Relation between Quasi-normed Ideals of Entropy Numbers and of Approximation Numbers

by

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ABSTRACT

Let $e_n(T)$ and $a_n(T)$ be the entropy numbers and approximation numbers of operator $T$ between Banach spaces, respectively. Let $\mathcal{S}_p$ be the operator ideal defined by

$$
\mathcal{S}_p := \{ T \in \mathcal{L}; \{ e_n(T) \} \in l_p \},
$$

and $\mathcal{S}_p$ be the operator ideal defined by

$$
\mathcal{S}_p := \{ T \in \mathcal{L}; \Sigma a_n(T)p < \infty \}.
$$

Then if $0 < p < 1$, and $1 \leq u, v \leq \infty$ we have $\mathcal{S}_p(l_u, l_v) \subseteq \mathcal{S}_p(l_u, l_v)$. 
العلاقة بين
مثاليات مؤثرات اعداد الانترولي
والاعداد التقريبية

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عنى هذا البحث بدراسة العلاقة بين مثالي كل من مؤثر اعداد الانترولي و

ـ مؤثر الاعداد التقريبية \( S_p \) في فراغات بناخ \( l_u, l_v \)

وفي حالة \( 1 > p > 0, 0 < U < 1, 0 < V < \infty \), فقد أثبتت العلاقة

\[ S_p (l_u, l_v) \subseteq \xi_p (l_u, l_v) \]
Introduction

In [1] the quasi-normed ideal $\xi_p$ has been introduced. The relation between $\xi_p$ and $S_p$ in Hilbert spaces has been studied and it has been found that $[1] S_p(l_2, l_2) = \xi_p(l_2, l_2)$ for all exponent $0 < p < \infty$. In particular, $\xi_2(l_2, l_2)$ is the ideal of Hilbert-Schmidt operators.

In Banach spaces a little is known about the relation between $\xi_p$ and $S_p$ [1]. The relation between $\xi_p$ and the ideal of p-nuclear operators $N_p$, $0 < p < 1$ has been previously estimated [7].

This work investigates the relation between $\xi_p(l_u, l_v)$ and $S_p(l_u, l_v)$ for all exponents $0 < p < 1$.

Basic Notations

In the following $L$ denotes the class of all bounded linear operators between arbitrary Banach spaces and $L(E,F)$ the set of all such operators between specific Banach spaces $E$ and $F$. The closed unit ball of $E$ is denoted by $U$ and of $F$ is denoted by $V$. $L_n(E,F)$ denotes the subspace of $L(E,F)$ of operators of rank $(T) < n$. Let $l_u$ denote the Banach space of all $u$-absolutely summable sequences provided with the norm

$$\|x\|_u = \left\{ \sum |\xi_i|_u \right\}^{1/u} \text{ if } 1 \leq u < \infty$$

and

$$\|x\| = \sup |\xi_i| \text{ if } u = \infty$$

respectively.

We mention that the ideals of entropy numbers, of p-nuclear and of approximation numbers are denoted by $[\xi_p, E_p]$, $[N_p, \nu_p]$ and $[S_p, \sigma_p]$, respectively ([1, 4], [2, 3] and [5, 6]).

Main Result

Our main result can be obtained through a series of theorems. We begin with a result of Pietsch [6].

Theorem 1

Each mapping $T \in S_p(E,F)$ with $0 < p \leq 1$ can be represented as

$$T x = \sum_{r \in R} \lambda_r \langle x, a_r \rangle y_r$$

with linear forms $a_r \in U^o$ and elements $y_r \in V$, such that the inequality

$$\left\{ \sum_{R} |\lambda_r|_p \right\}^{1/p} \leq 2^{2+3/p} \sigma_p(T)$$

holds for the numbers $\lambda_r$. 
In the next theorem we need the well-known Auerbach’s lemma.

**Lemma:** Let $M$ be an $n$-dimensional normed linear space. Then there exists a basis $\{x_1, \ldots, x_n\}$ for $M$ and a subset $\{u'_1, \ldots, u'_n\}$ of $M'$ (the dual of $M$) such that

$$x = \sum_{i=1}^{n} \langle x, u'_i \rangle x_i$$

for each $x \in M$,

with $\|u'_i\| = \|x_i\| = 1$ and $\langle x_i, u'_j \rangle = \delta_{ij}$, $i, j = 1, 2, \ldots, n$.

**Theorem 2**

Let $E$ and $F$ be Banach spaces, and $0 < p < 1$. Then we have

$S_p(E, F) \subset N_p(E, F)$

and

$$\nu_p(T) \leq 2^{2+3/p} \sigma_p(T)$$

for each $T \in S_p(E, F)$.

**Proof:** By definition of approximation numbers, for $n = 1, 2, \ldots$ there exists an $A_n \in L^{\frac{1}{p}-1}(E, F)$ such that

$$\|T - A_n\| \leq 2^{s_{2,1}(T)}.$$

We now put

$$B_n = A_{n+1} - A_n,$$

$$\dim R(B_n) = d_n (R(B_n) \text{ denotes the range of } B_n),$$

$$i_0 = 0, i_r = \sum_{n=1}^{r} d_n$$

and

$$I_r = \{ \text{the integers in } [n_{r-1} - 1 + n_r, n_r] \}, \quad r = 1, 2, \ldots$$

Then, since the sequence $\{a_j(T)\}$ is decreasing, we have

$$\|B_n\| \leq 4 a_{\frac{n}{2n}}(T).$$

And since

$$d_n < 2^{n+1} - 1 + 2^n - 1 < 2^{n+2},$$

we have

$$i_r < 2^{3} (2^r - 1) < 2^{3+r}, \quad r = 1, 2, \ldots$$

By Auerbach’s lemma, there exist $\{u'_n\}_{n \in I_r} \subset F'$ and $\{y_n\}_{n \in I_r} \subset R(B_n)$ such that $\|u'_n\| = 1$, $\|y_n\| = 1$ and

$$B_r x = \sum_{n \in I_r} (B_r x, u'_n) y_n, \quad r = 1, 2, \ldots$$

for each $x \in E$. Putting

$$x'_n = B_r u'_n / \|y_n\|,$$

$$\lambda_n = \|B_r u'_n\| / \|y_n\|$$

for $n \in I_r$, $r = 1, 2, \ldots$, we have

$$B_r x = \sum_{n \in I_r} \lambda_n \langle x, x'_n \rangle y_n, \quad r = 1, 2, \ldots$$

By making use of these $\{x'_n\}_{n \in I_r}, \{y_n\}_{n \in I_r}$, $r = 1, 2, \ldots$, we can write
\[ T x = \lim_{r \to \infty} A_{r+1} x = \sum_{r=1}^{\infty} B_r x \]

\[ = \sum_{r=1}^{\infty} \sum_{n \in I_r} \lambda_n \langle x, x'_n \rangle y_n \quad \text{for each } x \in E, \]

with \( \|x'_n\| = 1, \|y_n\| = 1, \quad n = 1, 2, \ldots \). Therefore, for \( 0 < p \leq 1 \), we get

\[ \left\{ \nu_p(T) \right\}^p \leq \sum_{r=1}^{\infty} \sum_{n \in I_r} \lambda_1^p \leq \sum_{r=1}^{\infty} \sum_{n \in I_r} \|B_r\|^p \]

\[ \leq \sum_{r=1}^{\infty} 2^{r+2} (4a_{x_{r-1}}^2(T))^p \]

\[ \leq 2^{3+2p} \sum_{r=1}^{\infty} \sum_{n=2^{r-1}}^{2^r-1} \|a_n(T)\|^p \]

\[ \leq 2^{3+2p} \left\{ \sigma_p(T) \right\}^p . \]

Hence

\[ \nu_p(T) \leq 2^{2+3/p} \sigma_p(T), \]

which finishes the proof.

**Theorem 3**

Let \( 0 < p \leq 1 \). Then there exist diagonal operators \( D \) from \( l_1 \) into \( l_1 \) such that \( D \in \mathcal{N}_p(l_1, l_1) \) and \( D \notin \mathcal{S}_p(l_1, l_1) \).

The following example shows that the above theorem is true

**Example:** Let \( \lambda_k = 1/k - 1/(k+1) = 1/k(k+1) \) and define an operator \( T \in L(l_1, l_1) \) by \( T \{ \xi_1 \} = \{ \xi_n \} \).

Since

\[ \sum_{k=1}^{\infty} \lambda_k^p = \sum_{k=1}^{\infty} 1/(k^2 + k)^p \leq \sum_{k=1}^{\infty} 1/k^2 < \infty \quad \text{for } 1/2 < p \leq 1, \]

then \( T \) is \( p \)-nuclear, \( 1/2 < p \leq 1 \).

But we have

\[ \sigma_1(T) = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} (1/k - 1/(k+1)) = \sum_{n=0}^{\infty} 1/(n+1) = \infty , \]

which proves that \( T \notin \mathcal{S}_1(l_1, l_1) \). So \( T \notin \mathcal{S}_p(l_1, l_1) \) for every \( 0 < p \leq 1 \), [5].

Theorems 2 and 3 prove the following
Theorem 4
Let $0 < p \leq 1$. Then
\[ \text{Sp} (l_u, l_v) \supseteq \text{N}_p (l_u, l_v) \]
It is known that $\text{N}_p (l_2, l_2)$ is identical with $\text{Sp} (l_2, l_2)$ for $0 < p \leq 1$, [3].

The relation between $\text{N}_p (l_u, l_v)$ and $\xi_p (l_u, l_v)$ for $0 < p < 1$ has been investigated in [7]. However, we state the main result of [7] as

Theorem 5
Let $0 < p < 1$. Then
\[ \text{N}_p (l_u, l_v) = \xi_p (l_u, l_v). \]

As a consequence of theorems 4 and 5, we get

Theorem 6
Let $0 < p < 1$. Then
\[ \text{Sp} (l_u, l_v) \subseteq \xi_p (l_u, l_v). \]

This result answers problem raised in [1] in case $0 < p < 1$. 

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REFERENCES