

ON AN INVERSE SCATTERING PROBLEM WITH A DISCONTINUOUS COEFFICIENT ON THE WHOLE LINE

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مسألة تشتتية عكسية ذات معامل غير متصل على خط الأعداد

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يختص هذا البحث بدراسة المسألة التشتتية العكسية لمعادلة شتورم ليوفيل المعدلة على خط الأعداد . وقد أعطيت حلول المسألة المدروسة مع سلوكها التقريبي واستنتجت معادلات بارسيغال من المفكوك بدلالة الدوال الذاتية . وتم الحصول على بيانات التشتت اليميني واليسري وحل مسألة التشتت العكسية للمسألة المعطاه .

Key Words: Scattering, Inverse problem, Discontinuous coefficient.

ABSTRACT

This paper is devoted to study the inverse scattering problem of a generalized Sturm-Liouville equation on the whole line. The solutions of the considered problem and their asymptotic behaviour are presented. The expansion by eigenfunctions of the considered problem and hence Parseval equations are given. The right and left scattering data are obtained. The inverse scattering problem is formulated and completely solved.

INTRODUCTION

Let us consider the modified Sturm - Liouville differential equation

$$-y'' + v(x)y = k^2 \rho(x)y \quad , -\infty < x < \infty \quad , \quad (1)$$

where $v(x)$ is a real valued potential which satisfies the condition

$$\int_{-\infty}^{+\infty} (1 + |x|)|v(x)|dx < \infty \quad (2)$$

and the density function $\rho(x)$ is discontinuous at $x = 0$ such that

$$\rho(x) = \begin{cases} 1, & x \geq 0 \\ -\alpha^2, & x < 0, \alpha^2 \neq -1. \end{cases} \quad (3)$$

It is well known that [5] the harmonic solution of the equations of an infinite stretched string with a different density are determined by equation (1).

This paper is devoted to study the inverse scattering problem for (1). Here, the inverse scattering problem for equation (1) can be stated as follows: Is it possible to define uniquely the scattering data for any given function $v(x)$ in which $v(x) \rightarrow 0$ as a certain sufficiently rapid rate as $|x| \rightarrow \infty$. If this is so can $v(x)$ and $\rho(x)$ be reconstructed from the scattering data? .

The inverse scattering problem of equation (1) was investigated in [7,8,9] as $\rho(x) = 1$. Also, the inverse scattering problem of an eigenvalue problems of Sturm - Liouville type with discontinuous coefficients was studied in [1,3,6,10].

In the following treatments, the solutions of equation (1) will be given. An expansion of a certain function by an eigenfunctions of (1) will be presented and hence Parseval equations will be deduced. Also, the right and left scattering data of (1) will be

obtained. Moreover, the inverse scattering problem of (1) will be formulated and completely solved.

PRELIMINARIES

In this section, we collect those preliminary notations and relations from [2,4,8] which we shall use in the subsequent sections.

Denote by $y_1(x, k)$ and $y_2(x, k)$ the solution of equation (1) as $x \geq 0$ and $x < 0$ respectively.

Lemma 1: For every value of k from the closed upper half plane, equation (1) has the solutions $y_1(x, k)$ and $y_2(x, k)$ which are represented in the forms

$$y_1(x, k) = \exp(ikx) + \int_x^{\infty} K^+(x, z) \exp(ikz) dz, \quad x \geq 0 \quad (4)$$

and

$$y_2(x, k) = \exp(\alpha kx) + \int_x^{\infty} \bar{K}(x, z) \exp(\alpha kz) dz, \quad x < 0 \quad (5)$$

Here, the kernels $k^\pm(x, z)$ are differentiable with respect to x and z and that satisfy the conditions

$$\frac{\partial^2 k^\pm(x, z)}{\partial x^2} - \frac{\partial^2 k^\pm(x, z)}{\partial z^2} = V^\pm(x) K^\pm(x, z)$$

and

$$K^+(x, x) = \frac{1}{2} \int_x^{\infty} V^+(z) dz, \quad K^-(x, x) = \frac{1}{2} \int_x^{\infty} V^-(z) dz. \quad (6)$$

Lemma 2: The solutions $y_1(x, k)$ and $y_2(x, k)$ have the following asymptotic behaviour

$$y_1(x, k) = \exp(ikx) \chi_{1+o(1)}, \quad y_1(x, -k) = \exp(-ikx) \chi_{1+o(1)} \quad (7)$$

as $x \rightarrow \infty$

and

$$y_2(x, k) = \exp(\alpha kx) \chi_{1+o(1)}, \quad y_2(x, -k) = \exp(-\alpha kx) \chi_{1+o(1)} \quad (8)$$

as $x \rightarrow -\infty$.

Moreover, the solutions $y_1(x, k)$ and $y_2(x, k)$ are defined on the forms

$$y_1(x, k) = \begin{cases} \exp(ikx) + \int_x^{\infty} k^+(x, z) \exp(ikz) dz, & x \geq 0 \\ -a(k) y_2(x, k) + b(k) y_2(x, -k), & x < 0 \end{cases}$$

and

$$y_2(x, k) = \begin{cases} a_1(k) y_1(x, k) + b_1(k) y_1(x, -k), & x \geq 0 \\ \exp(\alpha kx) + \int_x^{\infty} \bar{k}(x, z) \exp(\alpha kz) dz, & x < 0; \end{cases}$$

where

$$a(k) = (2k\alpha)^{-1} [y_2(o, -k) y_1'(o, k) - y_2'(o, -k) y_1(o, k)] \quad (9)$$

$$b(k) = (2k\alpha)^{-1} [y_2(o, k) y_1'(o, k) - y_2'(o, k) y_1(o, k)] \quad (10)$$

$$a_1(k) = (2ik)^{-1} [y_2(o, k) y_1'(o, -k) - y_2'(o, k) y_1(o, -k)] \quad (11)$$

and

$$b_1(k) = (2ik)^{-1} [y_1(o, k) y_2'(o, k) - y_1'(o, k) y_2(o, k)] \quad (12)$$

From these equations we find that $a_1(k)$ and $b_1(k)$ are analytical in the upper half plane of k that having the asymptotic behaviours

$$a_1(k) = \left(\frac{1-i\alpha}{2}\right) + O\left(\frac{1}{|k|}\right) \quad \text{and} \quad b_1(k) = \left(\frac{1+i\alpha}{2}\right) + O\left(\frac{1}{|k|}\right).$$

Combining (9), (10), (11) and (12) one obtains

$$u^-(x, k) = i\alpha T(k) y_1(x, k) = r^-(k) y_2(x, k) + y_2(x, -k) \quad (13)$$

and

$$u^+(x, k) = T(k) y_2(x, k) = r^+(k) y_1(x, k) + y_1(x, -k) \quad (14)$$

where

$$r(k) = -\frac{a_1(-k)}{b_1(k)}; \quad r^+(k) = \frac{a_1(k)}{b_1(k)} \quad (15)$$

$$\text{and} \quad T(k) = \frac{1}{b_1(k)}.$$

Accordingly, in view of (4), (5), (13) and (14) these solutions have the following asymptotics form

$$\bar{u}(x, k) = \begin{cases} i\alpha T(k) \exp(ikx) + o(1), & x \rightarrow \infty \\ r^-(k) \exp(\alpha kx) + \exp(-\alpha kx) + o(1), & x \rightarrow -\infty \end{cases}$$

and

$$u(x, k) = \begin{cases} r^+(k) \exp(ikx) + \exp(-ikx) + o(1), & x \rightarrow \infty \\ T(k) \exp(\alpha kx) + o(1), & x \rightarrow -\infty. \end{cases}$$

The functions $u^-(x, k)$ and $u^+(x, k)$ are called the eigenfunctions of the left and right scattering problem of (1) respectively and the coefficients $r(k)$; $r^+(k)$ and $T(k)$ are called the left and right reflection coefficients and the transmission coefficient respectively. In [2] it was shown that the discrete eigenvalues of equation (1) are the simple zeros $-\chi_n^2$ of the function b_1 in the upper half plane and

$$b_1(i\chi_n) = -i \int_{-\infty}^{\infty} y_1(x, i\chi_n) y_2(x, i\chi_n) \rho(x) dx$$

Also, the corresponding eigenfunctions are related by

$$\bar{u}^-(x, i\chi_n) = c_n^- u^+(x, i\chi_n) = y_2(x, k) \quad (16)$$

and

$$u^+(x, i\kappa_n) = c_n^+ u^-(x, i\kappa_n) = y_1(x, k), \text{ where } c_n^- c_n^+ = 1. \quad (17)$$

EIGENFUNCTION EXPANSION AND PARSEVAL EQUATIONS OF (1).

In this respect, we give an expansion of a certain function by eigenfunctions of equation (1) and hence we deduce Parseval equations. Also, we define the right and left scattering data of equation (1).

Lemma 3: Suppose that the function $f(x)$ has a continuous second derivative in $L_2(-\infty, \infty; \rho(x))$ and finite in the neighbourhood of $x = \pm\infty$. Then, the expansion of $f(x)$ by eigenfunctions of (1) can be written in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{r^+(k)y_1(x, k)y_1(\zeta, k) + y_1(x, k)y_1(\zeta, -k)\} \rho(\zeta) f(\zeta) d\zeta dk + \sum_{n=1}^{\lambda} (m_n^+)^2 \left[\int_{-\infty}^{\infty} y_1(x, i\kappa_n) y_1(\zeta, i\kappa_n) \rho(\zeta) f(\zeta) d\zeta \right],$$

$$\text{where } (m_n^+)^2 = \left\{ \int_{-\infty}^{\infty} |y_1(x, i\kappa_n)|^2 \rho(x) dx \right\}^{-1} \quad (18)$$

With the help of this lemma, the following Parseval equations are valid *

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \{r^+(k)y_1(x, k)y_1(\zeta, k) + y_1(x, k)y_1(\zeta, -k)\} dk + \sum_{n=1}^{\lambda} (m_n^+)^2 y_1(x, i\kappa_n) y_1(\zeta, i\kappa_n) = \delta(x-\zeta) \rho^{-1}(x) \quad (19)$$

and

$$\frac{1}{2\pi\alpha i} \int_{-\infty}^{\infty} \{r^-(k)y_2(x, k)y_2(\zeta, k) + y_2(x, k)y_2(\zeta, -k)\} dk + \sum_{n=1}^{\lambda} (m_n^-)^2 y_2(x, i\kappa_n) y_2(\zeta, i\kappa_n) = \delta(x-\zeta) \rho^{-1}(x).$$

* since $f(x) = \int_{-\infty}^{+\infty} f(\zeta) \delta(x - \zeta) d\zeta$, then comparing both sides of

(18) to have (19).

FORMULATION OF THE INVERSE SCATTERING PROBLEM OF (1).

In the preceding section, we determined the right and left scattering data for equation (1) in the form

$\{r^\pm(k), -\kappa_n^2, m_n^\pm, n = \overline{1, \lambda}\}$. In this section, we formulate the inverse scattering problem as follows: knowing the right or the left scattering data of equation (1); can we find:

- (i) $v(x)$ and $\rho(x)$ in equation (1) ?
- (ii) necessary and sufficient conditions for the set

$\{r^\pm(k), -\kappa_n^2, m_n^\pm, n = \overline{1, \lambda}\}$ to be the scattering data of (1)?

Now, in order to solve this problem we find the fundamental equation of the kernel $K^+(x, z)$ which plays an important role for solving the inverse scattering problem of equation (1).

Theorem 1: The kernel $K^+(x, z)$ of the formula (4) satisfies the fundamental equation

$$F^+(x+z) + k^+(x, z) + \int_{-\infty}^{\infty} F^+(z+\zeta) k^+(x, \zeta) d\zeta = 0, 0 < x \leq z < \infty, \quad (20)$$

where

$$F^+(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ r^+(k) - \frac{1-\alpha i}{1+\alpha i} \right\} \exp(ikx) dk + \sum_{n=1}^{\lambda} (m_n^+)^2 \exp(-\kappa_n x). \quad (21)$$

Proof: To derive the fundamental equation we use the identity (14) and (15). Thus,

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{r^+(k)y_1(x, k) + y_1(x, -k)\} \exp(ikz) dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[r^+(k) - \frac{1-\alpha i}{1+\alpha i} \right] \exp(ik(x+z)) dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1-\alpha i}{1+\alpha i} \exp(ik(x+z)) + \exp(-ik(x+z)) \right] dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} k^+(x, \zeta) \int_{-\infty}^{\infty} \left\{ r^+(k) - \frac{1-\alpha i}{1+\alpha i} \right\} \exp(ik(z+\zeta)) dk dz + \frac{1}{2\pi} \int_{-\infty}^{\infty} k^+(x, \zeta) \int_{-\infty}^{\infty} \left\{ \frac{1-\alpha i}{1+\alpha i} \exp(ik(z+\zeta)) + \exp(ik(z-\zeta)) \right\} dk dz = F_1^+(x+z) + k^+(x, z) + \int_x^{\infty} k^+(x, \zeta) F_1(z+\zeta) d\zeta + \delta(x-z),$$

where

$$F_1^+(x+z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ r^+(k) - \frac{1-\alpha i}{1+\alpha i} \right\} \exp(ik(x+z)) dk.$$

On the other hand

$$\exp(ikz) = y_1(z, k) + \int_z^{\infty} R(z, \zeta) y_1(\zeta, k) d\zeta.$$

Therefore,

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{r^+(k)y_1(x, k) + y_1(x, -k)\} \{y_1(z, k) + \int_z^{\infty} R(z, \zeta) y_1(\zeta, k) d\zeta\} dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{r^+(k)y_1(x,k)y_1(z,k) + y_1(x,-k)y_1(z,k)\} dk$$

$$+ \frac{1}{2\pi} \int_z^{\infty} R(z,\zeta) \int_{-\infty}^{\infty} \{r^+(k)y_1(x,k)y_1(\zeta,k) + y_1(x,-k)y_1(\zeta,k)\} dk d\zeta$$

$$= \delta(x-z) - \sum_{n=1}^{\lambda} (m_n^+)^2 y_1(x, i\kappa_n) y_1(z, i\kappa_n)$$

$$+ \int_z^{\infty} R(z,\zeta) \delta(x-\zeta) d\zeta - \sum_{n=1}^{\lambda} (m_n^+)^2 y_1(x, i\kappa_n) \int_z^{\infty} R(z,\zeta) y_1(\zeta, i\kappa_n) d\zeta$$

$$= \delta(x-z) + R(z,x) - \sum_{n=1}^{\lambda} (m_n^+)^2 \left[\exp(-\kappa_n(x+z)) + \int_x^{\infty} K^+(x,\zeta) \exp(-\kappa_n(z+\zeta)) d\zeta \right]$$

Since $R(z,x) = 0$ as $x < z$, it yields that

$$I = \delta(x-z) - \sum_{n=1}^{\lambda} (m_n^+)^2 \left[\exp(-\kappa_n(x+z)) + \int_x^{\infty} K^+(x,\zeta) \exp(-\kappa_n(z+\zeta)) d\zeta \right].$$

Thus, $F_1^+(x+z) + K^+(x,z) + \int_x^{\infty} K^+(x,\zeta) F_1(z+\zeta) d\zeta + \delta(x-z)$

$$= \delta(x-z) - \sum_{n=1}^{\lambda} (m_n^+)^2 \left[\exp(-\kappa_n(x+z)) + \int_x^{\infty} K^+(x,\zeta) \exp(-\kappa_n(z+\zeta)) d\zeta \right].$$

Therefore, $F_1^+(x+z) + K^+(x,z) + \sum_{n=1}^{\lambda} (m_n^+)^2 \exp(-\kappa_n(x+z))$

$$+ \int_x^{\infty} K^+(x,\zeta) \left[F_1(z+\zeta) + \sum_{n=1}^{\lambda} (m_n^+)^2 \exp(-\kappa_n(z+\zeta)) \right] d\zeta = 0.$$

Hence, we conclude that

$$F^+(x+z) + K^+(x,z) + \int_x^{\infty} F^+(z+\zeta) K^+(x,\zeta) d\zeta = 0, \quad (22)$$

where $F^+(x)$ is defined by (21).

Proceeding in a similar manner with identity (16), we obtain the equality

$$F^-(x+z) + K^-(x,z) + \int_x^{\infty} F^-(z+\zeta) k^-(x,\zeta) d\zeta = 0, \quad \zeta < x < 0 \quad (23).$$

in which the function

$$F^-(x) = \sum_{n=1}^{\lambda} (m_n^-)^2 \exp(i\alpha\kappa_n x) + \frac{1}{2\pi} \int_{-\infty}^x \left\{ r^-(k) + \frac{1-\alpha i}{1+\alpha i} \right\} \exp(ikx) dk$$

is completely specified by the left scattering data (23).

Note: It can be shown that [8,9] the functions $F_1^{\pm}(x)$ are absolutely continuous with the properties:

i) $\int_0^{\infty} F_1^+(x) dx < \infty$ and $\int_0^{\infty} \left| \frac{\partial}{\partial x} F_1^+(x) \right| dx < \infty.$

ii) $\int_{-\infty}^0 F_1^-(x) dx < \infty$ and $\int_{-\infty}^0 \left| \frac{\partial}{\partial x} F_1^-(x) \right| dx < \infty.$

Next, let us consider two arbitrary collections

$$\{r^+(k), i\kappa_n^+, m_n^+\} \text{ and } \{r^-(k), i\kappa_n^-, m_n^-\}, n = \overline{1, \lambda}; \kappa_n^+ > 0$$

and $m_n^+ > 0$ such that the functions $r^+(k)$ satisfy the previous conditions (i) - (ii). From these collections we construct the fundamental equations (20) and (22) for the unknown functions $k^{\pm}(x,z)$ such that these equations are considered in $L_1(x, \infty)$ and $L_1(-\infty, x)$, respectively.

Theorem 2 :- For every fixed $x \in [0, \infty)$ the fundamental equation (20) has a unique solution $K(x,z)$ in $L_2(x, \infty)$.

Proof : To prove the theorem, it is enough to show that the

$$\text{homogenous equation } f(z) + \int_x^{\infty} f(\zeta) F(\zeta+z) d\zeta = 0 \quad (24)$$

has only the zero solution in $L_1(x, \infty)$. Assume the contrary, that is, the homogeneous equation (24) has a non zero solution $f(z)$ in $L_2(x, \infty)$. Multiply (24) by $f(z)$ scalarly and integrating with

respect to z on (x, ∞) to have $\int_x^{\infty} f^2(z) dz + \int_x^{\infty} \int_x^{\infty} f(\zeta) F^*(z+\zeta) d\zeta dz = 0.$

Substituting for $F^*(z+\zeta)$ from (21) to get

$$\int_x^{\infty} f^2(z) dz + \int_x^{\infty} \int_x^{\infty} f(\zeta) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[r^+(k) - \frac{1-\alpha i}{1+\alpha i} \right] \exp(i(z+\zeta)) \right.$$

$$\left. + \sum_{n=1}^{\lambda} (m_n^+)^2 \exp(-\kappa_n(z+\zeta)) + \delta(\zeta-z) - \delta(\zeta-z) \right\} f(z) d\zeta dz = 0.$$

Thus, we have

$$\int_x^{\infty} f^2(z) dz + \int_x^{\infty} \int_x^{\infty} f(\zeta) F^*(z+\zeta) d\zeta dz = 0$$

Therefore

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \{r^+(k) |\psi(k)|^2 + |\psi(k)|^2\} dk + \sum_{n=1}^{\lambda} (m_n^+)^2 \psi^2(i\kappa_n) = 0, \text{ where}$$

$$\psi(k) = \int_x^{\infty} f(z) \exp(ikz) dz$$

From (18), we have $|\text{Re } r^+(k)| < 1$ as $\text{Im } k = 0$, and therefore $\psi(k) = 0$. Hence $f(z) = 0$. Accordingly, our assumption that the homogenous equation (24) has non zero solution is not true and the theorem is completely proved.

Similarly, as $x < 0$ the unique solvability of equation (22) is proved in a similar manner. Hence, we conclude that it is sufficient to know the function $F^+(x)$ to form the fundamental equation (20) and in its turn, to find $F^+(x)$ it is sufficient to know the right scattering data $\{r^+(k), i\kappa_n^+, m_n^+ = \overline{1, m}\}$. Equation (20) plays an important role in the solution of the inverse scattering problem on the interval $[0, \infty)$.

Thus, if the equation (20) is constructed by using the scattering data and has a unique solution $K^*(x,z)$ then equation (1) as $0 \leq x < \infty$, can be constructed.

Finally, we have been interested in the relationships between the left and right scattering data. It turns out that the left scattering data are uniquely determined by the right ones, and vice versa. In fact, it follows from formulas (16), (17) and (18) that

$$r^-(k) = -r^+(-k) \frac{b_1(-k)}{b_1(k)} \text{ and } (m_n^-)^{-2} = \frac{1}{\alpha_2} (m_n^+)^2 [b_1(x, i\kappa_n)] \quad (25)$$

Hence, we can find $v(x)$ as $x < 0$.

Theorem 3: In order for the collection

$\{r^+(k), i\kappa_n^+, m_n^+, n = \overline{1, \lambda}, \kappa_n^+ > 0, m_n^+ > 0\}$ to be the right scattering data of an equation of the form (1), it is necessary and sufficient that the following conditions be satisfied:

1) for real $k \neq 0$, the reflection coefficient $r^\pm(k)$ are continuous and having the properties:

(i) $r^+(-k) = \overline{r^+(k)}$; $|r^\pm(k)| < 1$;

(ii) $r^\pm(k) = \pm \frac{1-\alpha i}{1+\alpha i} + o\left(\frac{1}{k}\right)$ as $k \rightarrow \pm\infty$;

2) The function $F_1^+(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[r^+(k) - \frac{1-\alpha i}{1+\alpha i} \right] \exp(ikx) dk$

is absolutely continuous with continuous derivative such that

$$\int_0^{\infty} (1+|x|) \left| \frac{\partial}{\partial x} F_1^+(x) \right| dx < \infty;$$

3) The function

$$F_1^-(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \left[r^+(-k) \frac{b_1(-k)}{b_1(k)} - \frac{1-\alpha i}{1+\alpha i} \right] \exp(\alpha kx) dk$$

is absolutely continuous with continuous derivative such that

$$\int_{-\infty}^0 (1+|x|) \left| \frac{\partial}{\partial x} F_1^-(x) \right| dx < \infty.$$

Proof: The necessity of these conditions has been established previously. To prove their sufficiency, we construct from the given set a new set

$$\{r^-(k), i\kappa_n^-, m_n^-, n = \overline{1, \lambda}, \kappa_n^- > 0, m_n^- > 0\}.$$

Setting

$$r^-(k) = -r^+(-k) \frac{b_1(-k)}{b_1(k)} \text{ and } (m_n^-)^{-2} = \frac{1}{\alpha_2} (m_n^+)^2 [b_1(x, i\kappa_n^+)]$$

and then show that $\{r^+(k), i\kappa_n^+, m_n^+\}$ and $\{r^-(k), i\kappa_n^-, m_n^-\}$ are the right and left scattering data of one and the same equation

(1) with real valued potential $v(x)$ which subject to (2) and including a discontinuous coefficient $\rho(x)$ on the form (3).

The function $\rho(x)$ can be constructed from condition (1). Moreover, it follows from conditions 1,2 and 3 that for every fixed

$x \neq \pm\infty$, equations (20) and (22) which constructed from these sets, have unique solutions $k^\pm(x,z)$ and the functions $y_1(x,y)$, $y_2(x,k)$ defined by formulas (4) and (5), are solutions of equations

$$-y_1''(x,k) + v^+(x)y_1(x,k) = k^2 y_1(x,k), x \geq 0$$

and

$$-y_2''(x,k) + v^-(x)y_2(x,k) = -k^2 \alpha^2 y_2(x,k), x < 0$$

where

$$v^+(x) = -2 \frac{dK^+(x,x)}{dx}; v^-(x) = -2 \frac{dK^+(x,x)}{dx} \text{ and } \int_{-\infty}^{\infty} (1+|x|) |v^+(x)| dx < \infty$$

for all $a > -\infty$.

To prove the theorem, it obviously suffices to show that for real

values of k , the solutions $y_1(x,k)$ and $y_2(x,k)$ are connected by the relations

$$i\alpha [b_1(k)]^{-1} y_1(x,k) = r^-(k) y_2(x,k) + y_2(x,-k) \quad (26)$$

and

$$[b_1(k)]^{-1} y_2(x,k) = r^+(k) y_1(x,k) + y_1(x,-k) \quad (27)$$

Moreover,

$$(m_n^+)^{-2} = \int_{-\infty}^{\infty} \rho(x) |y_2(x, i\kappa_n^+)|^2 dx; (m_n^-)^{-2} = \int_{-\infty}^{\infty} \rho(x) |y_1(x, i\kappa_n^+)|^2 dx \quad (28)$$

Since, $F_1^+(x) \in L_2(-\infty, \infty)$, thus the functions

$$\psi^+(x,z) = F_1^+(x+z) + \int_x^{\infty} k^+(x,\zeta) F_1^+(z+\zeta) d\zeta, x \geq 0$$

and

$$\psi^-(x,z) = F_1^-(x+z) + \int_{-\infty}^x k^-(x,\zeta) F_1^-(z+\zeta) d\zeta, x < 0$$

belong to $L_2(-\infty, \infty)$ for every fixed x and

$$\lim_{R \rightarrow \infty} \int_R^k \psi^+(x,z) \exp(-ikz) dz = \left[r^+(k) - \frac{1-\alpha i}{1+\alpha i} \right] \left[\exp(ikx) + \int_x^{\infty} k^+(x,\zeta) \exp(ik\zeta) d\zeta \right]$$

$$= \left[r^+(k) - \frac{1-\alpha i}{1+\alpha i} \right] y_1(x,k).$$

Also,

$$\lim_{R \rightarrow -\infty} \int_R^k \psi^-(x,z) \exp(-\alpha k z) dz = \left[r^-(k) + \frac{1-\alpha i}{1+\alpha i} \right] \left[\exp(\alpha kx) + \int_{-\infty}^x k^-(x,\zeta) \exp(\alpha k\zeta) d\zeta \right]$$

$$= \left[r^-(k) + \frac{1-\alpha i}{1+\alpha i} \right] y_2(x,k).$$

On the other hand, using equations (20) and (22) to get

$$\psi^+(x,z) = -k^+(x,z) - \sum_{n=1}^{\lambda} (m_n^+)^2 \exp(-i\kappa_n x) y_1(x,k)$$

and

$$\psi^-(x,z) = -k^-(x,z) - \sum_{n=1}^{\lambda} (m_n^-)^2 \exp(i\alpha \kappa_n x) y_2(x,k).$$

Thus

$$\lim_{R \rightarrow \infty} \int_{-R}^R \Psi^+(x,z) \exp(-ikz) dz = \lim_{R \rightarrow \infty} \int_{-R}^R \Psi^+(x,z) \exp(-ikz) dz + \exp(-ikx) - y_1(x,-k) - \sum_{n=1}^{\lambda} \frac{(m_n^+)^2}{\kappa_n + ik} \exp(-\kappa_n x) \exp(-ikz) y_1(x, i\kappa_n)$$

and

$$\lim_{R \rightarrow \infty} \int_{-R}^R \Psi^-(x,z) \exp(akz) dz = \lim_{R \rightarrow \infty} \int_{-R}^R \Psi^-(x,z) \exp(akz) dz + \exp(akx) - y_2(x,-k) - \sum_{n=1}^{\lambda} \frac{(m_n^-)^2}{\alpha(\kappa_n + ik)} \exp(i\alpha\kappa_n x) \exp(akz) y_2(x, i\kappa_n)$$

Comparing these identities with the Fourier transformations of the functions $\Psi^{\pm}(x,z)$ we have the equalities

$$r^+(k) y_1(x,k) + y_1(x,-k) = [b_1(k)]^{-1} g^-(x,k) \tag{29}$$

and

$$r^-(k) y_2(x,k) + y_2(x,-k) = i\alpha [b_1(k)]^{-1} g^+(x,k), \quad -\infty < k < \infty, \tag{30}$$

where

$$g^+(x,k) = \frac{\exp(-\alpha k x)}{i\alpha} b_1(k) \left[1 + \lim_{R \rightarrow \infty} \int_x^R \Psi^-(x,z) \exp(\alpha k(z-x)) dz - \sum_{n=1}^{\lambda} \frac{(m_n^-)^2}{i\alpha(\kappa_n + ik)} \exp(i\alpha\kappa_n x) y_2(x, i\kappa_n) - \frac{1 - \alpha i}{1 + \alpha i} \exp(\alpha k x) y_2(x,k) \right]$$

and

$$g^-(x,k) = \exp(-ikx) b_1(k) \left[1 + \lim_{R \rightarrow \infty} \int_{-R}^x \Psi^+(x,z) \exp(-ik(z+x)) dz - \sum_{n=1}^{\lambda} \frac{(m_n^+)^2}{\kappa_n + ik} \exp(-\kappa_n x) y_1(x, i\kappa_n) + \frac{1 - \alpha i}{1 + \alpha i} \exp(ikx) y_1(x,k) \right]$$

Thus, as was shown in [8] we can find

$$\lim_{|x| \rightarrow \infty} \exp(ikx) g^-(x,k) = \lim_{|x| \rightarrow \infty} \exp(-\alpha k x) g^+(x,k) = 1, \text{ i.e., } g^+(x,k) = y_1(x,k)$$

and $g^-(x,k) = y_2(x,k)$

Upon substituting into (29), (30) it leads to the equalities (26), (27). Moreover, from the formulas of $g^{\pm}(x,k)$ we have

$$g^+(x,k) = \frac{-i}{\alpha^2} b_1(i\kappa_n) (m_n^-)^2 y_2(x, i\kappa_n) \text{ and } g^-(x,k) = i b_1(i\kappa_n) (m_n^+)^2 y_1(x, i\kappa_n).$$

Therefore,

$$y_1(x, i\kappa_n) = \frac{-i}{\alpha^2} b_1(i\kappa_n) (m_n^-)^2 y_2(x, i\kappa_n)$$

and

$$y_2(x, i\kappa_n) = i b_1(i\kappa_n) (m_n^+)^2 y_1(x, i\kappa_n)$$

Thus, we conclude that

$$\int_{-\infty}^{\infty} |y_1(x, i\kappa_n)|^2 \rho(x) dx = \left[\frac{1}{\alpha} b_1(i\kappa_n) m_n^- \right]^2$$

$$\text{and } \int_{-\infty}^{\infty} |y_2(x, i\kappa_n)|^2 \rho(x) dx = \left[\frac{1}{\alpha} b_1(i\kappa_n) m_n^+ \right]^2.$$

and hence in view of (25) formula (28) holds and this complete the proof of the theorem.

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