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Cleavability of Decomposition Spaces

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الفضاءات القابلة للفلق المتحللة

عدنان البصول
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افرض أن X و Y لهما نفس نوع الهوموتوبي، إذا كان X (على الترتيب Y) قابلا" للفلق فوق ρ، برهنا أن الفضاء المتحلل Y/g (على الترتيب X/f) قابل للفلق فوق ρ. حصلنا على نتيجة حول مسألة أرهانكلسكي وتيروني، وبعض النتائج حول القابلة للفلق والفضاءات القابلة للإستقامة (القوية). أعطيت بعض المسائل المفتوحة ذات العلاقة مع نظريات الهوموتوبي والهومولوجي.

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ABSTRACT

Assume that X and Y have the same homotopy type, if X (respectively Y) is cleavable over ρ, we have proved that the decomposition space of Y/g (respectively X/f) is cleavable over ρ. A result concerning Arhangel'skii - Teroni's problem is obtained, also some results concerning cleavable spaces and (strong) straightenable spaces are obtained. Some open problems related with homotopy a and homology theories are given.

1. Introduction

In 1985 Arhangel'skii [1] introduced the notion of cleavability to be as a generalization of one - to - one continuous mapping. More recently Arhangel'skii and Teroni [2] introduced the notion of straightenability spaces that concerned with embeddings of spaces in product of two spaces, and proposed a general problem, then, they obtained some partial results concerning their problem. In this paper some results concerning some separation axioms, homotopy theory are obtained, and a result concerning Arhangel'skii -Teroni's problem is obtained.

Let ρ be a class of topological spaces. A space X is said to be *straightenable over* ρ if for every non-closed subspace A of X there is a space $Y \in \rho$ such that A is homeomorphic to a closed subspace of $X \times Y$. A space X is *straightenable over a space* Z if it is straightenable over the class of all subspaces of Z. A space X is said to be *cleavable* over a class ρ of spaces if for any subset A of X there are $Y \in \rho$ and a continuous mapping f of X onto Y such that $f(A) \cap f(X \setminus A) = \Phi$ or, equivalently $f^{-1}(f(A)) = A$. A class ρ of spaces is called *hereditary* if whenever $X \in \rho$, every subspace of X is in ρ. The key connection between cleavability and straightenability is: if a space

X is cleavable over a hereditary class ρ of Hausdorff spaces, then for any subspace A of X and for any subspace B of X containing A, there is $Z \in \rho$ such that A is homeomorphic to a closed subspace of $B \times Z$. Also, Arhangel'skii and Teroni [2] introduced a slightly stronger version of straightenability in the following way: A space X is *strongly straightenable* over a class ρ of spaces if for any subspace A of X, there is a space $Z \in \rho$ such that A is homeomorphic to a closed subspace of $Y \times Z$, where Y is the closure of A in X. It is worth to mention here that Arhangel'skii and Teroni [2] proposed the following problem.

Question 1. *When a compact space is straightenable over a "nice" class of spaces?*

If A is a subset of the topological space (X, τ), then the closure of A will be denoted by $cl(A)$. The identity mapping from the set X into itself will be denoted by 1_X.

Notation. Let $f: Y \to X$ be a mapping. The decomposition space $Y/\sim$, where $y_1 \sim y_2$ in Y if and only if $f(y_1) = f(y_2)$ will be denoted by Y/f.

2. Cleavable Spaces

In this section we shall give a result concerning Question 1, if we consider nice class of spaces to be T_5-spaces. Some results concerning some separation axioms will be given. To give the next results, we need the following definition.

Definition 2.1.

a. A space X is called a *Urysohn space* iff for each pair of distinct points x, y in X, there exist open sets U and V such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \Phi$. A Urysohn space also is called a $T_{2\frac{1}{2}}$ *- space*.

b. A space X is called a *functionally Hausdorff* space iff for each pair of distinct points x, y in X, there exists a continuous function $f: X \to [0,1]$ such that $f(x) = 0$, and $f(y) = 1$.

c. A space X is called *completely normal* iff every subspace of X is normal. A completely normal T_1-space is called a *T_5-space*.

Theorem 2.2. *If X is cleavable over a class of Urysohn spaces, then X is a Urysohn space.*

Proof. Let x and y be two distinct points in X. Consider the set $A = \{y\}$, since X is cleavable over ρ, there exists a Urysohn space $Z \in \rho$ and a continuous surjective mapping f from X onto Z such that $f(X \setminus \{y\}) \cap f(\{y\}) = \Phi$. Since Z is a Urysohn space, there are disjoint open sets U and V in Z such that $f(x) \in U$ and $f(y) \in V$ with $cl(U) \cap cl(V) = \Phi$. Hence, $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$, since f is continuous, so $f^{-1}(U)$ and $f^{-1}(V)$ are open in X, and $cl(f^{-1}(U)) \cap cl(f^{-1}(V)) = \Phi$. □

Theorem 2.3. *If X is cleavable over a class ρ of functionally Hausdorff spaces, then X is functionally Hausdorff.* □

Theorem 2.4. *If X is cleavable over a class ρ of functionally Hausdorff (Urysohn) spaces, then X is (strongly) straightenable over ρ.*

Proof. Let $A \subseteq X$. Since X is cleavable over ρ, there exist $Z \in \rho$ and a continuous mapping f from X onto Z such that $f^{-1}(f(A)) = A$. Consider the graph of f,

$$G(f) = \{(x, f(x)) \mid x \in A\}.$$

Since X is Hausdorff, so $G(f)$ is closed in $X \times Z$, and hence is closed in $cl(A) \times Z$. □

It is obvious that, if X is cleavable over a hereditary class ρ of T_i - spaces; $i = 2,\ 2\frac{1}{2},\ 3,\ 3\frac{1}{2},\ 4,\ 5$, then X is strongly straightenable over ρ. Now, we shall give a partial answer of Question 1, in case of T_5 - spaces.

Theorem 2.5. *Let X be a compact space. If X is cleavable over a class ρ of T_5-spaces, then X is T_5.*

Proof. Let A and B be two subsets of X such that $cl(A) \cap B = A \cap cl(B) = \Phi$. For the subset $cl(A)$, there exist a T_5-space Z and a continuous surjective mapping f of X onto Z such that $f^{-1}(f(cl(A))) = cl(A)$. Since X is compact, thus $cl(A)$ and $cl(B)$ are compact subsets of X, by continuity of f, $f(cl(A))$ and $f(cl(B))$ are closed subsets of Z. Thus, $f(cl(A)) = cl(f(A))$ and $f(cl(B)) = cl(f(B))$. Now, it is easy to see that, $cl(f(B)) \cap f(A) = \Phi$ and $cl(f(A)) \cap f(B) = \Phi$.

Since Z is completely normal, there are two disjoint open sets G and H in Z such that $f(A) \subseteq G$ and $f(B) \subseteq H$. Hence, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint open sets in X containing A and B respectively. Therefore, X is T_5. □

Corollary 2.6. *Let X be a compact space. If X is cleavable over a hereditary class ρ of T_5-spaces, then X is (strongly) straightenable over ρ.* □

3. Cleavability and Same Homotopy Type

In this section we shall study the cleavability of quotient spaces obtained by functions. Also, we shall study the concept "same homotopy type" with cleavability. Some open problems will be given.

Theorem 3.1. *Let X be a cleavable space over a hereditary class ρ of spaces. If there is a continuous surjective mapping $f : Y \to X$ from a space Y onto X, then, the decomposition space Y/f is cleavable over ρ.*

Proof. We have the following diagram:

$$\begin{tikzcd} Y \arrow[rr, "P"] \arrow[dr, "f"'] & & Y/f \arrow[dl, "F"] \\ & X & \end{tikzcd}$$

where P is the natural map, given by $P(y) = [y]$, and $F : Y/f \to X$ is the map defined by $F([y]) = f(y)$. So, for each $y \in Y$, we have

$$(F \circ P)(y) = F(P(y)) = F([y]) = f(y)$$

hence $F \circ P = f$. Since f is continuous, so F is continuous. Now, if $[y_1] \neq [y_2]$ in Y/f, hence $[y_1] \cap [y_2] = \Phi$, thus $f(y_1) \neq f(y_2)$ and so $F([y_1]) = f(y_1) \neq f(y_2) = F([y_2])$, so F is one - to - one. Let $x \in X$, since f is surjective, there is $y_0 \in Y$ such that $f(y_0) = x$, so $x = f(y_0) = (F \circ P)(y_0) = F([y_0])$, and hence F is onto. By Theorem 3.1, Y/f is cleavable over ρ. □

Corollary 3.2. *Let X be a cleavable space over a hereditary class ρ of spaces. If there is a continuous bijective mapping $g : Y \to X$ from a space Y onto X, then Y is cleavable over ρ.* □

Now, it is obvious that *cleavability over a hereditary class ρ of spaces is a topological property.*

Definition 3. 3 [3]. Two spaces X and Y are of the *same homotopy type* if there exist continuous maps $f : X \to Y$, and $g : Y \to X$ such that gof is homotopic to 1_X and fog is homotopic to 1_Y. The maps f and g are called *homotopy equivalences.*

Theorem 3.4. *Assume that the spaces X and Y have the same homotopy type. If X is cleavable over a hereditary class ρ of spaces, and $g : Y \to X$ is a homotopy equivalence, then the decomposition space Y/g of Y is cleavable over ρ.*

Proof. Since X and Y have the same homotopy type, there are homotopy equivalences $f : X \to Y$, $g : Y \to X$. Consider the following commutative diagram:

$$\begin{tikzcd} Y \arrow[rr, "P"] \arrow[dr, "g"'] & & Y/g \arrow[dl, "G"] \\ & X & \end{tikzcd}$$

where P is the natural map, and the map $G : Y / g \to X$ is defined by

$$G([y]) = g(y) \text{ for all } [y] \in Y / g.$$

Then G is continuous, because g is continuous and $g = G \circ P$. Also, G is injective.

Now, let $B \subseteq Y / g$. Since X is cleavable over ρ, there exists $Z \in \rho$ and a continuous surjective mapping $h : X \to Z$ such that $h(G(B)) \cap h(X \setminus G(B)) = \Phi$.

Let $W = h(G(Y / g)) \subseteq Z$, so $W \in \rho$. Define a continuous surjective mapping $k : (Y/g) \to W$ by

$$k = h\,|_{G(Y/g)} \circ F = H \circ F,$$

where $F : (Y / g) \to G(Y / g)$, given by $F([y]) = G([y])$ for all $[y] \in Y / g$, and $H = h\,|_{G(Y/g)} : G(Y / g) \to W$, given by $H(x) = h(x)$ for all $x \in G(Y / g)$. In fact we have the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "P"] \arrow[d, "g"'] & Y/g \arrow[dl, "G"'] \arrow[d, "F"] \\
X \arrow[d, "h"'] \arrow[dr, "h|_{G(Y/g)}"] & G(Y/g) \arrow[l, "i"'] \arrow[d, "H"] \\
Z & W \arrow[l, "j"]
\end{tikzcd}$$

where i, and j are the inclusion maps. Hence, $k(B) \cap k((Y / g) \setminus B) = \Phi$. □

Corollary 3.5. *Assume that $f : X \to Y$, $g : Y \to X$ are homotopy equivalences. If one of the spaces X, and Y is cleavable over the hereditary class ρ of spaces, then the decomposition space of the other is cleavable over ρ, where the decomposition space of X or Y is the space X / f and Y / g, respectively.* □

Corollary 3.6. *Assume that X is cleavable over a hereditary class ρ of spaces. If there is a continuous mapping $f : Y \to X$. Then the decomposition space Y / f is cleavable over ρ.* □

Let us close this paper by the following general open problems:

Question 2. *Assume that X is cleavable over a (hereditary) class ρ of spaces and assume that X and Y have the same homotopy type. What are the conditions on ρ that makes Y cleavable over ρ?*

Question 3. *Assume that X is cleavable over a (hereditary) class ρ of spaces and assume that the first fundamental group $\pi_1(X, x_0)$ of X at some point $x_0 \in X$ is isomorphic to the first fundamental group $\pi_1(Y, y_0)$ of Y at some point $y_0 \in Y$. What are the conditions on ρ that makes Y cleavable over ρ?*

Question 4. *Assume that X is cleavable over a (hereditary) class ρ of spaces and assume that the nth singular homology group $H_n(X)$ of X is isomorphic to the nth singular homology group $H_n(Y)$ of Y for all $n \geq 1$. What are the conditions on ρ that makes Y cleavable over ρ?*

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