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EXPLICIT FORMS OF CONSERVATION LAWS OF A CLASS OF DIFFERENTIAL EQUATIONS

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صيغ صريحة لقوانين الثبات لفصل من المعادلات التفاضلية

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في هذا البحث نشتق الصور الصريحة لقوانين الثبات لفصل من المعادلات التفاضلية .

أولاً : باستخدام تحويلات باكلاند .

ثانيا : بإستخدام المعادلات الأساسية للطريقة العكسية .

Key Words: Conservation laws - Bäcklund transformation.

ABSTRACT

Explicit forms of conservation laws of a class of nonlinear partial differential equations are derived; firstly from Bäcklund transformation and secondly from the fundamental equations of the corresponding inverse method.

INTRODUCTION

This paper is devoted to the study of the relationship among the conservation laws, Bäcklund transformation, and the inverse method for the class of nonlinear partial differential equations:

$$u_t + a_1 u u_x + a_2 u u_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0 \quad (1-1)$$

where $\alpha_i \varepsilon R$, i = 1,2, ..., 6. The connection between these three properties was given in 1973 by Wadati *et al.* [6] for the Kdv equation which represents an element of the class of equation (1-1). So the method given herein is a generalization for the method that used in the above mentioned reference.

The conservation law associated with an equation such as (1-1) is expressed by an equation of the form [2], [3];

$$\frac{\partial}{\partial_t} T + \frac{\partial}{\partial_x} X = O \quad (1\text{-}2)$$

where T, the conserved density and X, the flux are polynomials of x, t, u, and the derivatives of u with respect to x and t In [4] it was proved that if a_1/a_2 is multiple root of order three of the algebraic equation

$$a_3 - a_4 K + a_5 K^2 - a_6 K^3 = O \quad (1\text{-}3)$$

then the class (1-1) has an infinite number of conservation laws of the form (1-2) (the class (1-1) under (1-3) is the Kdv subclass [2]). For example the first two conserved densities and fluxes are the following

$$T_1 = u + a_2 \frac{u^2}{2} + a_4 u_{xt} + a_6 u_{tt},$$

$$X_1 = u + a_1 \frac{u^2}{2} + a_3 u_{xx} + a_5 u_{xt} \quad (1-4)$$

$$T_2 = \frac{u^2}{2} + a_2 \frac{u^3}{3} - a_4 \frac{u_t^2}{2} + a_5 u u_{xt} + a_6 (u u_{tt} - \frac{u_t^2}{2}),$$

$$T_2 = \frac{u^2}{2} + a_1 \frac{u^3}{3} + a_3 (u u_{xx} - \frac{u_x^2}{2}) + a_4 u u_{xt} - a_5 \frac{u_t^2}{2} \quad (1-5)$$

The condition (1-3) was necessarily used to derive Bäcklund transformation of the class (1-1) in the form [5]:

$$a_1 (w + w')_x + a_2 (w + w')_t = -2\eta^2 + \frac{a_1^3}{12 a_3} (w + w')^2, \quad (1-6)$$

$$(w' - w)_t + (w' - w)_x = (a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial t})$$

$$[2 \frac{a_3}{a_1^3} (a_1^2 w_{xx} + 2 a_1 a_2 w_{xt} + a_2^2 w_{tt}) - \frac{1}{3} (w - w')$$

$$(a_1 w_x + a_2 w_t) + \frac{2}{3} \eta^2 (w - w')] \quad (1-7)$$

where η is arbitrary parameter and we put $u = -(a_1w_x + a_2w_t)$ and $u' = -(a_1w'_x + a_2w'_t)$.

The condition (1-3) was similarly used in the above mentioned reference to produce the fundamental equations of the inverse scatteirng method for the class (1-1) in the form:

$$(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial t})\psi_1-(\frac{a_1^3}{6a_3})^{1/2}\eta\psi_1=(\frac{a_1^3}{6a_3})^{1/2}\mu\psi_2, \quad (1-8a)$$

$$(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial t})\psi_2+(\frac{a_1^3}{6a_3})^{1/2}\eta\psi_2=(\frac{a_1^3}{6a_3})^{1/2}r\psi_1, \quad (1-8b)$$

$$(\frac{\partial}{\partial t}+\frac{\partial}{\partial x})\psi_1=(\frac{a_1^3}{6a_3})^{1/2}$$

$$[A(x,t,\eta)\psi_1+B(x,t,\eta)\psi_2] \quad (1-9a)$$

and

$$(\frac{\partial}{\partial t}+\frac{\partial}{\partial x})\psi_2=(\frac{a_1^3}{6a_3})^{1/2}$$

$$[C(x,t,\eta)\psi_1-A(x,t,\eta)\psi_2] \quad (1-9b)$$

where

$$r=-1, A=-\frac{2}{3}\eta^3-\frac{1}{3}\eta u-\frac{1}{6}(\frac{6a_3}{a_1^3})^{1/2}(a_1u_x+a_2u_t),$$

$$B=-\frac{a_3}{a_1^3}(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial t})^2u-\frac{1}{3}(\frac{6a_3}{a_1^3})^{1/2}\eta$$

$$(a_1u_x+a_2u_x)-\frac{2}{3}\eta^2u-\frac{1}{3}u^2,$$

$$C=\frac{2}{3}\eta^2+\frac{1}{3}u \quad (1-10)$$

In this work we shall use the Bäcklund transformation (1-6) and (1-7) to derive explicit form of conservation laws and to do the same thing by using the fundamental equations of the inverse method (1-8) and (1-9) with (1-10).

2- Derivation of explicit forms of conservation laws

In the following theorem, an explicit form of conservation laws of the class (1-1) is derived from the Bäcklund transformation.

Theorem 1: The conservation laws of the class (1-1) under the condition (1-3) can be derived from the Bäcklund transformation (1-6) and (1-7).

proof: Expanding (w-w′) in the form

$$w-w'=2(\frac{6a_3}{a_1^3})^{1/2}\eta+\sum_{n=1}^{\infty}f_n\eta^{-n} \quad (2-1)$$

and inserting (2-1) into (1-6) yields

$$\eta\sum_{n=1}^{\infty}f_n\eta^{-n}=(\frac{6a_3}{a_1^3})^{1/2}(a_1w_x+a_2w_t)-\frac{1}{2}(\frac{6a_3}{a_1^3})^{1/2}\sum_{n=1}^{\infty}$$

$$(a_1\frac{\partial f_n}{\partial x}+a_2\frac{\partial f_n}{\partial t})\eta^{-n}-\frac{1}{4}(\frac{a_1^3}{6a_3})^{1/2}(\sum_{n=1}^{\infty}f_n\eta^{-n})^2 \quad (2-2)$$

Equating the coefficients for higher power $1/\eta$ and replacing $-(a_1w_x + a_2w_t)$ by u, we obtain a recurrence formula for f_n, i.e.,

$$F_{n+1}=-(\frac{6a_3}{a_1^3})^{1/2}u\delta_{n,0}-\frac{1}{2}(\frac{6a_3}{a_1^3})^{1/2}(a_1\frac{\partial f_n}{\partial x}+a_2\frac{\partial f_n}{\partial t})$$

$$-\frac{1}{4}(\frac{a_1^3}{6a_3})^{1/2}\sum_{m=1}^{n-1}f_mf_{n-m} \quad (2-3)$$

This recurrence relation can be solved and give

$$f_1=-(\frac{6a_3}{a_1^3})^{1/2}u,$$

$$f_2=\frac{1}{2}(\frac{6a_3}{a_1^3})(a_1u_x+a_2u_t),$$

$$f_3=-\frac{1}{4}(\frac{6a_3}{a_1^3})^{3/2}(a_1^2u_{xx}+2a_1a_2u_{xt}+a_2^2u_{tt})-\frac{1}{4}$$

$$(\frac{6a_3}{a_1^3})^{1/2}u^2,$$

$$f_4=(\frac{1}{2})^3(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial t})$$

$$\left[(\frac{6a_3}{a_1^3})^2(a_1^2u_{xx}+2a_1a_2u_{xt}+a_2^2u_{tt})+2(\frac{6a_3}{a_1^3})u^2\right],$$

etc

Inserting equation (2-1) into the second part of the Bäcklund transformation i.e., (1-7), we have

$$\sum_{n=1}^{\infty}\frac{\partial f_n}{\partial t}\eta^{-n}+\sum_{n=1}^{\infty}\frac{\partial f_n}{\partial x}\eta^{-n}+(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial t})$$

$$[2\frac{a_3}{a_1^3}(a_1^2w_{xx}+2a_1a_2w_{xt}+a_2^2w_{tt})-\frac{2}{3}(\frac{6a_3}{a_1^3})^{1/2}\eta$$

$$(a_1w_x+a_2w_t)-\frac{1}{3}(a_1w_x+a_2w_t)\sum_{n=1}^{\infty}f_n\eta^{-n}$$

$$+\frac{4}{3}(\frac{6a_3}{a_1^3})^{1/2}\eta^3+\frac{2}{3}\eta^2\sum_{n=1}^{\infty}f_n\eta^{-n}]=O. \quad (2-4)$$

Then, equating the coefficient of the higher power of $1/\eta$ and replacing $-(a_1 w_x + a_2 w_t)$ by u, we obtain:

$$\frac{\partial}{\partial t} f_n + \frac{\partial}{\partial x} f_n + (a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial t})\left[\frac{1}{3} u f_n + \frac{2}{3} f_{n+2}\right] = O \quad (2-5)$$

This implies that

$$\frac{\partial}{\partial t}\left[f_n + \frac{a_2}{3} u f_n + \frac{2a_2}{3} f_{n+2}\right] + \frac{\partial}{\partial x}\left[f_n + \frac{a_1}{3} u f_n + \frac{2a_1}{3} f_{n+2}\right] = O \qquad (2-6)$$

which, together with (2-3), constitutes explicit formula of conservation laws and proves the theorem.

Once the explicit formula of conservation laws has been established from the Bäcklund transformation we turn to do the same thing by the fundamental equations of the inverse method, (1-8) and (1-9), with (1-10). For this purpose, the following theorem is introduced.

Theorem 2: The conservation laws of the class (1-1) with the condition (1-3) can be derived from the fundamental equations of the inverse method.

Proof: Inserting the variable

$$\phi = \frac{\psi_2}{\psi_1}$$

into the fundamental equations, (1-8) and (1-9), with r = -1 we obtain

$$2\left(\frac{a_1^3}{6a_3}\right)^{1/2}\eta(u\phi) = -\left(\frac{a_1^3}{6a_3}\right)^{1/2} u - \left(\frac{a_1^3}{6a_3}\right)^{1/2}(u\phi)^2 - u\left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial t}\right)\left(\frac{u\phi}{u}\right) \qquad (2-7a)$$

and

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)(u\phi) = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial t}\right)(A + B\phi) \qquad (2-7b)$$

Expanding $(u\phi)$ in the power series of $1/\eta$, i.e.,

$$u\phi = \sum_{n=1}^{\infty} g_n \eta^{-n} \qquad (2-8)$$

Thus, equation (2-7a) becomes

$$\eta \sum_{n=1}^{\infty} g_n \eta^{-n} = -\frac{1}{2}u - \frac{1}{2}\left(\sum_{n=1}^{\infty} g_n \eta^{-n}\right)^2 - \frac{1}{2}\left(\frac{6a_3}{a_1^3}\right)^{1/2} u\left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial t}\right)\left(\frac{\sum_{n=1}^{\infty} g_n \eta^{-n}}{u}\right)$$

Equating the terms of the same powers of $1/\eta$, we obtain a recursion relation for g_n:

$$g_{n+1} = \frac{1}{2}\left[-u\delta_{n,o} - \sum_{k=1}^{n-1} g_k g_{n-k} - \left(\frac{6a_3}{a_1^3}\right)^{1/2} u\left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial t}\right)\left(\frac{g_n}{u}\right)\right] \qquad (2-10)$$

from which we have

$$g_1 = -\frac{1}{2}u, g_2 = 0, g_3 = -\left(\frac{1}{2}\right)^3 u^2,$$

$$g_4 = \left(\frac{1}{2}\right)^4 \left(\frac{6a_3}{a_1^3}\right)^{1/2} (a_1 u u_x + a_2 u u_t),$$

$$g_5 = -\left(\frac{1}{2}\right)^5 \left[2u^3 + \frac{6a_3}{a_1^3}(a_1^2 u u_{xx} + 2a_1 a_2 u u_{xt} + a_2^2 u u_{tt})\right], \qquad (2-11)$$

and so on. We note that equation (2-7b) is the form of conservation law,

$$\frac{\partial}{\partial t}\left[(u\phi) - a_2\left(A + \frac{Bu\phi}{u}\right)\right] = \frac{\partial}{\partial x}\left[-(u\phi) + a_1\left(A + \frac{Bu\phi}{u}\right)\right] \qquad (2-12)$$

substituting (2-8), A and B from (1-10) into (2-12) and equating the terms of the same powers of $1/\eta$, we obtain

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) g_n + \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial t}\right)\left[\left(\frac{a_3}{a_1^3 u}\left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial t}\right)^2 u + \frac{1}{3}u\right) g_n + \left(\frac{1}{3}\left(\frac{6a_3}{a_1^3}\right)^{1/2} \frac{1}{u}(a_1 u_x + a_2 u_t)\right) g_{n+1} + \frac{2}{3} g_{n+2}\right] = 0 \quad (2-13)$$

Formula (2-13) with equations (2-11) yield conservation laws and the theorem is proved.

Example: The formula (2-13) for n=1 yields

$$(\frac{\partial}{\partial t}+\frac{\partial}{\partial x})u+(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial t})$$
$$\left[\frac{a_3}{a_1^3}(a_1^2u_{xx}+2a_1a_2u_{xt}+a_2^2u_{tt})+\frac{1}{2}u^2\right]=0$$

i.e.,

$$\frac{\partial}{\partial t}(u+a_2\frac{u^2}{2}+3\frac{a_2}{a_1}a_3u_{xt}+\frac{a_2^3}{a_1^3}a_3u_{tt})+\frac{\partial}{\partial x}(u+a_1\frac{u^2}{2}$$
$$+a_3u_{xx}+3\frac{a_2^2}{a_1^2}a_3u_{xt})=0 \qquad (2-14)$$

But since $\frac{a_1}{a_2}$ is multiple root of order three of (1-3 then (2-14) reduces to

$$\frac{\partial}{\partial t}(u+a_2\frac{u^2}{2}+a_5u_{xt}+a_6u_{tt})$$
$$+\frac{\partial}{\partial x}(u+a_1\frac{u^2}{2}+a_3u_{xx}+a_4u_{xt})=0$$

which is the first conservation law for the class (1-1) with (1-3) satisfied. Similarly when n = 2 we obtain a trivial one while for n = 3 one can establish the second conservation law, and so on.

CONCLUSION

The result of this work has many advantages. Firstly, it provides a procedure for establishing recurrence formulae for higher conservation laws. Secondly, it produces a relationship among conservation laws, Bäcklund transformation and the inverse method. Finally, it confirms that if the condition (1-3) on the coefficients of the class satisfied, then the class has infinite number of conservation laws since n = 1,2... and this observation agrees with the result in [4].

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