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ON FINITE GROUP ACTIONS ON THE SOLID KLEIN BOTTLE

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ABSTRACT

In this paper we classify all G-actions on the solid Klein bottle when $G = Z_n$ and when $G = Z_2 \oplus Z_2$.

Let G be a group and M a topological space. An action of G on M is a map Θ: GxM --> M such that (i) Θ (g, Θ ((h,x)) = Θ ((gh,x)) for all g,h $\in$ G and x $\in$ M, and (ii)Θ((e,x)) = x for all x $\in$ M, where e is the identity of G. Θ (g,x)) is denoted by g(x). The action Θ is called effective if it is injective. Two G-actions Θ on M and Φ on N are weakly conjugate if there exists a group automorphism A:G-->G and a homeomorphism t:M-->N (called the connected homeomorphism) such that tΘ((g,x))=Φ(Axt)((g,x)), i.e. tg(x)=A(g) (t(x)). If A(g)= g, then Θ and Φ are conjugate.

In this paper we consider the classification of the G-actions on the solid Klein bottle SK. We give complete classifications when $G = Z_n$, the finite cyclic group, and when $G = Z_2 \oplus Z_2$. We extend the results of Natsheh (4).

Throughout the paper we work in the PL category (our results are valid for Diff-category without any changes). We divide the paper into three sections. In section 1 we prove theorem 1, the product theorem and state theorem 2, the involutions on SK. In section 2, we classify all Z_n-actions on SK, up to weak conjugation. In section 3, we classify the $Z_2 \oplus Z_2$-actions on SK.

Let G be an Abelian group acting effectively on a connected space M. Let g,h $\in$ G and q:M --> M/g be the orbit map induced by g. Then there exists a homeomorphism $\bar{h}$ on M/g uniquely determined by h such that $\bar{h}$=qq. $\bar{h}$ is called the action on M/g induced by h.

Throughout the paper S^n, D^n, and P^n denote the n-sphere, the n-cell and the n-dimensional projective plane, respectively. Mb denotes a Mobius band. C(X) denotes the cone over the space X. S^1 is viewed as the set of complex numbers x with norm 1. The closed unit interval is denoted by I. T= S^1xS^1.

$D^2 = \{rx: 0 \leq r \leq 1, x \in S^1\}$
$SK = RxD^2/\sim$, $(s,rx) \sim (s+1, C(rx))$, where $C(rx) = r\bar{x}$.

Section 1.

In this section we make use of recent results of Dunwoody (1) and Meeks and Scott (3); Moreover we write down theorem 2 which was proved in (4).

Theorem 1. Let G be a finite group acting effectively on the solid Klein bottle SK. Then the action is conjugate to an action which preserves the product structure, i.e. for every $g \in G$ $g([s,rx]) = [\alpha(s), \beta(rx)]$, up to conjugation.

Proof. Let $g \in G$, M= SK and M′ be a disjoint copy of M with a corresponding g′ action, g′: M′, g′(x′) = (g(x))′. Consider the double of M, 2M= $S^1\bar{x}S^2$, the non-orientable two-sphere bundle over S^1 obtained from M and M′ by identifying them along their boundary by the identity map. Then g and g′ define an action $\bar{g}$ on 2M and hence G-acts on 2M. By Dunwoody (1), there exists a two sphere S properly embedded in 2M which does not bound a 3-cell such that for every $g \in G$ $g(S)=S$ or $g(S) \cap S = \phi$. Now since each of M and M′ are invariant under the G-action and $S \cap M = D^2$ it follows that for every $g \in G$, $g(D^2) = D^2$ or $g(D^2) \cap D^2 = \phi$. Now by Meeks and Scott (3) the result follows.

The following theorem may be found in (4). It is an easy consequence of theorem 1 and Kims result (2).

Theorem 2. Let h be an involution on SK, then h is conjugate to exactly one of the following involutions with fixed point sets $Fix(h_i)$ and orbit spaces M^*_i

1. $h_1([s,rx]) = [s,rx]$
   $Fix(h_1) = s^1xI$
   $M^*_1 = S^1xD^2$
2. $h_2([s,rx]) = [s,-rx]$
   $Fix(h_2) =$ Mb
   $M^*_2 =$ SK
3. $h_3([s,rx]) = [s,-rx]$
   $Fix(h_3) = S^1$
   $M^*_3 =$ SK
4. $h_4([s,rx]) = [1-s,r\bar{x}]$
   $Fix(h_4) = D^2 U I$
   $M^*_4 = D^3$
   $h'_4([s,rx]) = [1-s,r\bar{x}]$
5. $h_5([s,rx]) = [1-s,-rx]$
   $Fix(h_5) = I\ U\ \{*\}$
   $M^*_5 = C(P^2)$
   $h'([s,rx]) = [1-s,-r\bar{x}]$

Remark. It is easy to see that h_4, h'_5, are conjugate to h'_4, h'_5, respectively by taking the connecting homeomorphism t: SK-->SK t([s,rx])= [s+½,rx] .

Section 2.

In this section we classify all Z_n-actions on SK.

Theorem 3. Let h be a generator of a Z_n-action on SK. Then h is weakly conjugate to one of the following maps, with quotient spaces M*.

1. $h_1([s,rx]) = [s+\frac{1}{n}, rx]$, n is odd
   $Fix(h^i_1) = \Phi, 0 < i < n$
   M*= SK

2. $h_2([s,rx]) = [s+\frac{1}{k}, rx]$, n= 2k,
   $h^k_2([s,rx]) = [s+1, rx] = [s, rx]$
   $Fix(h^k_2) = S^1 x\ I$
   $M^* = S^1\ x\ D^2$

3. $h_3([s,rx]) = [s+\frac{1}{k}, -rx]$, n= 2k, k is even,
   $h^k_3([s,rx]) = [s+1, rx] = [s, rx]$
   $Fix(h^k_3) = S^1 x\ I$
   M*= SK

4. $h_4([s,rx]) = [s+\frac{1}{k}, -rx]$, n= 2k, k is odd
   $h^k_4([s,rx]) = [s+1, -rx] = [s, -rx]$
   $Fix(h^k_4) = Mb$
   M*= SK

5. $h_5([s,rx]) = [s+\frac{1}{k}, -rx]$, n= 2k, k is odd
   $h^k_5([s,rx]) = [s+1, -rx] = [s, -rx]$
   $Fix(h^k_5) = S^1$
   M*= SK

6. $h_6([s,rx]) = [1-s, rx]$, n= 2
   $Fix(h_6) = D^2 U\ I$
   $M^* = D^3$

7. $h_7([s,rx]) = [1-s, -rx]$, n= 2
   $Fix(h_7) = I\ U\ \{*\}$
   $M^* = C(P^2)$

Proof. Let h be a generator of a Z_n-action on SK. It follows from theorem 1 that, up to conjugation h is given by either

$$h([s,rx]) = [s+\frac{1}{m}, g(rx)]$$

where m divides n, g is a homeomorpism on D^2 such that Cg= gC and $g^n = C^{n/m}$, or

$$h([s,rx]) = [1-s, g(rx)]$$

where n is even $n=2k$, g is a periodic map on D^2 with period n or k, and $CgC = g$.

First let n be odd, then m is also odd and $g^n(r\bar{x}) = r\bar{x}$ or rx, hence $g(rx) = r\bar{x}$, from which we have $h([s,rx]) = [s+\frac{1}{m}, r\bar{x}]$ and $h^m([s,rx]) = [s+1, r\bar{x}] = [s,rx]$, and hence $n=m$. Therefore $\bar{h}$ is given by h_1, up to weak conjugation.

Second let n be even, $n=2k$ and h is given by
$$h([s,rx]) = [s+\tfrac{1}{m}, g(rx)]$$

we have the following cases:

Case 1. $h^k([s,rx]) = [s,r\bar{x}]$, up to conjugation. Then $SK/h^k = S^1xD^2$ and $Fix(h^k) = S^1x\ I \subset S^1\ x\ \partial D^2$. h induces a periodic map $h^\cdot:(S^1xD^2, S^1\ xI) \text{-->} (S^1xD^2, S^1xI)$ which preserves the product structure. Hence up to weak conjugation $\bar{h}^\cdot((s,rx)) = (s+\frac{1}{k}, \bar{g}^\cdot(rx))$ where $g(rx) = rx, -rx, r\bar{x}^\cdot$ or $-r\bar{x}^\cdot$. Therefore, up to weak conjugation, h is given by $h([s,rx]) = [s+\frac{1}{k}, g(rx)]$, where $g(rx) = rx, -rx, r\bar{x}^\cdot$ or $-r\bar{x}^\cdot$.

If $g(rx) = rx$, then up to weak conjugation h is given by h_2. If $g(rx) = rx^\cdot$, then k is even and $h = h_2^{k-1}$; therefore h is weakly conjugate to h_2. If $g(rx) = -rx$, then k is even and $\bar{h} = h_3$, up to weak conjugation. Finally if $g(rx) = -rx^\cdot$, then k is even and $\bar{h} = h_3^{k+1}$, hence h is weakly conjugate to h_3.

Case 2. $h^k([s,rx]) = [s,-rx]$, up to conjugation. $SK/h^k = SK$ and $Fix(h^k) = Mb$. $h(MB) = Mb$ and Mb is two-sided in SK, hence h interchanges the two sides of Mb and k is odd. We finish this case as we did in Case 1 to conclude that h is weakly conjugate to h_4.

Case 3. $h^k([s,rx]) = [s,-rx]$, up to conjugation. $SK/h^k = SK$ and $Fix(h^k) = S^1$ is a fiber contained in (SK/h^k). h induces $h^\cdot:(SK/h^k, S^1) \text{-->} (SK/h^k, S^1)$, where h has period k. We finish this case as in Case 1 to conclude that h is weakly conjugate to h_5.

Third let n be even, $n=2k$ and h is given by
$$h([s,rx]) = [1-s, g(rx)].$$

If $g(rx) = rx\,\omega$, where ω is a primitive root of unity, then $\overline{g(rx)} = g(r\bar{x}^\cdot)$, hence $rx\bar{\omega} = rx\,\omega$ and $\bar{\omega} = \omega$ from which we have $\omega = 1$ or -1. Therefore $g(rx) = rx, -rx, r\bar{x}^\cdot$ or $-r\bar{x}^\cdot$ and $n=2$. If $g(rx) = rx$, h is given by h_6, up to conjugation. If $g(rx) = r\bar{x}^\cdot$, then it is easy to check that h is conjugate to h_6. Similarly if $g(rx) = -rx$ or $g(rx) = -rx^\cdot$, then h is conjugate to h_7.

Section 3.

In this section we classify the $Z_2 \oplus Z_2$-actions on SK.

Theorem 4. Let $Z_2 \oplus Z_2$ -act effectively on SK, then the action is weakly conjugate to one of the following actions.

1) $G_1 = \{e,h_1,h_2,h_3\}$, 2) $G_2 = \{e,h_1,h_4,h'_4\}$, 3) $G_3 = \{e,h_1,h_5,h'_5\}$, 4) $G_4 = \{e,h_2,h_4,h_5\}$ or 5) $G_5 = \{e,h_3,h_4,h_5\}$. Where the h_i,s are the involutions on SK given in theorem 2.

Proof. Let h be a generator of a $Z_2 \oplus Z_2$-action on SK, then h is an involution. First let h be given by h_1, up to conjugation. Let g be the second generator, then g is also an involution. If $g=h_2$ (or h_3) then $Z_2 \oplus Z_2 = G_1$ up to weak conjugation. If $g=h'_4$, then $Z_2 \oplus Z_2 = G_2$ up to weak conjugation. If $g=h_5$, then $Z_2 \oplus Z_2 = G_3$ up to weak conjugation. Second if $h=h_2$, up to conjugation, then if $g=h_1$ or h_2 we get G_1. If $g=h_4$ then $Z_2 \oplus Z_2 = G_4$, up to weak conjugation. If $g=h_5$, then $Z_2 \oplus Z_2 = G_4$, up to weak conjugation where the connected homeomorphism t: SK --> SK, $t([s,rx]) = [s+½,rx]$ makes this action and the preceeding one weakly conjugate. Third if $h=h_3$, then for $g=h_4$ we have $Z_2 \oplus Z_2 = G_5$, up to weak conjugation.

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عن المجموعة المحدودة للأفعـــال في قنّينة كلـيْن

محمـــد عرفـــات النتشـــة

في هذا البحـــث تصنيف لجميع actions — G على قنينـة كلـــين Klein عندمـــا يكـــون : $G = Z_n$ وعندما يكون $G = Z_2 \oplus Z_2$.