Filtered Dehomogenization Theory

For The Micro-structure sheaves OY

By
Abdel-Aziz E. Radwan
Department of Mathematics, Faculty of Science,
Ain Shams University, Cairo, EGYPT

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المقادح لحزمة البناء الدقيق Dehomogenization

عبد العزيز العزب رضوان

وذلك باستخدام حزم البناء الدقيق المقامة OY

والمعرفة على الفراغ الهندسي (G(R))، و Y = sec (G(R)) المتlimits لها معني عند استخدام النسخة الكمية بالتقيد إلى الجزاءات ذات الدرجة الصفرية.

Key Words: Micro-structure sheaves, Filtered sheaves, Quantum sheaves, Dehomogenization

ABSTRACT

In this work we study the theory of dehomogenization at the level of filtered sheaves using the micro-structure sheaves. For the sheaf of quantum section the results are still true.
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0. INTRODUCTION

In projective algebraic geometry homogeneous coordinate rings appear together with a suitable dehomogenization. For example, if \( V(I) \) is a projective variety determined by a homogenous ideal \( I \) of the polynomial ring \( k[x_0, \ldots, x_n] \) and \( R \) is the graded coordinate ring \( k[x_0, \ldots, x_n]/I \) then \( R_{n-1}R \) is isomorphic to the coordinate ring of the open affine subvariety complementary to the hyper plane at infinity in \( V(I) \). In a similar way every determinantal ring is a dehomogenization of a Schubert cycle (being the graded coordinate ring of a Schubert variety) and this dehomogenization principle is the basis for the study of determinantal rings. We now extend this to sheaf level. We study the theory of dehomogenization at the level of filtered sheaves using the micro-structure sheaf \( \tilde{\mathcal{O}} \) (see [1], [3], [131]) in this studying.

As in section 2 we prove that if \( \mathcal{F} \) is a dehomogenization of its associated Rees sheaf \( \tilde{\mathcal{O}}_{\mathcal{F}} \). For the sheaf of quantum sections (see [1], [131]) \( \tilde{\mathcal{O}}_{\mathcal{F}} = 0 \) the result is still true. In section 3 we do the opposite, i.e., we show that by taking a suitable dehomogenization a graded sheaf may be made into Rees sheaf associated to Zariski filtered sheaf. Again, the opposite is true for the sheaf of quantum sections.

So in this way, one also gets that \( \tilde{\mathcal{O}}_{\mathcal{F}} \) behaves as well as \( R \) with respect to this dehomogenization theory.

1. PRELIMINARIES

In fact we use the same preliminaries of [1] and let us recall them again.

All rings considered are associative rings with unit. Modules are left modules and ideals are two-sided ideals unless otherwise stated. A filtration \( \mathcal{F} \) on a ring \( R \) is given by an ascending chain \( 0 = F_{-1} \subset F_{0} \subset F_{1} \subset \ldots \) of additive subgroups \( F_n, n \in \mathbb{Z} \) satisfying:

\[
1 \in F_0, \quad F_nF_m \subset F_{n+m} \quad \text{for } n, m \in \mathbb{Z}, \quad \bigcup_{m \in \mathbb{Z}} F_m = R.
\]

A filtered \( R \)-module is an \( R \)-module \( M \) with filtration \( FM \) given by an ascending chain of additive subgroups \( c \subset F_{n-1} \subset F_{n} \subset \ldots \) satisfying: \( F_nR \subset F_{n+m} \) for \( n, m \in \mathbb{Z} \). We write \( R \text{-filt} \) for the (non-abelian) category of filtered modules with degree preserving morphisms.

To the filtration \( \mathcal{F} \) there corresponds the associated graded ring \( \mathcal{G}(R) = \bigoplus_{n \in \mathbb{Z}} F_nR/F_{n-1}R \) and similarly an associated graded module. \( \mathcal{G}(M) = \bigoplus_{n \in \mathbb{Z}} F_nM/F_{n-1}M \) is associated to \( FM \).

If \( x \in F_n^{-1}F_{-1} \) for some \( n \in \mathbb{Z} \) then we say that \( x \) has filtration degree \( n \) and the principle symbol map \( \sigma \) is defined by putting \( \sigma(x) = x \bmod F_{n-1} \) when \( x \) has filtration degree \( n \) and \( \sigma(x) = 0 \) when \( x \in F_n^{-1}M \) for all \( n \in \mathbb{Z} \).

A filtration \( \mathcal{F} \) is separated when \( \bigcap_{n \in \mathbb{Z}} F_n = 0 \); for a separated filtration \( \sigma(x) = 0 \) if and only if \( x = 0 \) holds.

There is another graded ring that may be associated to \( \mathcal{F} \), namely \( \bigoplus_{n \in \mathbb{Z}} F_n \). We may identify this so called Rees ring of \( \mathcal{F} \) to the subring \( \bigoplus_{n \in \mathbb{Z}} F_n \) in \( R[X, X^{-1}] \).

A filtration \( \mathcal{F} \) is said to be a good filtration if for every \( n \in \mathbb{Z} \), \( F_n = \sum_{m=0}^{d_n} F_{n-m} \) for certain \( d_0, \ldots, d_n \in \mathbb{Z} \) and some \( \mathcal{F}_n \). For a filtered inclusion \( i : M \to N \) strictness of \( i \) just states that \( M \) is considered with the filtration induced by \( FN \) on
M. Note however that a good filtration FN need not induce a good filtration on a submodule M. Recall from [5], the following.

1.1. Lemma. With conventions and notation as above:

a. $R/xR \cong G(R)$, $R/(1-x)R \cong R$, $R_x \cong R[X, x^{-1}]$.

b. $R/\mathfrak{m} \cong G(R)$, $R/(1-x)R \cong M$, $R_x \cong M[X, x^{-1}]$

where $(-)_X$ denotes the (graded) localization at the multiplicative central set \( \{ 1, X, X^2, \ldots \} \).

c. The functor $\mathcal{R}_1 \to \mathcal{R}_2$ defines an equivalence of categories between $\mathcal{R}_1$ and $\mathcal{R}_2$.

1.2. Lemma. With notation as before:

a. $FM$ is good if and only if $M$ is finitely generated.

b. A filtered morphism $f: M \to N$ is strict exactly then when $\text{Coker } f \in F_1$. When $f$ is a morphism in $F_1$, a strict sequence in $R$-filt transforms to an exact (graded) sequence in $R$-filt.

c. $FR$ is faithful if and only if $FR$.

d. $FR$ has the property that good filtration induce good filtrations on submodules if $i$ is left Noetherian.

We say that $FR$ is a Zariskian filtration when $\mathcal{F} \subseteq \mathcal{S}$ is a Zariski filtration that $G(R)$ is a commutative domain, this situation is general enough in the sense that it allows application of the results to most of the important examples: enveloping algebras of Lie algebras, Weyl algebras, any rings of differential operators as well as the classical commutative Zariski rings that show up in singularity theory.

On the topological space $Y=\text{Proj } G(R)$ with its Zariski topology having the $Y(I)=\{P \in \text{Proj } G(R), P \supseteq I\}$, $I$ varies over the principle graded ideals of $G(R)$, for a basis, we may define $O_Y$ as before. Up to the final section the results we establish are insensitive to the definition of $Y$ as $\text{Proj } G(R)$ or as $\text{Spec } G(R)$. Moreover, now as in [2], it is possible that $\text{Proj } G(R)=\text{Spec } G(R)$. So unless otherwise specified $Y$ is either $\text{Proj } G(R)$ or $\text{Spec } G(R)$; it is clear in the positively graded case $\text{Spec } G(R)$ is a somewhat strange topological space so that in this case we will automatically assume $Y=\text{Proj } G(R)$, [4].

Associating to an open set $Y(f) \in \mathcal{B}$ the
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micro-localization $Q^\mu(R)$, resp. $Q_f^\mu(R)$, we obtain sheaves $\tilde{Q}^\mu_Y$, resp. $\tilde{Q}_f^\mu_Y$, having as the completed stalks at $p \in Y$ the rings $Q^\mu(R)$, resp. $Q_f^\mu(R)$ see [13]. Note that $\tilde{Q}^\mu_Y$ is a sheaf of Zariski rings.

Replacing $R$ by any filtered $R$-module $M$ in the above leads to the construction of an $\tilde{Q}^\mu_Y$-module $\tilde{M}$ over the ringed space $\tilde{Q}^\mu_Y$ and also an $\tilde{Q}_f^\mu_Y$-module $\tilde{M}_f$ over the ringed space $\tilde{Q}_f^\mu_Y$, where we have written $\tilde{R}^\mu_Y$ and $\tilde{R}_f^\mu_Y$ for the sheaves defined by the presheaves constructed (note: in case $M$ is such that $G(M)$ is absolutely torsionfree then the presheaves of micro-localization are sheaves). The sheaves of quantum section for $G(R)$, resp. $G(M)$, are degree zero in the graded sense for the corresponding Rees sheaves for some basic properties and definition.

In [5], [8] and [6], [12], the use of graded techniques in $\tilde{R}$-$\tilde{G}$ allowed to obtain the desired results for filtered objects. Here we plan to use the same strategy when studying coherent sheaves of modules and their relations with structure sheaves of modules of sections, but we have to modify the technique somewhat when dealing with micro-localization and micro-structure sheaves because the elements of sections and stalks are not fractions. By a theorem from [5] we know that $Q^\mu(R) = (S^{-1}_X R)\wedge$ where $S_f = \{ r \in R, \sigma(r) \in \{ 1, f, \ldots \} \}$ and $\wedge$ stands for completion of the ring of fractions with respect to the localized filtration. Therefore it makes sense to use $\lim_{\leftarrow}{\downarrow}^n\tilde{R}$ and the "slicing-up of $\tilde{R}$" in terms of $\tilde{R}^N_X$ for every $n \in N$ in order to restate all problems to the graded case and consideration of homogeneous fractions $Q^\mu_{S_f}(\tilde{R}^N_X)$ for every $n \in N$.

When looking at sheaves of filtered and modules over some sheaf of filtered rings it may happen that the restriction morphisms in the sheaf change the filtration degree of some elements even if these morphisms are filtered morphisms, in other words it may not be possible to define a "principle symbole" globally on the sheaf of filtered modules even though the sheaf of associated graded modules is perfectly well-defined. This technicality is avoided if we define a filtered sheaf $\tilde{M}$ of modules over a filtered sheaf of rings $\tilde{R}$ by assuming that we have subsheaves of additive groups $\tilde{F}_n \tilde{R}$, $\tilde{F}_n \tilde{M}$ for $n \in \mathbb{Z}$ satisfying the expected conditions. Of course $\tilde{Q}^\mu_Y$ is such a filtered sheaf of rings because the restriction morphisms on principle Zariski open sets are strict filtered morphisms and under assumptions they are even strict filtered monomorphisms. A filtered sheaf morphism between filtered sheaves, $f: \tilde{M} \rightarrow \tilde{N}$, say, is then one such that for all $n \in \mathbb{Z}$, $f(\tilde{F}_n \tilde{M}) \subseteq \tilde{F}_n \tilde{N}$ and we say that $f$ is a strict sheaf morphism if for all stalks the induced stalk-morphism is strict. For coherent and filtered sheaves of modules we now introduce the equivalent of a good filtration and follows. If $\tilde{M}$ is a coherent filtered sheaf of (filtered) $\tilde{R}$-modules then we say that $\tilde{M}$ is coherently filtered if for every open $U$ we have an exact sequence of sheaf morphisms

$$(\tilde{R}|_U)^{\tilde{M}} \rightarrow (\tilde{R}|_U)^{\tilde{N}} \rightarrow \tilde{M}|_U \rightarrow 0$$

for some $n$, $m \in \mathbb{N}$, where $\tilde{f}_u$ and $\tilde{g}_v$ are now supposed to be a strict sheaf morphism.

When $\tilde{M}$ is a filtered sheaf of $\tilde{Q}^\mu_Y$-modules then we may define a graded sheaf $\tilde{M}^\mu_Y$ by putting

$\tilde{M} = \bigoplus_{n \in \mathbb{Z}} \tilde{F}_n \tilde{M}$

where for all $\tilde{M}(U) = \bigoplus_{n \in \mathbb{Z}} (\tilde{F}_n \tilde{M})(U)$

$\tilde{M}^\mu_Y = \bigoplus_{n \in \mathbb{Z}} \tilde{F}_n \tilde{M}$ for all Zariski opens $U \subseteq Y$ and every $y \in Y$.

We say that $\tilde{M}$ is the (graded) Rees sheaf of $\tilde{M}$ and the sheaf determined by $\tilde{M}(U) = \bigoplus_{n \in \mathbb{Z}} (\tilde{F}_n \tilde{M})(U)$ is called the associated graded sheaf of $\tilde{Q}^\mu_{G(R)}$-modules. It is not a too difficult exercise to verify that the structure sheaf $\tilde{M}_Y$ constructed before is in fact a coherently filtered sheaf of complete $\tilde{Q}^\mu_Y$-modules.

For a general filtered sheaf $\tilde{Q}^\mu_Y$-modules $\tilde{M}$ we still have: $\tilde{M}/\tilde{M}^\mu_Y \cong G(\tilde{M})$ and $\tilde{M}/(1-\tilde{X})\tilde{M} \cong \tilde{M}$ as sheaves where $X$ is the "constant" global section. Again it makes sense to say that $\tilde{M}$ is $X$-torsion-free and as in [1]. The coherence of $\tilde{M}$ implies the coherence of $\tilde{M}$ and the coherence of $\tilde{M}$. Under some conditions, cf. P. Shapira's book [11], the coherence of $G(\tilde{M})$ alone will
implies the coherence of \( M \), but we do not need this here.

We need too some basic theory of dehomogenezations but we refer to [7], and [9].

2. GRADED SHEAVES FROM FILTERED SHEAVES AND DEHOMOGENIZATION

There are two associated functors from the category of coherently filtered sheaves \( \mathcal{O}_Y^H \)-modules to the category of graded sheaves \( G(\mathcal{O}_Y^H) \)-modules and the category of graded sheaves \( \check{\mathcal{O}}_Y^H \)-modules respectively.

That is, if \( \mathcal{F}_Y \in \mathcal{O}_Y^H \)-filt is coherently filtered that

\[ \check{\mathcal{O}}_Y = \check{\mathcal{O}}_Y^H \text{-Gr} \quad \text{and} \quad G(\mathcal{F}_Y) \in G(\mathcal{O}_Y^H) \text{-Gr}. \]

As in [8] denote by \( X \) the global central regular section of \( \mathcal{O}_Y^H \) determined by \( \chi \mathcal{O}_Y^H(\mathfrak{m}) \) for each \( \mathcal{F}_Y \in \mathcal{O}_Y^H \). In fact that \( X \) is the image of the unit section \( 1 \in \mathbb{F}_1 \mathcal{O}_Y^H \) in \( \mathcal{O}_Y^H \). \( \mathcal{F}_Y \in \mathcal{O}_Y^H \)-Gr is said to be \( X \)-torsionfree.

2.1. Lemma. For every \( \mathcal{F}_Y \in \mathcal{O}_Y^H \)-filt that is coherently filtered. It follows that \( \check{\mathcal{F}}_Y \) is \( X \)-torsionfree.

Proof. From the above definition and [12], [13].

These \( \check{\mathcal{F}}_Y \) with \( \mathcal{F}_Y \in \mathcal{O}_Y^H \)-filt such that \( \mathcal{F}_Y \) is coherently filtered, form a full subcategory of \( \mathcal{O}_Y^H \)-Gr denoted by \( \mathcal{F} \in \mathcal{O}_Y^H \).

Denote by \( \mathcal{I}_X^H = \check{\mathcal{O}}_Y^H \) the sheaf of graded ideals in \( \check{\mathcal{O}}_Y^H \), i.e., for every \( \mathcal{F}_Y \in \mathcal{O}_Y^H \), \( \mathcal{I}_X^H(\mathfrak{m}) \) which is a graded ideal in \( \mathcal{O}_Y^H(\mathcal{F}_Y) = \check{\mathcal{O}}_Y^H(\mathfrak{m}) \).

2.2. Theorem. With notation as above:

a. The functor \( \bigwedge \) defined above determines an equivalence of categories coherently filtered \( \mathcal{O}_Y^H \)-modules in \( \mathcal{F} \in \mathcal{O}_Y^H \). In particular, every coherent \( X \)-torsionfree \( \mathcal{O}_Y^H \)-module \( \mathcal{F}_Y \) is of the form for some \( \mathcal{H}_Y \in \mathcal{O}_Y^H \)-filt which is coming from a \( \mathcal{O}_Y^H \)-filt with good filtration FM.

b. The localization of \( \mathcal{O}_Y^H \) at the multiplicative closed set of global sections \( \{1, x, x^2, \ldots \} \) equals to \( \mathcal{O}_Y^H[X, x^{-1}] \) denoted by \( \mathcal{O}_Y^H[X, x^{-1}] \).

Proof. a. It follows from Lemma 2.1. above and Theorem 2.6 in [1].

b. To prove these statements we need some ideas about localization of sheaf of modules, but for this we refer to [14]. Now the statements are local hence for each \( Y(f) \) we see that

\[ \mathcal{O}_Y^H[X, x^{-1}] \mathcal{F}_Y = \mathcal{O}_Y^H[X, x^{-1}] \mathcal{F}_Y = \mathcal{O}_Y^H[X, x^{-1}] = \mathcal{O}_Y^H[X, x^{-1}] - \mathcal{O}_Y^H[X, x^{-1}] \mathcal{F}_Y = \mathcal{O}_Y^H[X, x^{-1}] - \mathcal{O}_Y^H[X, x^{-1}] \mathcal{F}_Y. \]

Hence \( \mathcal{O}_Y^H[X, x^{-1}] = \mathcal{O}_Y^H[X, x^{-1}] \) for each \( \mathcal{F}_Y \in \mathcal{O}_Y^H \).

Similarly, we may prove that \( \mathcal{O}_Y^H[X, x^{-1}] = \mathcal{O}_Y^H[X, x^{-1}] \).
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2.4. Remark. The foregoing Theorems 2.2 and 2.3 are, in fact, the basis for many results on filtered sheaf theory (in particular, on Zariskian filtered sheaf theory). From both it is clear that \( Q_\mu^Y \) is a dehomogenization of its associated graded Rees sheaf \( \tilde{Q}_\mu^Y \).

Given the micro-structure sheaf \( Q_\mu^Y \) and \( M_\mu^Y \in Q_\mu^Y \text{-filt} \) that is coherently filtered. Hence by theorem 2.2, we see that

\[
Q_\mu^Y = Q_\mu^Y/(1-\mu)Q_\mu^Y
\]

are filtered sheaves defined on \( Y \) with respect to the filtrations

\[
\gamma_n(Q_\mu^Y/(1-\mu)Q_\mu^Y) = \gamma_n(Q_\mu^Y)\quad \gamma_n(M_\mu^Y/(1-\mu)M_\mu^Y) = \gamma_n(M_\mu^Y)
\]

respectively. There are sheaf isomorphisms

\[
\begin{aligned}
\gamma_n(Q_\mu^Y/(1-\mu)Q_\mu^Y) &= \gamma_n(Q_\mu^Y) \\
\gamma_n(M_\mu^Y/(1-\mu)M_\mu^Y) &= \gamma_n(M_\mu^Y)
\end{aligned}
\]

3. FILTERED SHEAVES FROM GRADED SHEAVES AND DEHOMOGENIZATION

In this section we do the opposite, i.e. we show that by taking a suitable dehomogenization many graded sheaves of rings may be made into Rees sheaves of rings associated to Zariskian filtered sheaves of rings. So in this way one also gets information for a given graded sheaf from its associated filtered sheaf.

Let \( H_Y = \bigoplus_{n \in \mathbb{Z}} H^Y_n \) be any \( \mathbb{Z} \)-graded sheaf of rings defined on a topological space \( Y \). Associate to \( Y \) a base of open subsets \( \beta \). Let \( X \) be a homogeneous global section of degree 1 in \( H_Y \). This \( X \) determines for each \( U \in \beta \) a global section \( x_U = x_{H^Y(U)} \) whereas

\[
H_Y(U) = \bigoplus_{n \in \mathbb{Z}} H^Y_n(U) = \bigoplus_{n \in \mathbb{Z}} (H^Y(U))_n.
\]

For every homogeneous section \( h_n \) of \( H^Y_n \) over \( U \) and \( t \geq 0 \) we have \( h^t = h^t_n \). The sheaf \( D_Y = H^Y/(1-X)H^Y \) is a sheaf of ideals defined on \( Y \), may be made into a filtered sheaf of rings by endowing it with the filtration:

\[
\mathcal{F} D_Y = (H^Y_n + (1-X)H^Y)/(1-X)H^Y, \quad n \in \mathbb{Z}.
\]

We can see that the filtration \( \mathcal{F} D_Y \) on \( D_Y \) defined above is such that \( U_n \in \mathcal{F} D_Y \) and \( 1 \in \mathcal{F} D_Y \).

3.1. Lemma. With notations as above, if \( X \) is a regular homogeneous global section in \( H_Y \) then

a. \( (1-X)H_Y \cap H_Y, n = 0 \) : the zero sheaf for all \( n \in \mathbb{Z} \).

b. If \( X \) is also a normal global section \( (X H_Y = H_Y X) \) then \( X \) is a central global section if and only if and only if \( (1-X)H_Y \) is a sheaf of ideals in \( H_Y \).


b. To prove that \( (1-X)H_Y \) is a sheaf of ideals in \( H_Y \), let \( U \in \beta \), \( (1-X)H_Y(U) = (1-X)(H_Y(U)) \) which is an ideal in \( H_Y(U) \). Similarly \( H_Y(U) \) is regular homogeneous normal central in \( H_Y(U) \). To prove that \( X \) is regular homogeneous normal central in \( H_Y(U) \). But to prove that \( X \) is central it will be sufficient to be central over \( U \), \( U \in \beta \). Let us consider the sheaf morphism \( H_Y \to H_Y/(1-X)H_Y \), hence for \( U \in \beta \) we have \( X|_U = X_{H^Y(U)} \), \( (1-X)(H^Y(U))/(1-X)H^Y(U) \).

Now \( H_Y(U) \) is a graded ring and \( X_{H^Y(U)} \) is normal. Let us assume that \( X = H_Y \), \( s \in H_Y(U) \) and \( s \neq s \). This implies that the quantum of \( Q_\mu^Y \) is a dehomogenization of its shifted associated quantum Rees sheaf.

3.2. Theorem. Let \( H_Y \) be a graded sheaf of graded rings \( X \) a regular central homogeneous global section of degree 1 in \( H_Y \), with notations as above then

a. \( \tilde{H}_Y = H_Y \) as graded sheaf on \( Y \).
b. \( G(Y) \equiv H_Y / X_H \) as graded sheaf.

Proof. a. Since
\[
\overline{D}_Y = \oplus_{n \in \mathbb{Z}} \overline{D}_Y = \oplus_{n \in \mathbb{N}^+} \overline{D}_Y = \oplus_{n \in \mathbb{N}^+} \frac{H_{Y,n} + X H_{Y,n}}{X H_{Y,n}}
\]
and
\[
H_Y = \oplus_{n \in \mathbb{Z}} H_Y = \oplus_{n \in \mathbb{N}^+} H_Y.
\]
For each \( n \) one may define a sheaf isomorphism of sheaves of groups:
\[
\phi_n: \frac{H_{Y,n} + (1-X)H_Y}{(1-X)H_Y} \rightarrow H_Y,n
\]
as follows: let \( U \in \beta \) be a basic open subset of \( Y \), we have the group homomorphism
\[
h_n: (1-X)H_Y(U) \rightarrow H_Y,n(U)
\]
which is surjective with kernel
\[
\ker \phi_n(U) = \{ h \in (1-X)H_Y(U) : h \neq 0 \} / (1-X)H_Y(U).
\]
So
\[
\frac{(H_{Y,n} + (1-X)H_Y(U))}{(1-X)H_Y(U)} \rightarrow H_Y,n(U)
\]
is a group isomorphism. Hence \( \phi_n \) is an isomorphism for each \( U \in \beta \), so \( \phi_n \) is a sheaf isomorphism. Combine all \( \phi_n \) to obtain the required sheaf isomorphism. This proves \( G(Y) \equiv H_Y \) as graded sheaves defined on \( Y \).

b. By definition
\[
G(D_Y) = \oplus_{n \in \mathbb{N}^+} \frac{(H_{Y,n} + (1-X)H_Y(U))}{(1-X)H_Y(U)} = \oplus_{n \in \mathbb{N}^+} \frac{(H_{Y,n} + (1-X)H_Y(U))}{(1-X)H_Y(U)}
\]
and
\[
H_Y / X_H Y = \oplus_{n \in \mathbb{N}^+} \frac{(H_{Y,n} + X H_Y(U))}{X H_Y(U)}
\]
For each \( n \), one may define a sheaf isomorphism of sheaves of additive groups:
\[
\phi_n: \frac{H_{Y,n} + (1-X)H_Y(U)}{(1-X)H_Y(U)} \rightarrow H_Y,n + X H_Y
\]
as follows, for each \( U \in \beta \) define
\[
\phi_n(U): \frac{(H_{Y,n}(U)) + (1-X)H_Y(U)}{(1-X)H_Y(U)} \rightarrow \frac{(H_{Y,n}(U)) + X H_Y(U)}{X H_Y(U)}
\]
by
\[
h_n + H_{Y,n-1}(U) + (1-X)H_Y(U) \rightarrow h_n + X H_Y(U)
\]
From; since for every \( h_{n-1} \in H_{Y,n-1}(U) \), we have
\[
h_{n-1} + (1-X)h_{n-1} \rightarrow h_{n-1} + X H_Y(U)
\]
Clearly, \( \phi_n(U) \) is a surjective map. Moreover, if
\[
\text{if } h \in H_Y(U) \text{ then } h = h_{n-1} + h_{n-1} \in H_{Y,n-1}(U)
\]
then
\[
\phi_n(U) \text{ is injective and this shows that } \phi_n(U) \text{ is a group isomorphism.}
\]
\[
\phi_n(U) \text{ is a group isomorphism for each } U \in \beta \text{ then } \phi_n \text{ is a sheaf isomorphism. We shall obtain the required sheaf isomorphism if we combine all } \phi_n.
\]

3.3. Proposition. Consider the same hypothesis in theorem 3.2. then
\[
X E \in \mathbb{J} G(Y) \text{ if and only if } \mathbb{J} G(Y) \in \mathbb{J} G(Y)
\]
Proof. The statement is local so, let \( U \in \beta \) be a basic open subset of \( Y \), we have to show that
\[
\exists 1_{\mathbb{J} G(Y)}(U) \text{ if and only if } \mathbb{J} G(Y) \in \mathbb{J} G(Y)
\]
Now let \( \exists 1_{\mathbb{J} G(Y)}(U) = \mathbb{J} G(Y)(U) \), \( H_Y(U) \) graded ring it follows that
\[
\exists 1_{\mathbb{J} G(Y)}(U) = \mathbb{J} G(Y)(U)
\]
for each \( U \in \beta \). Hence \( \exists 1_{\mathbb{J} G(Y)} \in \mathbb{J} G(Y) \) as sheaves of groups. Conversely, since for each \( U \in \beta \) we have
\[
\exists 1_{\mathbb{J} G(Y)}(U) = 1_{\mathbb{J} G(Y)}(U)
\]
then it follows from \( h = X h \). Hence \( \exists 1_{\mathbb{J} G(Y)}(U) = \mathbb{J} G(Y)(U) \).

3.4. Corollary. With notation and conventions as above the localization of \( H_Y = \oplus_{n \in \mathbb{Z}} H_Y \) at the Ore sheaf set \( \{ 1, X, X^2, \ldots \} \) exists, it is denoted by \( (H_Y)_{\mathbb{J} G(Y)} \), which is a graded sheaf of rings defined on \( Y \) such that there is a commutative diagram of sheaf morphism
\[
H_Y \rightarrow (H_Y)_{\mathbb{J} G(Y)}
\]
Proof. One can define \( (H_Y)_{\mathbb{J} G(Y)} \) as follows: for each \( U \in \beta \),
\[
(H_Y)_{\mathbb{J} G(Y)}(U) = (H_Y(U))_{\mathbb{J} G(Y)} \text{ where } X = X_{H_Y(U)}, \text{ the localization of the graded ring } H_Y(U) \text{ at the Ore set } \{ 1, X, X^2, \ldots \}. \]
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(\(\mathcal{H}_Y(U)\)) is a graded ring hence (\(\mathcal{H}_Y\)) is a graded sheaf on \(Y\) such that 

\[
(\mathcal{H}_Y) = \bigoplus_{n \in \mathbb{Z}} (\mathcal{H}_Y(U))_n;
\]

where \(n \leq 0\).

For each \(U \in \beta\) one may define a homomorphism of rings: 

\[
\phi_Y(U) : \mathcal{H}_Y(U) \rightarrow \mathcal{D}_Y(U); \quad x \mapsto h + (1-x) \mathcal{H}_Y(U)
\]

where \(h \in \mathcal{H}_Y, n \leq 0\), \(j \in \mathbb{Z}\)

and this define a sheaf morphism 

\[
(\mathcal{H}_Y) \rightarrow \mathcal{H}_Y/(1-x)\mathcal{H}_Y.
\]

3.5. Example. Let \(\mathcal{H}_Y\) be any \(\mathbb{Z}\)-graded sheaf of graded rings defined on \(Y\). Then \(\mathcal{H}_Y(T)\) is a graded sheaf; for each \(U \in \beta\): 

\[
\mathcal{H}_Y(U) = \mathcal{H}_Y(U) \oplus \mathcal{D}_Y(U) / (1-x)\mathcal{H}_Y(U)\]

is a filtered sheaf (i.e., sheaf of filtered rings), has an associated graded sheaf \(\mathcal{H}_Y(T) \cong \mathcal{H}_Y\) as graded sheaves and a Rees graded sheaf isomorphic to \(\mathcal{H}_Y(T)\) as graded sheaves.

3.6. Remark. Todo the opposite of the result concerning the quantum case, let \(\mathcal{H}_Y\) be any coherent graded sheaf defined on \(Y\). Hence 

\[
\left(\frac{\mathcal{H}_Y}{(1-x)\mathcal{H}_Y}\right)_0 \cong \mathcal{H}_Y(\ast) \quad (*)
\]

where \(\mathcal{H}_Y(1-x)\mathcal{H}_Y = \mathcal{D}_Y\) may be made as above into a filtered sheaf by 

\[
\mathcal{D}_Y = ((\mathcal{H}_Y)_n + (1-x)\mathcal{H}_Y)/(1-x)\mathcal{H}_Y.
\]

But from (*) we can conclude the required result. So the theory dehomogenization can be applied to the quantum level.

A filtered sheaf \(\mathcal{D}_Y\) defined on \(Y\), that has a base \(\beta\) as above, is said to be a Zariski filtered sheaf on \(Y\) if its associated Rees sheaf \(\mathcal{D}_Y\) is a Noetherian graded sheaf on \(Y\) and \(\mathcal{D}_Y < \mathcal{J}(\mathcal{H}_Y)\) as sheaves on \(Y\).

Now we are ready to prove the final result of this note that mentioned above may be made into a Zariski filtered sheaf defined on \(Y\).

3.7. Corollary. With notation and considerations as before, if \(x \in \mathcal{D}(\mathcal{H}_Y)\) and \(\mathcal{D}_Y\) is a Noetherian graded sheaf on \(Y\) then \(\mathcal{D}_Y\) will be Zariski filtered on \(Y\).

Proof. From Theorem 3.2, we have seen that \(\mathcal{D}_Y = \mathcal{H}_Y\), hence \(\mathcal{D}_Y\) is Noetherian graded on \(Y\). From Proposition 3.3, we have seen that \(x \in \mathcal{J}(\mathcal{H}_Y)\) is equivalent to 

\[
\mathcal{F}_1 \mathcal{D}_Y < \mathcal{J}(\mathcal{F}_0 \mathcal{D}_Y) \quad \text{as sheaves on } Y,
\]

hence the result follows.

3.8. Final remark. This is an important dehomogenization theory. For more cases and results we may continue to find applications. We shall do this in the near future.

REFERENCES

Abdel-Aziz E. Radwan
