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REMARKS ON S-CLOSED SPACES

M. E. ABD EL-MONSEF and A. M. KOZAE

Department of Mathematics, Faculty of Science

Tanta University, Tanta, Egypt

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ABSTRACT

In this paper, several characterizations for s-closed spaces are obtained using regular semiclosed sets and some sets having two properties of near openness and near closedness in the same time. Images of s-closed spaces under some noncontinuous mappings are investigated. The relations between s-closedness and near compactness, co-compactness, almost co-compactness, light compactness, mild compactness are obtained. S-closed subsets relative to a semi-T_2-spaces are also discussed.

INTRODUCTION AND PRELIMINARIES

Throughout this paper, X and Y mean topological spaces on which no separation axioms are defined unless otherwise stated explicitly. X is s-closed Thompson (1975) if any semi-open cover of X has a finite subfamily, the closures of whose members cover K. For a subset $A \subset X$, A^-, A^o and A^c denotes, the closure, the interior and the complement of A, respectively. An open set A is co-open (Mashhour and Atia 1974), if A^- is open and the complement of a co-open set is Ic-closed. (Mashhour and Atia 1974). A subset $S \subset X$ is regular open, R. o, (resp. α-open, α. o. (Niastad 1965), semi-open, s. o. (Levine 1963), preopen, p. o. (Mashhour *et. al* 1982), ß-open, ß. o. (Abd El-Monsef *et al.* 1982) if $S = S^o$ (resp. $S \subset S^{o-o}$, $S \subset S^{o-}$, $S \subset S^{-o}$, $S \subset S^{-o-}$) each of these sets is called nearly open. The complement of a R. o. (resp. α. o., s. o., p. o., ß. o.) is called regular closed (resp. α –closed, semi closed, preclosed, ß–closed). Each of these sets is called nearly closed. The symbol RO (X) (resp. αO (X), SO (X), PO (X), ßO (X)) indicates the family of all R. o. (resp. α. o., s. o., p. o., ß. o.) subsets of X. A subset A of X is regular semi–open, R. s. o. (Cameron 1983) if there exists a regular open set $U \subset X$ such that $U \subset A \subset U^-$. A subset A of X is regular semi closed, R, s. c. if there exists a regular closed set $F \subset X$ such that $F^o \subset A$ F. X is an extremally disconnected space if the closure of every open subset of X is open. X is H (i) if any open cover of X has a finite subfamily, the closures of whose members cover X. X is co–compact if any co–open cover of X has a finite subcover. X is almost co–compact if any co–open cover of X has a finite subfamily the closures of whose members cover X. X is lightly compact if any countable open cover of X has a finite subfamily the closures of whose members cover X. X is mildly compact if any countable open cover of X has a finite subfamily the interiors of the closures of whose members cover X. X is a semi–$\acute{T}_2$–space if for each $x, y \in X, x \neq y$, there exist $U, V \in SO(X), x \in U$, $y \in V$ such that $U^- \cap V^- = \emptyset$. A function $f : X \longrightarrow Y$ is semi–continuous (Levine 1963) if $f^{-1}(U) \in SO(X)$ for every U is open in Y.

CHARACTERIZATIONS FOR S–CLOSED SPACES

Theorem 2.1. The following statements are equivalent for a space X.

(1) X is S–closed.

(2) For any family $\{ F_i : i \in I \}$ of regular semi–closed sets of X for which $\bigcap_{i \in I} F_i = \emptyset$, there exist a finte subfamily $I_o \subset I$ such that $\bigcap_{i \in I} F^o = \emptyset$.

(3) $\bigcap_{i \in I} B_i \neq \emptyset$, where $\{ B_i : i \in I \}$ is a family of regular semi–closed subsets of X for which $\bigcap_{i \in I_o} B^o \neq \emptyset$ for a finite subfamily I_o of I.

(4) $\bigcap_{i \in I} A_i \neq \emptyset$, where $\{ A_i : i \in I \}$ is a family of semiclosed subsets of X for which $\bigcap_{i \in I_o} A^o \neq \emptyset$ for a finite subfamily I_o of I.

(5) $\bigcap_{i \in I} R_i \neq \emptyset$, where $\{ R_i : i \in I \}$ is a family of regular open subsets of X for which $\bigcap_{i \in I_o} R_i \neq \emptyset$ for a finite subfamily I_o of I.

(6) For any family $\{ F_i : i \in I \}$ of ß–closed and semi–open subsets of X for which $\bigcap_{i \in I} F_i = \emptyset$, there exists a finite subfamily I_o of I such that $\bigcap_{i \in I_o} F_o = \emptyset$.

(7) $\bigcap_{i \in I_o} F_i \neq \emptyset$, where $\{ F_i : i \in I \}$ is is a family of ß–closed and semi–open subsets of X for which $\bigcap_{i \in I_o} F^o \neq \emptyset$ for a finite subfamily I_o of I.

(8) Any ß–open and α –closed cover of X has a finite subcover.

(9) Any ß–open and semi closed cover of X has a finite proximate subcover.

Proof. The pattern of the proof will be 1 $\leftrightarrow$ i, $i \in \{ 2, 3, 4, 5, 6, 7, 8, 9 \}$.

1 $\longrightarrow$ **2.** Let $\{ F_i : i \in I \}$ be a regular semi closed family of subsets of X for which $\bigcap_{i \in I} F_i = \emptyset$, then $\bigcup_{i \in I} F_i^c = X$. Thus, $\{ F^c : i \in I \}$ is a regular semi–open cover of X which is s–closed, then there exists a finite subfamily I_o of I such that $X = \bigcup_{i \in I_o} F^{c-} = \bigcup_{i \in I_o} F_i^{oc} = (\cap F_i^o)^c$. Hence, $\bigcap_{i \in I} F_i^o = \emptyset$.

2 $\longrightarrow$ **1.** Let $\{ U_i : i \in I \}$ be a regular semi–open cover of X, then $X = \bigcup_{i \in I} U_i$ and $\bigcap_{i \in I} U_i^c\ \emptyset$. Thus, $\{ U_i^c : i \in I \}$ is a family of regular semi closed subsets of X for which $\bigcap_{i \in I} U_i^c = \emptyset$, then there exists a finite subfamily I_o of I such that $\bigcap_{i \in I_o} (U^c)^o_= \emptyset$. Since $\bigcap_{i \in I_o} (U_i^c)^o = \bigcap_{i \in I_o} U_i^{-c} = (\bigcup_{i \in I_o} U_i^-)^c = \emptyset$. Then, $X = \bigcap_{i \in I_o} U_i^-$ and X is s–closed.

1 $\longrightarrow$ **3.** Suppose the inverse, i. e., $\bigcap_{i \in I} B_i = \emptyset$, then $\bigcup_{i \in I} B_i^c = X$ and so,

$\{ B^c : i \in I \}$ is a regular semi–open cover of X which is s–closed, then there exists a finite subfamily I_o of I such that $X = \bigcup_{i \in I_o} B_i^{c-} = \bigcap_{i \in I_o} B_i^{oc} = (\bigcap_{i \in I_o} B_i^o)^c$. Thus, $\bigcap_{i \in I} B_i^o = \emptyset$, a contadiction. Hence, $\bigcap_{i \in I} B_i \neq \emptyset$.

3 $\longrightarrow$ **1.** Assume the inverse, i. e., X is not s–closed, then there exists a regular semi–open cover $\{ U_i : i \in I \}$ which does not have any finite proximate subcover, thus $\bigcup_{i \in I_o} U^- \neq X$ for any finite subfamily I_o of I. So, $\emptyset \neq \bigcap_{i \in I} U^{-c} = \bigcap_{i \in I_o} U^{co} \subset \bigcap_{i \in I_o} U_i^c = (\bigcup_{i \in I_o} U_i)^c \subset (\bigcup_{i \in I} U_i)^c$. Hence, $X \neq \bigcup_{i \in I} U_i$, a contraduction. Therefore, X is s–closed.

1 $\longrightarrow$ **4.** Using Thompson's definition for s–closedness and by the same manner of 1 $\leftrightarrow$ 3, the result follows.

1 $\longrightarrow$ **5.** Assume the inverse, i. e., $\bigcap_{i \in I} R_i = \emptyset$, then $\bigcup_{i \in I} R_i^c = X$ and $\{ R^c : i \in I \}$ is a regular closed cover of X which is s–closed. So, there exists a finite subfamily I_o of I such that $X = \bigcup_{i \in I_o} R_i^c = (\bigcap_{i \in I_o} R_i)^c$ implies $\bigcap_{i \in I_o} R_i = \emptyset$, a contradiction. Hence, $\bigcap_{i \in I} R_i \neq \emptyset$.

5 $\longrightarrow$ **1.** Assume the inverse, i. e., X is not s–closed, then there exists a regular closed cover $\{ U_i : i \in I \}$ which does not have any finite subcover. So, for any finite subfamily I_o of I, $\bigcup_{i \in I_o} U_i \neq X$ which implies $\bigcap_{i \in I_o} U_i^c \neq \emptyset$. then, $\bigcap_{i \in I} U_i^c \neq \emptyset$ implies $\bigcup_{i \in I} U_i \neq X$, a contradiction. Hence, X is s–closed.

1 $\longrightarrow$ **6.** Since each ß–closed and semi–open set is semi closed (Abdel –Mosef *et. al* 1982), the result follows.

1 $\longrightarrow$ **7.** Obvious.

1 $\longrightarrow$ **8.** Using the fact that "any ß–open and α –closed set is regular closed", the result is obtained.

1 $\longrightarrow$ **9.**Obvious.

Theorem 2.2. If X is an s-closed space, then any preopen cover of X has a finite proximate subcover.

Proof. Let $\{ P_i : i \in I \}$ be a preopen cover of X, then for every $i \in I$, we have $P_i \subset P_i^{-o}$ and so, $P_i^- \subset P_i^{-o-}$, thus $\{ P_i^- : i \in I \}$ is a semi-open cover of X which is s-closed. Then there exists a finite subfamily I_o of I such that $X = \bigcup_{i \in I} P_i^-$.

Corollary 2.1. If X is s-closed, then any preclosed family $\{ P_i : i \in I \}$ for which $\bigcap_{i \in I} P_i = \emptyset$ has a finite subfamily $\{ P_i : i \in I_o \}$, $I_o \subset I$ such that $\bigcap_{i \in I_o} P_i = \emptyset$.

Corollary 2.2. If X is s-closed, then $\bigcap_{i \in I} P_i^o \neq \emptyset$, where $\{ P_i : i \in I \}$ is a family of preclosed subsets of X for which $\bigcap_{i \in I_o} P_i^o \neq \emptyset$ for any finite subfamily I_o of I.

Corollary 2.3. If X is s-closed, then any regular open cover of X has a finite proximate subcover.

Corollary 2.4. If X is s-closed, then $\bigcap_{i \in I} R_i \neq \emptyset$, where $\{ R_i : i \in I \}$ is any family of regular closed subsets of X for which $\bigcap_{i \in I_o} R_i^o \neq \emptyset$, for any finite subfamily I_o of I.

Corollary 2.5. If X is s-closed, then for any regular closed family $\{ F_i : i \in I \}$ for which $\bigcap_{i \quad I} F_i = \emptyset$, there exists a finite subfamily I_o of I such that $\bigcap_{i \in I_o} F^o = \emptyset$.

Remark The authors need a counters example illustrates that the converse of Theorem 2.2 is not true.

Theorem 2.3 For a space X having the property that the interiors of the members of any cover of X is also a cover of X, X is s-closed if any preopen cover has a finite proximate subcover.

Proof. Let $\{ U_i : i \in I \}$ be a semi–open cover of X, then $U_i \subset U_i^{o-}$ for each $i \in I$. So, $U_i^o \subset U_i^{o-o}$ and $U_i^o \in PO(X)$. Thus, $\{ U_i^o : i \in I \}$ is a preopen cover of X, then there exists a finite subfamily I_o of I such that $X = \bigcup_{i \in I_o} U_i^{o-}$. Since, $U_i^{o-} \subset U_i^-$, then $X = \bigcup_{i \in I} U_i^-$ and X is s–closed.

3 – S–CLOSED SETS RELATIVE TO A SPACE AND S–CLOSED SUBSPACES.

Theorem 3.1. A semi–closed open subset of an s–closed Space X is an s–closed subspace of X.

Proof. Follows from Corollary 3.2 in (Noiri, 1978) since every semi–closed and open set is regular open.

Theorem 3.2. The interior of a semi–open subset A is an s–closed subspace of X iff A^o is s–closed relative to X.

Proof. Follows from Theorem 1.2 in Noiri (1977).

Theorem 3.3. The closure of a preopen set $A \subset X$ is an s–closed subspace of X if A^- is s–closed relative to X.

Proof. Follows from Corollary 3.5 and Theorem 3.5 of Noiri, (1978) since the closure of a preopen set is regular closed.

Theorem 3.4. Let a be an s–closed subset relative to a semi $\acute{T}_2$–space X and $p \in X-A$, then there exist a semi–open set V and a closed set U such that $p \in U, A \subset V$ and $U \cap V = \emptyset$.

Proof. Let $a \in A$, then there exist $G_a, F_a \in SO(X)$ such that $p \in G_a$, $a \in F_a$ and $G_a^- \cap F_a^- = \emptyset$ because X is semi–$\acute{T}_2$. Thus, $\{ F_a : a \in A \}$ is a semi–open cover of A which is s–closed relative to X. Then there exists a family

$\{ F_{a_1}, F_{a_2}, \ldots, F_{a_n} \}$ such that $A \subset \bigcup_{i=1}^{n} F^-_{a_i} = W^- = V$, where $W = \bigcap_{i=1}^{n} F_{a_i}$ is a semi-open set, since it is the union of semi-open sets. Thus, $V \subset V^{o-}$ is a semi-open set containing A. Also, for every $a_i \in A$, $i \in \{1, 2, \ldots, n\}$, there exist $F_{a_i}, G_{a_i} \in SO(X)$, $p \in G_{a_i}$ and $a_i \in F_{a_i}$ such that $G^-_{a_i} \cap F^-_{a_i} = \emptyset$, then $(\bigcap_{i=1}^{n} G^-_{a_i}) \cap (\bigcup_{i=1}^{n} F^-_{a_i}) = \emptyset = \bigcap_{i=1}^{n} G^-_{a_i} \cap V$. Put $U = \bigcap_{i=1}^{n} G^-_{a_i}$, then U is closed set containing p and $U \cap V = \emptyset$. This complete the proof.

Theorem 3.5. The semi continuous image of an s–closed space X, in which every semi–open set is preclosed, into a Hausdorff space Y is closed.

Proof. Follows from Theorem 5.2 of Noiri (1980).

Definition 3.1. A mapping $f: X \longrightarrow Y$ is M–ß–continuous if the inverse image of each ß–open set in Y is ß–open in X.

Theorem 3.6. Let $f: X \longrightarrow Y$ be an M–ß–continuous mapping from an s–closed space X, in which every ß–open set is semi–closed, into a space Y. Then, f(X) is s–closed relative to Y.

Proof. To prove that f(X) is s–closed relative to Y, let $\{ U_i : i \in I \}$ be a cover of f(X) by semi–open subsets of X. So, $X = \bigcup_{i \in 1} f^{-1}(u_i)$ and thus $\{ f^{-1}(U_i) : i \in I \}$ is a family of ß–open sets of X. Since every ß–open set in X is semi–closed. then $\{ f^{-1}(U_i) : i \in I \}$ is a semi–open cover of X which is s–closed, then there exists a finite subfamily I_o of I such that, $X = \bigcup_{i \in I_o} (f^{-1}(U_i))^- = \bigcup_{i \in I} f^{-1}(U_i) = f^{-1}(\bigcup_{i \in I_o} U_i)$. Therefore, $f(X) \subset \bigcup_{i \in I_o} U_i \subset \bigcup_{i \in I_o} U^-_i$ and so, f(X) is s–closed relative to Y.

4 – RELATIONS BETWEEN S–CLOSEDNESS AND SOME TYPES OF COMPACTNESS

Theorem 4.1. Each s–closed space is almost co–compact.

Proof. Since each s–closed space is H (i) space Cameron (1983), Thompson (1975) and each H (i) space is almost co–compact Mashhour and Atia (1974), then the result follows.

Theorem 4.2 Each extremally disconnected and almost co–compact space is s–closed.

Proof. Let $\{ U_i : i \in I \}$ be a regular closed of X, then $U_i = U_i^{o-}$ and from extrema disconnectedness, we have $U_i = U_i^{o-o}$ which implies $U^o = U_i$ is open for each $i \in I$. Also, $U_i^- = U^{o-} = U_i = U_i^{o-o} = U_i^o$, i. e., the closure of any open set U_i is open and hence $\{ U_i : i \in I \}$ is a co–open cover of X which is almost co–compact. So, there exists a finite subfamily I_o of I such that $X = \bigcup_{i \in I_o} U_i^-$ and X is s–closed.

Corollary 4.1. Each extremally disconnected co–compact space is s–closed.

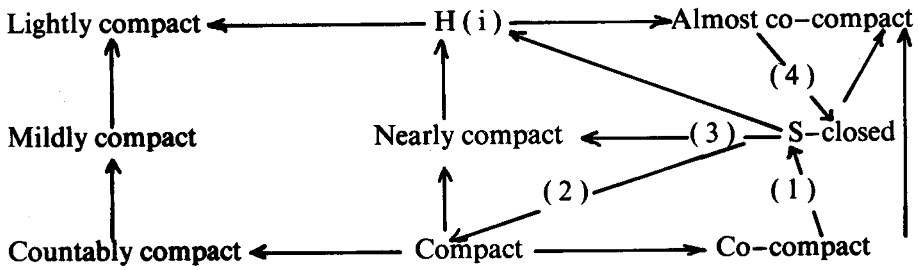
Proof. Obvious since each co–compact space is almost co–compact.

Theorem 4.3. An s–closed space in which every open set is co–open is nearly compact.

Proof. Let $\{ U_i : i \in I \}$ be an open cover of X, then $\{ U_i : i \in I \}$ is a semi–open cover of X which is s–closed, then there exists a finite subfamily I_o of I such that $X = \bigcup_{i \in I_o} U_i^-$. Since U_i^- is co–open, $U_i = U^{-o}$ for each $i \in I$. Hence $X = \bigcup_{i \in I} U_i^{-o}$. Thus, X is nearly compact.

Corollary 4.2. An s–closed space in which every open set is co–open is mildly compact.

We introduce the following diagram for a space X.



The implications 1, 2, 3 and 4 take place under the following conditions.

(1) X is extremely disconnected.

(2) Every open set in X is co-open and semi-closed, or X is regular.

(3) Every open set in X is co-open.

(4) X is extremaly disconnected.

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ملاحظـات عن فـراغات س - المغلقـة

محمد عزت عبد المنصف و عبد المنعم محمد قوزع

في هذا البحث بعض الخواص الجديدة لفراغات س - المغلقة التي قدمها طومسون عـام ١٩٦٧ ، وذلك باستخدام بعض المجموعات التي لها خاصية قريبة الإنفتاح وقريبة الإنغلاق في آن واحـد ، كما تم منـاقشـة صور الفراغـات س - المغلقـة تحت تـأثير بعض الرواسم غير المتصلـة ، كما تضمن البحث دراسـة عن علاقـة الإنغلاق السيني ببعض أنواع الأصاط ، وإحتـوى البحث كـذلـك على منـاقشـة للمجموعات الجزئية السينية المغلقة بالنسبة لفراغ T2 النصفي .