. The two vertices p and q are called the **end points** of the path. The path is called a **circuit** if p = q. A path of **length O,** is the path that joins a point p with itself. The two vertices p and q are said to be **connectable** if there is a path from p to q. A **component** of a graph G is made up of all vertices (and the edges incident on them) which can be connected by paths to a (fixed) vetex of G. A graph G is said to be **connected** if every pair of distinct vertices can be joined by at least one path. It is clear that any graph decomposes into disjoint, connected components. Let p and q be two connectable vertices in G. Then the **distance** between p and q in G is the minimum length of all paths from p to q. If p and q are not connectable, then the distance is said to be infinite. The distance between p and q is denoted by | p,q |. A **tree** is a connected graph without circuits. Two vertices p and q of G are called **adjacent** if they are joined with an edge of G (i.e. | p,q | = 1). Let E′ be a subset of vertices E of G. Let K′ be a subject of the set of edges K of G, such that all end of the edges of K′ are in E′ . Then: G′ = E′ UK′ is a graph. This graph is called a **partial** graph of G and is denoted by G′ ⊆ G. The graph G and the empty graph are partial graphs of G. There can be more than one partial graph (with E′ as a set of vertices) to the same subset E′ of E. If we take all edges in G with end points in E′ to be also in K′, the graph E′ UK′ is called a **subgraph** or more exact, the **subgraph of G which is spanned by E′.** This graph is denoted by G(E′).

Fig. 3



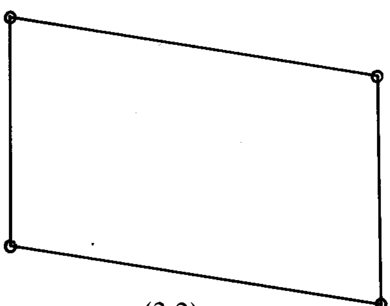
In the graph of Fig. 3, the graph with the vertices p_1, p_2, p_3 and the edges k_1, k_2 is a partial graph but not a subgraph of G. Two vertices p and q of G are called not adjacent if | p, q | > 1. Let k be a new edge (i.e. an edge which is not in G) which joins two not adjacent vertices in G. The graph which results from adding k to G, is denoted by G U k. A graph is called complete if it is not possible to add any edges to it. A complete graph is denoted by < n > where n is the number of its vertices. < n > is said to be the complete graph which is spanned by n vertices. The following graphs are complete graphs with n = 1, 2,...,5.



<1> <2> <3> <4> <5>

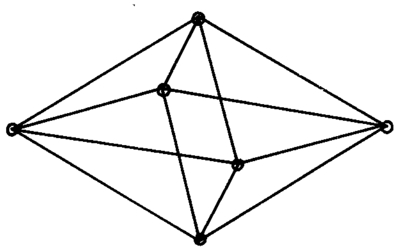
Fig. 4

Suppose that E is decomposed into disjoint subsets and that two vertices of E are adjacent if each of them belongs to a different subset of E. Let K be the set of all edges that join different vertices of E. Then the graph E U K is called a **bipartite** graph. If E is finite and the decomposition of E into disjoint subsets is: $E_1, E_2, \ldots E_n$ where the numbers $a_1, a_2, \ldots . a_n$ of vertices of $E_1, \ldots . E_n$ are then ordered such that: $a_1 \leqslant a_2 \leqslant \ldots . \leqslant a_n$ the bipartite graph is denoted by: $(a_1, a_2, \ldots . a_n)$. If $a_1 = a_2 = a_3 = \ldots . = a_n = 1$, then the bipartite graph is a complete graph with n vertices. The bipartite graph (a) is made up of only one vertex. (2,2) is a quadruple and (2,2,2) is an octader (see figures):



(2,2)

Fig. 5



(2,2,2)

Fig. 6

3. Some theorem and a method of proof in Graph Theory

We give now, a list of some theorems in graph theory. As an example of a method of proof, we give a proof of the five colours theorem.

Embedding and contraction of planar graphs

Definition: A graph is said to be planar if it can be embedded in the plane.

Definition: A graph is said is to be planar triangulated graph if the border of each region into which it divides the plane is a triangle.

Theorem: A graph is planar if it contains neither $< 5 >$ nor (3,3) as a partial graph.

Theorem: Suppose that T is a prime triangulated graph, e_1 and e_2 two arbitrary not adjacent vertices of T and e a vertex in T such that $| e, e_1 | > 1$ or (and) $| e, e_2 | > 1$. Then there is an edge k in T such that k is not incident on e_1 or e_2 and the graph which results from the contraction of k is again prime.

Basis of a graph:

Definition: Let g and G′ be two graphs with the set of vertices E and E′ respectively. Let $\mathcal{E}$ be a mapping from E onto E′. $\mathcal{E}$ is called homomorphic mapping from G onto G′ if the following hold:

1. $\forall$ $e' \in E'$ is $G(\mathcal{E}^{-1}(e'))$ a connected graph.

2. $|e_1', e_2'| = 1 \Rightarrow e_1 \in \mathcal{E}^{-1}(e_1')$ and $e_2 \in \mathcal{E}^{-1}(e_2')$ such that $| e_1, e_2 | = 1$.

Definition: G is called homomorphic to G′ in symbols: $G \succ G'$ if there is a homomorphic mapping from G onto G′.

Definition: A set S of (finite) graphs is called homomorphism class if: for each two graphs G_1 and G_2 with $G_1 \in S$ and $G_1 \succ G_2$ it follows that $G_2 \in S$. Let A be a fixed (finite) graph. We denote the set of all (finite) graphs, which are not homomorphic to A by H (A). That is: $H(A) = \{G/G \succ A \text{ is false}\}$. It follows easily that:

Theorem: H(A) is a homomorphic class.

Definition: A graph $\overline{G} \in H(A)$ is called a maximal graph of H(A) if for each two different vertices p,q of $\overline{G}$ with $| p,q | > 2$ by the addition of a new edge k such that k joins p and q it follows that $G \cup k$ does not belong to H(A). The set of all maximal graphs of H(A) is denoted by H(A).

Definition: G is called prime graph if $G \neq G_1 \cup G_2$ with $G_1 \cap G_2 = < n >$.

Definition: A prime graph in $\overline{H}(A)$ is called a basis element of H(A) (or shortly of A). The set of all basis elements of H(A) is called the basis of H(A) (in short of A). It is denoted by B(A).

Theorem: Let A be a graph with basis B(A). Consider $G^* = \bigcup_{u=1}^{m} G_U$ such that $G_u \cap \bigcup_{v=1} G_V = < n >$ (u = 2, ...,m) and $G_U \in B(A)$ for u = 1,....,m. Then each $G^* \in H(A)$ is contained in at least one of such G^* as a partial graph.

It has been possible to characterise the basis elements of the following graphs: (1,1,1,3), (1,1,2,2), (1,2,3), L, 5, (3,3) and a part of the planar basis elements of (1,1,1,1,2).

Paths and circuits:

Definition: A connected graph is called Eulerian if there exists a closed path which includes every edge of G.

Definition: If there exists a closed path which passes exactly once through each vertex of a graph G, then G is said to be Hamiltonian.

Theorem: A connected graph G is Eulerian if and early if the degree of every vertex of G is even

Definition: The degree (or valency) of a vertex v of G is the number of edges incident to v.

Theorem: If G is a graph with n vertices and the valency $\alpha(v) \geq \frac{1}{2}n$ for every vertex v, then G is Hamiltonian.

Theorem: A graph G is bipartite if every circuit of G has even length.

The colouring of graphs:

Definition: If G is a graph then G is said to be k - colourable if to each of its vertex we can assign one of k colours in such a way that no adjacent vertices have the same colour.

Definition: A graph G is said to have a chromatic number ∅ (G) if ∅ (G) is the minimum number of colours which can be used to colour G.

Hadwiger's conjecture: H_n: $\varnothing(G) \geq n \Rightarrow G \succ < n >$.

It has been proved that:

i H_n is true for $n \leqslant 4$ (trivial)

ii H_5 is equivalent with the 4 colour problem

iii $H_{n+}{}^{1} \succ H_n$

iv $\varnothing (G) \geqslant 5 \Rightarrow G \succ < 5 > - k$

v $\varnothing (G) \geqslant 6 \Rightarrow G \succ < 6 > - k$

The 5-colour theorem: Every planar graph is 5 colourable.

Proof. (by mathematical induction)

The theorem is true for all graphs with number of vertices $\leqslant 5$. Hence we can assume that it is true for all planar graphs with number of vertices $< n$. Let G be a planar graph with n vertices. Since G is planar it must have a vertex p with valency $\propto (p) \leqslant 5$. Let $\propto (p) \leqslant 4$. Let G-p be the graph which results from G after deleting from it the vetex p together with the edges that are incident on p in G. By induction hypothesis the graph G-p can be coloured with 5 colours. p is adjacent to at most 4 vertices and these get by the colouring of G-p at most 4 colours. Hence one of the 5 colours remains which can be used to colour p in G.

This leaves only the case to consider in which the valency$\propto (p) = 5$ and five colours are used for the vertices of Gadjacent with p. Let $p_1, p_2, \ldots. p_5$ be the end points different from p of the five edges which meet p. Since G is planar there are two of these say p_1, and p_2 which are not adjacent in G. Let G' be the graph which results from G after deleting the edges (p,p_3), (p,p_4), (p,p_5) and contracting the path p_1pp_2 (of length 2) to a single point (i.e. p_1 is identified with p_2). G' can be coloured with 5 colours. By this colouring of G' : the two vertices which are different and adjacent in G' are coloured differently; the two vertices p_1 and p_2 both get the same colour since they are identified with each other in G'. But this implies that $p_1, \ldots. p_5$ get at most 4 colours. There remains again one of the five colours which can be used to colour p. Hence G can be coloured with five colours.

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