ON DIRICHLET PROBLEM WITH SINGULAR INTEGRAL BOUNDARY CONDITION

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ABSTRACT

In this paper a Dirichlet problem, with singular integral condition, is studied. It is shown that such problem can be reduced to a regular equation which allows us to construct the solution.

Let $G$ be a simply connected bounded region with a simple smooth contour $C$. Consider the equation

$$\Delta u = 0$$  \hspace{1cm} (1)

where $\Delta$ is the Laplace's operator and $u$ is a vector vector function which takes values in Banach algebra with involution $R$. In this paper we are concerned with the regular solution of equation (1) in $G$ which satisfies, on $C$, the singular integral condition:

$$a(t)u(t) + \frac{b(t)}{r_1} \int_{C} \frac{u(T)}{T-t} \, dT + \int_{C} k(t, T)u(T) dT = f(t)$$ \hspace{1cm} (2)

where $a(t), b(t), k(t, T)$ and $f(t)$ assume values in $R$, and each of them satisfies a Holder condition.

In the case $a(t) = 1, b(t) = 0$ and $k(t, T) = 0$, condition (2) takes the form:

$$u(t) = f(t)$$ \hspace{1cm} (2')

while in the case $a(t) = 0, b(t) = 1$ and $k(t, T) = 0$,

condition (2) takes the form

$$\frac{1}{r_1} \int_{C} \frac{u(T)}{T-t} \, dT = f(t), \text{which implies that } u(t) = \frac{1}{r_1} \int_{C} \frac{f(T)}{T-t} \, dT$$ \hspace{1cm} (2'')

The problem of finding the regular solution of (1) in $G$ prescribed by (2') or (2''), on $C$, is the usual Dirichlet problem for a vector function. The more general problem (1) - (2), mentioned earlier also called Dirichlet problem when (2) is solvable with respect to $u$. 

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The operator form of condition (2) is
\[ au + bSu + Ku = f \]  \hspace{1cm} (3)
where S is a singular operator satisfying the conditions:
1) \( S^2 = I \), 2) \( \forall v \in \mathbb{R} \) the operator \( Sv - vS \) is regular \[ 1 \]

**Lemma:** If S is the singular operator mentioned above, then \( \forall v,w \in \mathbb{R} \), we have
(i) \( Svw = vSw + Nw \)
(ii) \( SvSw = vw + Qw \)

where N and Q are regular operators.

**Proof:** (i) Since \( Svw = Sv \circ v - v \circ S \)
\[ = vSw + (Sv - vS)w \]
then from 2), it follows
\[ Svw = vSw + Nw. \]

(ii) Since \( SvSw = Sv \circ Sv - v \circ v \)
then it follows from 1)
\[ SvSw = vw + SvSw - S'vw \]
\[ = vw + S(vS - Sv)w \]
Since S in singular and \( vS - Sv \) is regular,
then \( S(vS - Sv) \) is regular and thus we have
\[ SvSw = vw + Qw. \]

**Theorem:** Problem (1)-(2) can be reduced to the regular equation
\[ q \omega + L \omega = h \]
which is completely defined in the space \( \mathbb{R}^2 \).

**Proof:** Every regular solution of (1) takes the form
\[ u = \phi (z) + \overline{\phi}(z) \]  \hspace{1cm} (4)
where \( \phi (z) \) is an arbitrary vector function in \( G \) and \( \overline{\phi}(z) \) is its involution.

Thus,
\[ u|_{C} = \phi (t) + \overline{\phi}(t) \]  \hspace{1cm} (5)
Substituting from (5) in (3) we find
\[ a(\phi + \overline{\phi}) + bS(\phi + \overline{\phi}) + K(\phi + \overline{\phi}) = f \]  \hspace{1cm} (6)
The unique integral representation on I.N. Vekua, [2], for the function \( \phi (z) \) takes the form
\[ \phi (z) = \int_{c} \frac{T \partial T \delta (T)}{T - z} \text{d}T + ik \]  \hspace{1cm} (7)
where $\delta(T) = \delta(T)$ and satisfies Holder condition, $k_1 = \bar{k}_1$ (constant).

From (7), we obtain [2]

$$\mathbf{c}(t) = \bar{n} \bar{\mu} t \delta(t) + \int_c \frac{T \delta(T)}{T - t} dT + i k_1$$

(8)

hence

$$\mathbf{c}(t) = -\bar{n} \bar{\mu} t \delta(t) + \int_c \frac{T \delta(T)}{T - t} dT - i k_1$$

(9)

Since $\frac{dT}{T - t} = \frac{dT}{T - t} + d\ln \left( \frac{T - t}{T - t} \right)$, [2], equation (9) takes the form

$$\mathbf{c}(t) = \bar{n} \bar{\mu} t \delta(t) + \int_c \frac{T \delta(T)}{T - t} dT + \int_c h_1(t, T) dT - i k_1$$

(10)

where the regular kernel $h_1(t, T)$ is given by:

$$h_1(t, T) = T \frac{d}{dT} \ln \left( \frac{T - t}{T - t} \right).$$

Equations (8) and (10), can be written respectively in the form

$$\mathbf{c}(t) = \bar{n} \bar{\mu} t \delta(t) + \frac{\bar{n} \bar{\mu} t}{\bar{n} \bar{\mu}} \int_c \frac{\delta(T)}{T - t} dT + \int_c \frac{T \delta(T)}{T - t} dT + ik_1$$

(11)

$$\mathbf{c}(t) = \bar{n} \bar{\mu} t \delta(t) + \frac{\bar{n} \bar{\mu} T}{\bar{n} \bar{\mu}} \int_c \frac{\delta(T)}{T - t} dT + \int_c \frac{T \delta(T)}{T - t} dT$$

$$+ \int_c h_1(t, T) \delta(T) dT - i k_1$$

(12)

Substituting from (11) and (12) in (5) we have

$$u_{|c} = \alpha(t) \delta(t) + \frac{\beta(t)}{\bar{n} \bar{\mu}} \int \frac{\delta(T)}{T - t} dT + \int_c M(t, T) \delta(T) dT$$

(13)

where $\alpha(t) = \bar{n} \bar{\mu} (t' t - t' t)$,

$$\beta(t) = \bar{n} \bar{\mu} (t' t + t' t),$$

and

$$M(t, T) = \frac{\bar{T} \bar{T} t' t}{T - t} + \frac{\bar{T} \bar{T} - t' t}{T - t} + h_1(t, T).$$

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The operator form of (13) is given by
\[ u_c = \alpha \delta + \beta S \delta + M \delta \] (15)
Consequently from (15) and (16), we obtain:
\[ a(\alpha \delta + \beta S \delta + M \delta) + bS(\alpha \delta + \beta S \delta + M \delta) + K(\alpha \delta + \beta S \delta + M \delta) = f \] (16)
\[ a(\alpha \delta + \beta S \delta + bS \delta + bS \beta S \delta + \delta) = f \] (17)
where \( T_0 \) is a regular operator
\[ T_0 \delta = K(\alpha \delta + \beta S \delta + M \delta) + aM \delta + bSM \delta \]
using the lemma, the following equation is obtained from (17)
\[ a(\alpha \delta + \beta S \delta + bS \delta + bS \beta S \delta + \delta) = f \] (18)
where \( P_1 \) and \( Q_1 \) are regular and completely defined by the following:
\[ S \alpha \delta = \alpha S \delta + P_1 \delta \]
\[ S \beta S \delta = \beta S \delta + Q_1 \delta \] (19)
Let,
\[ m = a \alpha + b \beta, \]
\[ n = a \beta + b \alpha, \]
and the regular operator \( Y \)
\[ Y = bP_1 + bQ_1 + T_0 \] (20)
we obtain from (18)
\[ m \delta + nS \delta + Y \delta = f \] (21)
and therefore, problem (1)-(2) can be reduced to the singular operator equation (21)
which is reshaped in a regular form as follows [3]:
From (21)
\[ Sm \delta + SnS \delta + SY \delta = Sf \] (22)
since \( S^2 = 1 \), we obtain from (22)
\[ SmS \delta + SnS \delta + SYSS \delta = Sf \] (23)
let \( \delta = \varphi_1 \) and \( S \delta = \xi_2 \), consequently from (21) and (23) the following system of equations is produced
\[ m \varphi_1 + n \varphi_2 + Y \varphi_1 = f \] (24)
\[ Sm \varphi_2 + SnS \varphi_1 + SYS \varphi_2 = Sf \]
Using the lemma, the above system takes the form
\[ n \varphi_1 + m \varphi_2 + Y \varphi_1 = f \]
\[ n \varphi_1 + m \varphi_2 + N_1 \varphi_1 + N_2 \varphi_2 + SYS \varphi_2 = Sf \] (25)
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where $N_1$ and $N_2$ are regular operators, which are completely defined by the following

\[
\begin{align*}
S_n S q_1 &= n q_1 + N_1 q_1, \\
S_m S q_2 &= m q_2 + N_2 q_2
\end{align*}
\]

(26)

Define,

\[
q = \begin{pmatrix} m & n \\ n & m \end{pmatrix}
\]

(27)

\[
L = \begin{pmatrix} Y & 0 \\ N_1 & N_2 + \text{SYS} \end{pmatrix}
\]

(28)

\[
h = (f, S f)
\]

\[
\omega = (\varphi_1, \varphi_2)
\]

(29)

(30)

The system (25), takes the form

\[
q \omega + L \omega = h
\]

(31)

which is regular and completely defined in the space $R^2$, the proof is complete.

If $q$ possess a bounded inverse $q^{-1}$ equation (31) has

\[
\omega + L_O \omega = h_O
\]

(32)

which is a regular equation with regular operator $L_O = q^{-1} L$ in the space $R^2$ and therefore, if $\delta$ is the solution of equation (21), then $(q_1, q_2) = (\delta, S \delta)$ is the solution of equation (32) and conversely if $\omega = (q_1, q_2)$ is the solution (32) then, $\delta = \frac{q_1 + S q_2}{2}$ is the solution (21) which allows us to construct the solution of the problem

\[(1) - (2).

REFERENCES


عن مسألة درشلت ذات الشرط الحدي التكامل الشاذ

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في هذا البحث درست مسألة درشلت ذات شرط حدي تكامل شاذ والتي امكن تحويلها الى معادلة ذات مؤثر منتظم مما أدى الى تركيب الحل المطلوب.