A SHORT NOTE ON THE POINT-WISE SUMMABILITY OF THE CONJUGATE SERIES OF A FOURIER SERIES IN THE NOLUND SENSE

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Key words: Conjugate Fourier series. Norlund summability.

ABSTRACT

In this note we consider an analogous theorem of [1] in the sense of point-wise summability of the conjugate series of a Fourier series in the Norlund sense.

INTRODUCTION

1. Let \( f(t) \) be a periodic function with period \( 2\pi \), integrable in the sense of Lebesgue over \((-\pi, \pi)\). Let \( \tilde{S} [f] \) denote the conjugate series of the Fourier series of \( f \). Let

\[
\phi(t) = f(x+t) + f(x-t) - 2f(x), \quad \psi(t) = f(x+t) - f(x-t),
\]

\[
\bar{\psi}(t) = \int_0^t |\psi(u)| \, du, \quad \text{and} \quad \bar{\phi}(t) = \int_0^t |\phi(u)| \, du.
\]

In [1] we have:

Theorem 1.1: Let \( q_n = \frac{1}{(n+1)^\alpha} \), \( 0 \leq \alpha < 1 \). Then \( \tilde{S} [f] \) is summable \( S(N, q_n) \) to \( f(x) \) at each point \( x \) where \( \bar{\phi}(t) = o(t) \).

(The case \( \alpha = 0 \) is Lebesgue's theorem [5].)

In this note we prove the following analogous theorem of theorem 1.1 above.

Theorem 1: Let \( q_n = \frac{1}{(n+1)^\alpha} \), \( 0 \leq \alpha < 1 \). Then \( \tilde{S} [f] \) is summable \( S(N, q_n) \) to

\[
\frac{1}{\pi} \int_0^\pi \frac{\psi(t)}{2 \tan \frac{1}{2} t} \]

at each point \( x \)

where \( \bar{\psi}(t) = o(t) \).

Proof: Let \( \tilde{S}_n(x) \) denote the sequence of partial sums of the conjugate series of a Fourier series, and let \( t_n(x) \) be the corresponding Norlund mean. Then clearly ([3], and [5]).
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\[ t_n(x) = \sum_{k=0}^{n} q_k \frac{1}{2\pi Q_n} \int_{0}^{\pi} \phi(t) \cos \left( \frac{t}{2} \cos \left( n-k+\frac{1}{2} \right) \right) t \sin \left( \frac{t}{2} \right) \]

Hence

\[ t_n(x) = \frac{1}{\pi} \int_{0}^{\pi} \phi(t) \left( \frac{1}{2 \tan \frac{t}{2}} \right) \left( \frac{1}{2 \pi Q_n} \right) \left( \sum_{k=0}^{n} \cos \left( n-k+\frac{1}{2} \right) t \right) \sin \left( \frac{t}{2} \right) dt = -\int_{0}^{\pi} \phi(t) K_n(t) \, dt, \]

where

\[ K_n(t) = \frac{1}{2 \pi Q_n} \sum_{k=0}^{n} \cos \left( n-k+\frac{1}{2} \right) t \sin \left( \frac{t}{2} \right). \]

In order to prove the theorem we show that:

\[ \int_{0}^{\pi} \phi(t) K_n(t) \, dt = o(1) \quad \text{as} \quad n \to \infty. \]

Now

\[ \int_{0}^{\pi} \phi(t) K_n(t) \, dt = \left( \int_{0}^{\pi} \frac{1}{n} \phi(t) K_n(t) \, dt \right) \left( \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} \right) \phi(t) K_n(t) \, dt \]

\[ = \bar{I}_1 + \bar{I}_2 + \bar{I}_3, \]

say

First by [2] above we have:

\[ \bar{I}_1 = \int_{0}^{\pi} \frac{1}{n} \phi(t) K_n(t) \, dt \]

\[ = o \left( \int_{0}^{\pi} n |\phi(t)| \, dt \right) \]

\[ = o(1) \quad \text{as} \quad n \to \infty. \]

Second, clearly the method \( S(N, \frac{1}{(n+1)^\alpha}) \) is regular. Hence by

the Riemann Lebesgue theorem, and the regularity of the method \( S(N, \frac{1}{(N+1)^\infty}) \)

we have:

\[ \bar{I}_3 = \int_{\pi}^{1} \phi(t) K_n(t) \, dt = o(1) \quad \text{as} \quad n \to \infty. \]
Third by [4] above we have:

\[ I_2 = \frac{1}{n} \int \frac{1}{n} \Phi (t) \ K_n (t) \ dt = o \left( \frac{1}{Q_n} \int \frac{1}{n} | \Phi (t) | \frac{Q \tau}{t} \ dt \right). \]

Now

\[ \frac{1}{Q_n} \frac{1}{n} \int \frac{1}{n} \Phi (t) \ |Q \tau| \ dt = \left( \int + \int + \ldots + \int \right) | \Phi (t) | \frac{Q \tau}{t} \ dt \]

Hence by integration by parts and simplifying we obtain:

\[ \frac{1}{Q_n} \frac{1}{n} \int \frac{1}{n} \Phi (t) \frac{Q \tau}{t} \ dt - o \left( \frac{1}{Q_n} \int \Phi (t) \frac{Q \tau}{t} \ dt \right) \]

Now

\[ \frac{1}{Q_n} \frac{1}{n} \int \frac{Q \tau}{t} \ dt = O \left( \frac{1}{Q_n} \right) + o \left( \frac{1}{Q_n} \frac{1}{Q_n} \right) = o \left( \frac{1}{Q_n} \right) \text{ as } n \to \infty \]

This completes the proof of theorem 1.

REFERENCES

Ali, Ziad, R.A. A generalization of a theorem of Lebesgue (is being considered for publication).


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