FIXED POINT THEOREMS IN TOPOLOGICAL SPACES

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ABSTRACT

A couple of theorems for a pair of self-maps on a topological space have been obtained using which certain fixed point theorems in the literature for pairs of maps satisfying contractive type conditions have been deduced.

In this paper our objective is to bring under a single banner various fixed point theorems for a single or a pair of self-maps on a metric or a or a topological space satisfying contractive type conditions under constrains such as the satisfaction of certain inequations by the independent variables. We have partially fulfilled our objective by generalising the results of Sastry, Naidu, Rao & Rao [4], and Sastry and Babu [2 & 3] (Theorems 1 & 2) while the latter, in turn, are generalisations of the results of Sehgal [5] and those of Khan, Swaleh & Sessa [1].

Throughout this paper unless otherwise stated, IN stands for the set of all natural numbers and IR⁺ for the set of all non-negative real numbers.

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**Fixed point theorems in topological spaces**

**Theorem 1**: Let $X$ be a topological space, $\rho$ be a bounded below real valued function on $X \times X$ and $f$ and $h$ be self-maps on $X$. Suppose that there is an $x_0$ in $X$ such that $\{ht^n x_0\}_{n=1}^{\infty}$ has a cluster point $z$ in $X$. Let $A = \{ht^n x_0 : n \in \mathbb{N}\}$ and $E$ be a subset of $X$. Suppose that $f$, $h$ and $fh$ are sequentially continuous at $z$, $\rho$ is sequentially continuous at $(z, f z)$ and $(h z, fh z)$, and that

$$\rho(h x, fh x) < \rho(x, fx) \quad \ldots (1)$$

for all $x$ in $E$. Then either $A \cap E = \emptyset$ or $z \in E \cap \phi$.

**Proof**: Suppose that $A \cap E = \emptyset$. Then $ht^n x_0 \in E$ for all $n \in \mathbb{N}$. Hence $\rho(ht^n x_0)_{n=1}^{\infty}$ is a monotonically decreasing sequence of real numbers. It is bounded below since $\rho$ is bounded below on $X \times X$. Hence it converges to a real number, say, $\alpha$. Since $z$ is a cluster point of $\{ht^n x_0\}_{n=1}^{\infty}$, there exists a convergent subsequence $\{ht^n x_0\}_{n=1}^{\infty}$ of it which converges to $z$. Since $f$, $h$ and $fh$ are sequentially continuous at $z$, the sequences $\{fht^n x_0\}_{n=1}^{\infty}$ and $\{fht^n x_0\}_{n=1}^{\infty}$ converge to $\rho(z, f z)$ and $\rho(z, f z)$ and $\rho(h z, fh z)$ respectively. But both of these sequences are subsequences of the convergent sequence $\rho(ht^n x_0, fht^n x_0)$_{n=1}^{\infty}$ with limit $\alpha$. Hence $\rho(h z, fh z) = \alpha = \rho(z, f z)$. Since inequality (1) is true for $x \in E$, $z \notin E$.

**Corollary 1**: (Theorem 2.1 of Sastry, Naidu, Rao & Rao [4]) Let $X$ be a topological space, $F: X \times X \rightarrow IR^*$ be symmetric and sequentially continuous with $F(x, x) = 0 \forall x \in X$. Let $f$ and $g$ be self-maps on $X$ such that

1. $F$ and $gf$ are sequentially continuous on $X$
2. $F(fx, gy) < \max \{F(x, y), F(fx, fx), F(fy, gy)\}$ for all distinct $x, y$ in $X$ and
3. $\{gf^n x_0\}_{n=1}^{\infty}$ has a cluster point for some $x_0$ in $X$.

Then either $f$ or $g$ has a fixed point in $X$.

**Proof**: From condition 2 and the symmetry $F$ we have $F(gfx, gfx) < F(fx, gfx) + F(fx, gx)$ if $gfx \neq fx$ and $fx$ and $F(fx, gfx) < F(fx, fx)$ if $x \neq fx$ so that $F(fgx, gfx) < F(fx, fx)$ if $gfx \neq fx$ and $fx \neq x$. On taking $\rho = F$, $h = gf$ and $E = \{x \in X : fx \neq x \}$ and $gfx \neq fx \}$ in Theorem 1 it follows that either $f$ has a fixed point in $B$ or $g$ has a fixed point in $f(B)$, where $B$ is the closure of $A$ in $X$.

**Remark 1**: In Corollary 1 the condition: "$F(x, x) = 0 \forall x \in X$" is redundant.

**Corollary 2**: Let $X$ be a $T_1$-topological space, $\rho$ be a nonnegative real valued sequentially continuous function on $X \times X$ with $\rho(x, x) = 0 \forall x \in X$, $f$ be a self-map on $X$ and $x_0 \in X$ be such that $\{fx_0\}_{n=1}^{\infty}$ has a cluster point $z$ in $X$. Suppose that $f$ and $f^2$ are sequentially continuous at $z$ and that

$$\rho(fx, fy) < \max \{\rho(x, y), (\max \{\rho(x, fx), \rho(y, fy)\} + \rho(fx, fy)\)^{1/2} \quad \ldots (2)$$

for all distinct $x, y$ in $X$. Then $z$ is the only fixed point of $f$ in $X$.

**Proof**: On taking $y = fx$ in inequality (2) we obtain $\rho(fx, ffx) < \rho(x, fx) \forall x \in X$ with $fx \neq x$. On taking $h = f$ and $E = \{x \in X : fx \neq x \}$ in Theorem 1 we conclude that either $f^{m}x_0 = f^{m}x_0$ for some $m \in \mathbb{N}$ or $f^{m}x_0 = f^{m}x_0 \forall n \geq m$ so that in view of the hypothesis that $z$ is a cluster point of $\{fx_0\}_{n=1}^{\infty}$ and $X$ is $T_1$ it follows that $z = fm x_0$. Hence $z$ is a fixed point of $f$ in either case. From inequality (2) it is evident that $f$ has at most one fixed point in $X$. Hence $z$ is the only fixed point of $f$ in $X$.

**Corollary 3**: (Theorem 2.2 of Sastry & Babu [2]) Let $(X, d)$ be a metric space, $f$ be a continuous self-map on $X$ and $\Psi: IR^* \rightarrow IR^*$ be a continuous map with $\Psi(t) = 0$ if $t = 0$. Suppose that for some $x_0 \in X$ the sequence $\{fx_0\}$ has a cluster point $z$ in $X$ and that there are nonnegative constants $a, b, c$ such that $a + b \leq 1, a + c \leq 1$ and $\psi(dx, fy) < a \psi(dx, y) + \psi(fx, fy) + c\psi(fx, fy) \psi(fx, fy)^{1/2}$ for all distinct $x, y$ in $X$, then $z$ is the unique fixed point of $f$ in $X$.

**Proof**: Let $\rho = \psi \circ \omega$. Then $\rho$ is a nonnegative real valued continuous function on $X \times X \times X$ with $\rho(x, x) = 0 \forall x \in X$, and for all distinct $x, y$ in $X$ we have

$$\rho(fx, fy) < \rho(x, xy) + \max \{\rho(fx, fx), \rho(\Psi(fx, fy), \Psi(fx, fy)) + c[\rho(fx, fy)]^{1/2}$$

$$\leq \rho(x, xy) + b \max \{\rho(x, fx), \rho(\Psi(fx, fy)) + c[\rho(\Psi(fx, fy), \Psi(fx, fy))^{1/2}$$

$$\leq \rho(x, xy) + (1 - a) \max \{\rho(fx, fx), \rho(xy, fy)\} + (1 - a)$$

$$\rho(fx, fy) \psi(dx, fy) \psi(fx, fy)^{1/2}$$

$$\leq \max \{\rho(fx, fy), \max \{\rho(fx, fx), \rho(xy, fy)\} + [\rho(fx, fy)]^{1/2}$$

$$\leq \max \{\rho(fx, fy), \max \{\rho(fx, fx), \rho(xy, fy)\} + \rho(fx, fy)]^{1/2}$$

$$\leq \max \{\rho(fx, fy), \max \{\rho(fx, fx), \rho(xy, fy)\} + \rho(fx, fy)]^{1/2}$$

$$(\because 0 \leq a \leq 1).$$

Hence Corollary 3 follows from Corollary 2.

**Corollary 4**: (Theorem 2.1 of Sastry & Babu [3]) Let $f$ be a continuous self-map on a metric space $(X, d)$ such that for
some \( x_0 \to X \) in the sequence \( \{f(x)\} \) has a cluster point \( z \) in \( X \),
and \( p: IR^* \to IR^* \) be a continuous map with \( p(t) = 0 \) iff \( t = 0 \).
Suppose that

\[
\psi(d(fx, fy)) < \max \\{\psi(d(x, y)), \psi(d(x, fx)), \psi(d(y, fy))\}
\]

\( \cdots (3) \)

for all distinct \( x, y \) in \( X \). Then \( z \) is the unique fixed point of \( f \) in \( X \).

**Proof:** Corollary 4 follows from Corollary 2 on taking \( p = \psi \).

**Note 1:** As observed by Sastry & Babu [3] Theorem 5 of Sehgal [5] follows from Corollary 4 on taking \( p = \psi \).

**Theorem 2:** Let \( X \) be a topological space, \( \rho \) be a bounded below real valued function on \( X \times X \) and \( f \) and \( h \) be self-maps on \( X \). Suppose that there is an \( x_0 \) in \( X \) such that \( \{h^n(x_0)\}_n \) has a cluster point \( z \) in \( X \). Let \( A = \{h^n(x_0) : n \in \mathbb{N}\} \) and \( E_1, E_2 \) be subsets of \( X \). Suppose that \( f \) and \( h \) are sequentially continuous at \( z \), \( \rho \) is sequentially continuous at \( (z, fz) \) and \( (fz, hz) \),

\[
\rho(fz, hz) = \rho(z, fz)
\]

\( \forall x \in E_1 \) and

\[
\rho(hz, fz) < \rho(fz, hz)
\]

\( \forall x \in E_2 \). Then either \( A = \{h^n(x_0) \}_n \) is nonempty or \( z \notin E_1 \).

**Proof:** Suppose that \( A = \{h^n(x_0) \}_n \) is empty. Then \( h^n \) is a continuous function on \( X \times X \) for all \( n \in \mathbb{N} \). Hence from inequalities (4) and (5) we have

\[
\rho(h^n(x_0), h^m(x_0)) < \rho(h^n(x_0), h^m(x_0)) \forall n, m \in \mathbb{N}.
\]

Hence \( \rho(h^n(x_0), h^m(x_0)) \sim^\omega \) is a monotonically decreasing sequence of real numbers. It is bounded below since \( \rho \) is bounded below on \( X \times X \). Hence it converges to a real number, say, \( \alpha \). Hence \( \rho(h^n(x_0), h^m(x_0)) \sim^\omega \) also converges to \( \alpha \). Since \( z \) is a cluster point of \( \{h^n(x_0)\}_n \), there exists a convergent subsequence \( \{h^m(x_0)\}_m \) of it which converges to \( z \). Since \( f \) and \( h \) are sequentially continuous at \( z \), the sequences \( \{f(h^n(x_0))\}_n \) and \( \{h^n(x_0)\}_n \) converge respectively to \( fz \) and \( hz \). Since \( \rho \) is sequentially continuous at \( (z, fz) \) and \( (fz, hz) \) it follows that the sequences \( \{\rho(h^n(x_0), f(h^n(x_0)))\}_n \) and \( \{\rho(f(h^n(x_0), h^n(x_0)))\}_n \) converge to \( \rho(z, fz) \) and \( \rho(fz, hz) \) respectively. But the first one is a subsequence of \( \{\rho(h^n(x_0), h^n(x_0)))\}_n \) which converges to \( \alpha \) and the second one is a subsequence of \( \{\rho(f(h^n(x_0), h^n(x_0)))\}_n \) which also converges to \( \alpha \). Hence \( \rho(fz, hz) = \alpha = \rho(z, fz) \).

Since inequality (4) is true for all \( x \in E_1 \), \( z \notin E_1 \).

**Remark 2:** Corollary 1 with the additional conclusion (as stated in [4]) that any cluster point of \( \{(gf)^n(x_0)\}_n \) is a fixed point of \( f \) if \( x_0 \notin x_{n+1} \forall n \in \mathbb{N} \), where \( x_{n+1} = f(x_n) \) and \( x_0 = x_{n+1} \) (\( n = 0, 1, 2, \ldots \)), can be deduced from Theorem 2 by taking \( p = \psi, h = gf, E_1 = \{x \in X: x \neq x_0\} \) and \( E_2 = \{x \in X: gx \neq f(x)\} \).

**Corollary 5:** (Theorem 3.1 of Sastry & Babu [3] Let \( f \) and \( g \) be self-maps on a metric space \((X, d)\) such that for some \( x_0 \in X \) the sequence \( \{(gf)^n(x_0)\} \) has a cluster point \( z \) in \( X \) and \( f \) and \( g \) are continuous at \( z \). Let \( p: IR^* \to IR^* \) be a continuous map with \( p(t) = 0 \) iff \( t = 0 \). Suppose that

\[
\psi(d(fx, gy)) < \max \\{\psi(d(x, y)), \psi(d(x, fx)), \psi(d(y, fy))\}
\]

\( \forall x, y \in X \). Then either \( f \) or \( g \) has a fixed point in \( X \). If both \( f \) and \( g \) have fixed points, then their fixed point sets are singletons and are equal.

**Proof:** Let \( \rho = \psi \cdot d \). Then \( \rho \) is a symmetric nonnegative real valued continuous function on \( X \times X \) with \( \rho(x, x) = 0 \) for all \( x \in X \) and

\[
\rho(fx, gy) < \min \{\rho(x, y), \rho(x, fx), \rho(y, gy)\}
\]

\( \forall x, y \in X \). On taking \( y = fx \) in inequality (6) we obtain

\[
\rho(fx, gfx) < \rho(fx, fx) \quad \text{if} \quad fx \neq x
\]

and on replacing \( x \) with \( gfx \) and \( y \) with \( fx \) in inequality (6) and on using the symmetry of \( \rho \) we obtain

\[
\rho(gfx, gfx) < \rho(fx, gfx) \quad \text{if} \quad gfx \neq x
\]

On taking \( h = gf, E_1 = \{x \in X: x \neq x_0\} \) and \( E_2 = \{x \in X: gfx \neq f(x)\} \) in Theorem 2 we can conclude that either \( f \) or \( g \) has fixed point in \( X \). The second conclusion of the corollary is evident from inequality (6).

**Note 2:** Corollaries 4 & 6 are the generalisations obtained by Sastry & Babu [3] for a theorem of Khan, Swaleh & Sessa [1].

**Remark 3:** In Corollaries 3, 4 & 5 the condition: "\( \psi(t) = 0 \) iff \( t = 0 \)" may be replaced by the weaker condition: "\( \psi(t) = 1 \)".

Sastry & Babu [3] posed the following

**Problem:** Find an example of a metric space \( X \), a continuous function \( \psi: IR^* \to IR^* \) which vanishes only at zero and a self-map \( f \) on \( X \) with a fixed point \( z \) satisfying inequality (3) for all distinct \( x, y \) in \( X \) such that for no \( x \) in \( X \) the sequence \( \{f(x)\} \) has a convergent subsequence.

The following example provides an answer to the above problem with \( \psi(t) = t \forall t \in IR^* \).

**Example:** Let \( X = \{0, 1, 2, \ldots \} \). Define \( d \) on \( X \times X \) as

\[
d(x, y) = \frac{1}{2^{|x-y|}}
\]
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d(x, y) = 0 if x = y, d(0, n) = d(n, 0) = 1 + ∀ n ∈ IN
and d(m, n) = 1 + \frac{1}{2^n} + \frac{1}{2^n} if m, n ∈ IN are distinct.

Define f: X → as f0 = 0 and fn = n + 1 ∀ n ∈ IN.

Then d is a metric on X, d (fx, fy) < d(x, y)

for all distinct x, y in X, 0 is the only fixed point of f in X and
for no x in X\{0}the sequence {fx} has a convergent subsequence.

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REFERENCES


