Qatar Univ. Sci. J. (1996), 16(2): 193-199

IMPROVEMENT OF THE PIECEWISE APPROXIMATION METHODS

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ملخص

نقدم في هذا البحث طريقة تقريب جزئي للمنحنيات في المستوي، والتي تحسن رتبة الطرق التقليدية المستعملة . هذه الطريقة تستغل حرية اختيار التابع وترفع رتبة التقريب إلى $\frac{4m}{3}$ حيث $m \geqslant 6$ هي درجة كثيرة الحدود المستعمله في التقريب .

عندما $m = 5$ نجد كثيرة الحدود من الدرجة الخامسة والتي تقرب المنحني حتى الرتبة الثامنة ، بالإضافة إلى ذلك نبرهن بشكل عام إمكانية إيجاد كثيرة حدود تقرب من الرتبة 2 m لمجموعة غير خالية من المنحنيات .

Keywords: Piecewise approximation, planar curves, splines, geometric smoothness, computer aided geometric design.

ABSTRACT

A piecewise approximation method is described for planar curves. The order of classical piecewise approximations is improved. The method exploits the freedom in the choice of the parametrization and rises the approximation order up to $\frac{4m}{3}$, where $m \geq 6$ is the degree of the approximating polynomial parametrization. For the case of $m = 5$, a quintic polynomial curve is constructed which approximates with order 8. Moreover, we show for a class of curves of non-zero measure that the optimal rate 2m is achieved.

INTRODUCTION

Parametric polynomials are used in CAGD for approximation and interpolation purposes or, more generally, for geometric modelling applications.

The question to approximate curves and surfaces within a certain tolerance by polynomials and splines arises often in working with CAGD, various error estimations have been obtained, see [1, 6, 11]. In this paper we describe a piecewise approximation procedure for planar curves which significantly rises the standard piecewise approximation rate up to order 2m, where m is the degree of the approximating polynomial. A quintic piecewise approximant is described in section 4. The quintic approximant has approximation order of 8 and a third order contact at each knot of the segment. The improvement of the Hermite approximation to order of $\frac{4m}{3}$, where m ≥ 6 is the degree of the approximating segment, is given in section 2 with proof. In section 3, the highest possible approximation order of 2m is achieved for a class of curves of non-zero measure.

Let

$$C : t \to \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

be a regular smooth planar curve. We want to approximate C in the interval $[t_0, t_1]; t_0, t_1 \in R$; $h = t_1 - t_0$, using information at the end points t_0 and t_1 by a polynomial curve

$$p : t \to \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix},$$

where X(t) and Y(t) are polynomials of degree m.

If we choose for X(t) and Y(t) piecewise approximation polynomials of degree m, then, as known r approximates C with order m + 1, i.e.

$$\{f(t) - X(t)\} = O(h^{m+1}), \{g(t) - Y(t)\} = O(h^{m+1}).$$

The improvement over the standard order m + 1 is possible because the parametrization of a curve is not unique. The first improvement was obtained in [3] by interpolating the position, the tangent, and the curvature at each knot. This construction yields a 6th order of accuracy. In [14] and [15] approximation methods are described for planar curves which improve the standard rate obtained by local classical approximation and achieve the order $\frac{4m}{3}$. The methods give not only quantitative improvements but also qualitative improvements over the Taylor approximations.

Without loss of generality we may assume that

$$t_0 = 0, t_1 = 1, (f(0), g(0) = (0,0), \quad (f^1(0), g^1(0) = (1, 0),$$

so that for small t we can parametrize C in the form

$$C : t \to X(t) \to \begin{pmatrix} X(t) \\ \phi(X(t)) \end{pmatrix}, t \in R.$$

p approximates C with order $\alpha_1 + \alpha_2$; $\alpha_1, \alpha_2 \in N$, iff

$$\phi(X(t)) - Y(t) = O(t^{\alpha_1+\alpha_2}),$$

i.e. iff

$$\left(\frac{d}{dt}\right)^j \{\phi(X(t)) - Y(t)\}|_{t=0} = 0, j = 0,1,\dots,\alpha_1 - 1,$$

$$\left(\frac{d}{dt}\right)^j \{\phi(X(t)) - Y(t)\}|_{t=0} = 0, j = 0,1,\dots,\alpha_2 - 1, \quad (1)$$

and

$$X(1) = 1, X(0) = 0.$$

If one normalizes the polynomial approximation by choosing $X'(0) = 1$, then the polynomial approximation is determined by 2m - 1 free parameters.

Conjecture: A smooth regular planar curve can at two points, in general, be approximated by a polynomial parametrization of degree ≤ m with order 2m.

This conjecture is illustrated in [15] by constructing a cubic polynomial curve r, which approximates the curve C with order 6. They have a second order of contact.

Improvement of the order of Hermite approximation

Theorem 1 improves the standard order of Hermite approximation by $\frac{m}{3}$. By an approximate choice of the parameters X, $\frac{4m}{3}$ equations of (1) are solved.

Theorem 1: For m > 5, define

$$n_1 = \left[\frac{m+2}{3}\right] \text{and } n_2 = m + n_1 - \left[\frac{m+n_1}{2}\right]$$

Then, for almost all $(\phi_1, \phi_2, \dots, \phi_{m+n_1}) \in R^{m+n_1}$ there is a solution for the first n_2 equations at t = 0 and the first $m + n_1$ equations at t = 1 in (1).

Proof of Theorem 1

For the proof, the following formula for the derivatives of φ(X(t)), will be needed:

$$\left(\frac{d}{dt}\right)^l \{\phi(X(t))\}|_{t=0,1} = \sum_{\substack{i=1 \\ p_1+\dots+p_i=l}}^{l} c(l, |p_1,\dots,p_l)\phi_i, X_{p_1} \dots X_{p_i} \quad (2)$$

For $l = 1, 2, \dots$, m, where ϕ_i and X_i are the i^{th} derivatives of φ and X at t = 0 or t = 1 as indicated and

$c(l, i \mid p_1, ..., p_i) = c(l, i)$ are strictly positive constants which depend on i, $p_1, ..., p_i$ and l.

To simplify the proof, we take $m = 6k + 1$; it is easy to prove the other cases, only with a change of the indices. To prove Theorem 1, (1) must be solved for j = 1, ..., 4k at the left side of interpolation; i.e. t = 0 and for j = 0, 1, ..., 4k at the right side of interpolation; i.e t = 1. We proceed as follows: (1) is solved at the left side for $Y_i(0)$ for i = 1, ..., 4k and substituted into (1) at the right side for j = 0, 1, ..., 4k and then solve it for $Y_j(0)$ for j = 4k +1, ..., 6k + 1 and for $X_j(0)$ with j = 4k + 1, ..., 6k + 1.

To simplify the arguments of the proof, the following notations are used

$$X_L^0 = X_1(0), ..., X_{2k}(0),$$
$$X_L^1 = X_{2k+1}(0), ..., X_{4k}(0),$$
$$Y_L^0 = Y_1(0), ..., Y_{2k}(0),$$
$$Y_L^1 = Y_{2k+1}(0), ..., Y_{4k}(0),$$
$$X_R^0 = X_1(1), ..., X_{2k}(1),$$
$$X_R^1 = X_{2k+1}(1), ..., X_{4k}(1),$$
$$Y_R^0 = Y_1(1), ..., Y_{2k}(1),$$
$$Y_R^1 = Y_{2k+1}(1), ..., Y_{4k}(1).$$

Choosing for the left side of interpolation

$$X_1(0) = 1, X_2(0) = ... = X_{2k}(0) = 0 \qquad (3)$$

and assuming that

$$\phi_1(0), ..., \phi_{4k}(0) \neq 0.$$

Substituting (3) into (1) one gets

$$Y_L^0 = C_0(\phi_L)$$

and

$$Y_L^1 = C_2(\phi_L)\ X_L^1 + C_1(\phi_L),$$

where $C_0(\phi_L)$ and $C_1(\phi_L)$ are vectors, and $C_2(\phi_L)$ is a matrix which contains only derivatives of ϕ at t = 0. There are only terms of this form. Suppose there exists a term of the other form

$$cX_{r_1}(0)X_{r_2}(0);$$

$r_1 + r_2 \leq 4k$ which contradicts $r_1, r_2 > 2k$.

Choosing for the right side

$$X_1(1) = 1, X_2(1) = ... = X_{2k}(1) = 0 \qquad (4)$$

and assuming that

$$\phi_1(1), ..., \phi_{4k}(1) \neq 0.$$

Substituting (4) into (1) yields

$$Y_R^0 = B_0(\phi_R)$$

and

$$B_2(\phi_R)\ X_R^1 + B_1(\phi_R) - Y_R^1 = 0, \qquad (5)$$

where $B_0(\phi_R)$ and $B_1(\phi_R)$ are vectors, and $B_2(\phi_R)$ is a matrix which contains only derivatives of ϕ at t = 1.

The interpolation problem is uniquely determined by 4k + 1 Hermite points at the left side and 2k + 1 points at the right side of interpolation, i.e.

$$\begin{aligned} Y_R^1 &= A_1Y_L^0 + A_2Y_L^1 + A_3Y_R^0 \\ &= A_1Y_L^0 + A_2\{C_2(\phi_L)\ X_L^1 + C_1(\phi_L)\} + A_3Y_R^0 \\ &= C(\phi) + A_2C_2(\phi_L)X_L^1 \end{aligned}$$

Substituting this into (5) yields

$$B_2(\phi_R)X_R^1 + B_1(\phi_R) - C(\phi) - A_2C_2(\phi_L)X_L^1 = 0.$$

Because

$$X_L^1 = D_0X_L^0 + D_1X_R^0 + D_2X_R^1,$$

gives

$$B_2(\phi_R)X_R^1 + B_1(\phi_R) - C(\phi) - A_2C_2(\phi_L)\{D_0X_L^0 + D_1X_R^0 + D_2X_R^1\} = 0$$

i.e.

$$B(\phi) + \{B_2(\phi_R) - A_2C_2(\phi_L)D_2\}X_R^1 = 0.$$

The matrix of coefficients of X_R^1 has only elements of the form

$$M_{i,j} = \left\{\begin{matrix} e_{i,j}; & j>i \\ e_{i,j} + d(i,j)\phi_{i-j+1}; & j \leq i \end{matrix}\right\}, \quad i = 1,...,2k,\ j = 1,...,2k,$$

where $e_{i,j}$ are independent from the derivatives of ϕ at t = 1 and d (i,j) are positive constants.

The determinant of this matrix has the form

$$\prod_{i=1}^{2k} d(i,i)\}\phi_1^{2k}(1) + c\phi_1^{2k-1}(1) + \ldots$$

This polynomial of ϕ_1 is not identically zero, because the coefficient of $\phi_1^{2k}(1)$ is not equal to zero, and the other terms contain $\phi_1(1)$ only with lower degree than 2k, which completes the proof of Theorem 1.

Optimal order for a special case

We cannot prove the conjecture, in general. Here the conjecture is confirmed for a special case, by finding a set of coefficients $(\phi_1, \ldots, \phi_{2m-1})$ of non-zero measure, for which the optimal piecewise approximation of order 2m is attained. To this end, equations 1, 2, ..., m-1 in (1) at t = 1 are viewed as a non-linear system

$$F(\phi,V) = \left(\frac{d}{dt}\right)^l \{\phi(X(t) - Y(t)\}|_{t=1} = 0, l = 1,2,\ldots,m-1$$

with

$$V = (X_{m-1}(1), \ldots, X_1(1))$$

and

$$\Phi = (\phi_1(0), \ldots, \phi_{m-1}(0), \phi(1), \phi_1(1), \ldots, \phi_{m-1}(1)).$$

We show that this system is solvable in a neighbourhood of a particular solution (ϕ^*, X^*).

Theorem 2 If

$$X_1(0) = 1, X_j^*(1) = 0, \phi_j^*(0) = 0, \text{ for } j = 1, \ldots, m-1$$

$$\phi^*(1) = 0 \text{ and } \phi_j^*(1) = \leq \begin{cases} 1 & \text{for } j=1 \\ 0 & \text{otherwise} \end{cases}$$

then (ϕ^*, X^*) is a solution of $F(\Phi, V) = 0$, where

$$X^* = (X_{m-1}^*(1), \ldots, X_1^*(1)$$

and

$$\Phi^* = (\phi_1^*(0), \ldots, \phi_{m-1}(0), \phi^*(1), \phi_1^*(1), \ldots, \phi_m^*(1)).$$

Moreover, there exists a neighbourhood of Φ^* such that the non-linear system (1) is uniquely solvable.

Proof of Theorem 2

The implicit function theorem will be applied for the proof of Theorem 2. First, it is verified that (ϕ^*, X^*) is a particular solution of $F(\phi, V) = 0$ and show that the Jacobi matrix $F_V(\phi^*, X^*)$ is invertible. Hence, X could be written as a function of Φ in the neighbourhood of $\Phi = \Phi^*$.

Recall that

$$\phi_j^*(0) = 0, j = 1, \ldots, m \text{ and } \phi_j^*(1) = \begin{cases} 1 & \text{for } j=1 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

Substituting (6) into (2) for t = 0 for l = 1, 2, ..., m-1, the sum involves only zero terms. Substituting (6) into (1) for t = 0 for l = 1, 2, ..., m - 1, gives $Y_i(0) = 0$ for i = 1, ..., m - 1. Further substituting these in (1) for t = 1 for l = 0, gives $Y_m(0) = 0$, and substituting this and (6) into (2) for t = 1 for l = 1, 2, ..., m - 1, the l^{th} derivative involves only terms of the form

$$\phi_1(1)\, X_l(1) = 0.$$

Therefore, from (1) we get the system

$$F(\Phi^*, V) = AV = 0,$$

where

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & \phi_1(1) \\ 0 & 0 & \cdots & \phi_1(1) & 0 \\ 0 & 0 & .\cdot^{\cdot} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \phi_1(1) & \cdots & 0 & 0 \\ \phi_1(1) & 0 & \cdots & 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} X_{m-1}(1) \\ X_{m-2}(1) \\ \vdots \\ X_1(1) \end{bmatrix}$$

This system has the solution $V = X^*$ with

$$X_1^*(1) = X_2^*(1) = \ldots = X_{m-1}^*(1) = 0.$$

To verify the hypotheses of the implicit function theorem, the Jacobian is computed at the points

$$\Phi^* = (\phi_1^*(0), \ldots, \phi_{m-1}^*(0), \phi^*(1), \phi_1^*(1), \ldots, \phi_{m-1}^*(1))$$

and

$$X^* = (X_{m-1}^*(1), \ldots, X_1^*(1))$$

with

$$X_j^*(1) = 0, \text{ for } j = 1, \ldots, m-1$$

$$\phi_j^*(0) = 0, j = 1, \ldots, m \text{ and } \phi_j^*(1) = \begin{cases} 1 & \text{for } j=1 \\ 0 & \text{otherwise} \end{cases}$$

Since

$$F(\Phi^*, V) = A\Phi^*, V).\, V$$

then

$F_v(\Phi^*, V) = A_v(\Phi^*, V).V + A(\Phi^*,V)$.

So the Jacobi matrix is

$$Fv(\Phi^*, X^*) = A(\Phi^*, X^*) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \phi_1(1) \\ 0 & 0 & \cdots & \phi_1(1) & 0 \\ 0 & 0 & \cdot^{\cdot^{\cdot}} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \phi_1(1) & \cdots & 0 & 0 \\ \phi_1(1) & 0 & \cdots & 0 & 0 \end{bmatrix}$$

which is trivially invertible. Hence, the hypotheses of the implicit function theorem are satisfied at the (Φ^*, X^*), which completes the proof of Theorem 2.

Example

We shall illustrate this method by constructing a polynomial curve of degree 5. To illustrate the conjecture and achieve the high order approximation of 10 for quintic polynomial approximation, we will set $\alpha_1 = \alpha_2 = 4$ in (1) then we get

$X(0) = 0,$
$Y(0) = \phi(0) = 0,$
$\phi_1(0)\, X_1(0) - Y_1(0) = 0,$
$\phi_2(0)X\quad (0) + \phi_1(0)\, X_2(0) - Y_2(0) = 0,$
$\phi_3(0)X\quad (0) + 3\phi_2(0)\, X_1(0)X_2(0) + \phi_1(0)\, X_3(0) - Y_3(0) = 0,$
$\phi_4(0)X\quad (0) + 6\phi_3(0)X\quad X_2(0) + 3\phi_2(0)X\quad (0)$
$\qquad + 4\phi_4(0)X_1(0)X_3(0) + \phi_1(0)X_4(0) - Y_4(0) = 0,$

$X(1) = 1,$
$\phi(1) - Y(1) = 0,$
$\phi_1(1)\, X_1(1) - Y_1(1) = 0,$
$\phi_2(1)X\quad (1) + \phi_1(1)\, X_2(1) - Y_2(1) = 0,$
$\phi_3(1)X\quad (1) + 3\phi_2(1)\, X_1(1)X_2(1) + \phi_1(1)\, X_3(1) - Y_3(1) = 0,$
$\phi_4(1)X\quad (1) + 6\phi_3(1)X\quad X_2(1) + 3\phi_2(1)X\quad (1)$
$\qquad + 4\phi_4(1)X_1(1)X_3(1) + \phi_1(1)X_4(1) - Y_4(1) = 0.$

We write X_i, Y_i and ϕ_i for the i^{th} derivatives of $X(t)$, $Y(t)$ and $\phi(t)$ at the associated point 0 or 1. The first two conditions determine the constants a_0 and b_0 in the quintic polynomials

$$X(t) = \sum_{i=0}^{5} a_i t^i \text{ and } Y(t) = \sum_{i=0}^{5} b_i t^i,\ a_i,\ b_i \in R$$

Because the 10^{th} equation is quadratic in $X_1(1)$, it is not possible to solve it explicitly in one of the derivaties of X. To solve this system, an offering of some information must be made. Setting $X_1(0) = X_1(1) = 1$ will make this system easier to be solved. Which means that the information from the 6^{th} and the 12^{th} equations are lost and there is a contact of order 3 at both end points. Setting $X_1(0) = X_1(1) = 1$, the equations of the above system are reduced to

$\phi_1(0) - Y_1(0) = 0,$
$\phi_2(0) + \phi_1(0)X_2(0) - Y_2(0) = 0,$
$\phi_3(0) + 3\phi_2(0)\, X_2(0) + \phi_1(0)X_3(0) - Y_3(0) = 0,$

$X(1) = 1,$
$\phi(1) - Y(1) = 0,$
$\phi_1(1) - Y_1(1) = 0,$
$\phi_2(1) + \phi_1(1)X_2(1) - Y_2(1) = 0,$
$\phi_3(1) + 3\phi_2(1)\, X_2(1) + \phi_1(1)X_3(1) - Y_3(1) = 0.$

Now, there is only a linear system of equations, which can be solved for the remaining derivatives of $X(t)$ and $Y(t)$; i.e. $X_3(t)$, $X_4(t)$, $X_5(t)$, and $Y_1(t)$, $Y_2(t)$, $Y_3(t)$, $Y_4(t)$, $Y_5(t)$. Numerical examples are given. The approximations of parts of the circle are done using this method.

(a) Setting $X_1(0) = X_1(1) = .66195$ and approximating at the points (- 0.3826834325, 0.9238795325) and (.9238795325, .3826834325) one gets

$X(t) = -\,0.0555653928929941310\,94$
$\quad + 1.0237201933943977493\, t$
$\quad + 0.3672432488642681268\, t^2$
$\quad - 0.229105291052060038889\, t^3$
$\quad - 0.105089335530651\ 73344\, t^4$
$\quad - 0.0335461579293391753\,31\, t^5.$

$Y(t) = 0.9981865622770127167$
$\quad + 0.055472995706540722\,3534\, t$
$\quad - 0.5063194815751254484\,2\, t^2$
$\quad - 0.3517264556290138445\,6\, t^3$
$\quad + 0.0619938471052715804\,1\, t^4$
$\quad - 0.0035417702429852274\,387\, t^5.$

(b) Setting $X_1(0) = X_1(1) = .965$ and approximating at the points (-0.3826834325, 0.9238795325) and (.270598050, 0.962692420) one gets

$X(t) = 0.0010527489711903119357$
$\quad + 1.0157489230526967376\, t$
$\quad - 0.0294340326873184598\,29\, t^2$
$\quad - 0.1621426990939340498\,5\, t^3$
$\quad + 0.0096732854611436198\,324\, t^4$
$\quad + 0.0011885985329860970\,733\, t^5.$

$Y(t) = 1.0000004594258282467$
$\quad - 0.0010728371329699749\,934\, t$
$\quad - 0.5158740933053666586\,7\, t^2$
$\quad + 0.0295848058718263878\,28\, t^3$
$\quad + 0.0316097336029446706\,612\, t^4$
$\quad + 0.0004120243162521841\,6644\, t^5.$

(c) Setting $X_1(0) = X^1(1) = 0.9665$ and approximating at the points (-.3826834325, .9238795325) and (.270598050, .962692420) one gets

$$X(t) = 0.0010202622176091443005 + 1.0150337285916156277\,t - 0.0283738091064832800041\,t^2 - 0.15469569302896395223\,t^3 + 0.010167009562341893909\,t^4 - 0.00016835774738197016347\,t^5.$$

(d) Setting $X_1(0) = X_1(1) = .9895041$ and approximating at the points (-.3826834325, .9238795325) and (-.0560426913, .9984283734) one gets

$$X(t) = -\,0.00090892306252014866828 + 0.97752823829615332313\,t - 0.12094111658644028232\,t^2 - 0.16498095444518701443\,t^3 + 0.035997331744010975799\,t^4 - 0.00033765637776034683642\,t^5.$$

$$Y(t) = 0.99999958632476606015 + 0.00088844204299832907671\,t - 0.47789331926964423838\,t^2 + 0.11846026754612611498\,t^3 + 0.039347337459467338029\,t^4 - 0.000092565709995766803576\,t^5.$$

Fig. 1 shows the errors of approximating the circle given in (a) multiplied with 10^{-3}, in (b) mult. with 10^{-5}, in (c) mult. with 10^{-7}, and in (d) mult. with 10^{-7}.

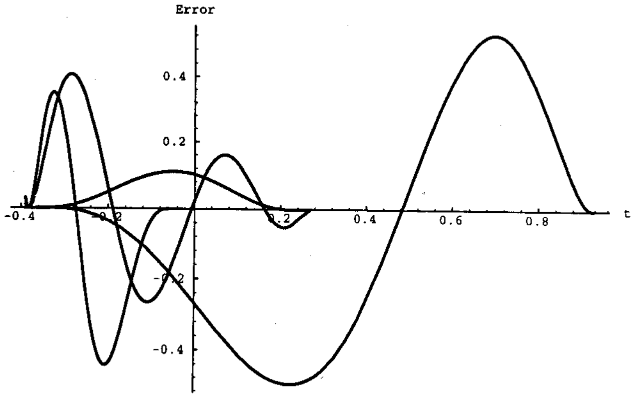

Fig. 1: Errors of app. circle at 4, 8, 8, 16 pts x 10^{-3}, 10^{-5}, 10^{-7}, 10^{-7} resp.

To calculate the order of approximation, we set the error of approximating the circle at n points $e_n = \beta n^{\alpha}$ where α can be estimated from the consecutive errors

$$\alpha = \frac{\ln(e_n) - \ln(e_m)}{\ln(n) - \ln(m)}$$

We use this formula and information from Fig. 1 to construct Table 1.

Table 1

Rate of quintic approximation

No. of points	Error	Rate
4	.52E-3	
8	.11E-5	-8.9
16	.42E-8	-8.1

Fig. 2 shows the error of approximating the circle given in (c) for all $t \in (-2, 2.9)$.

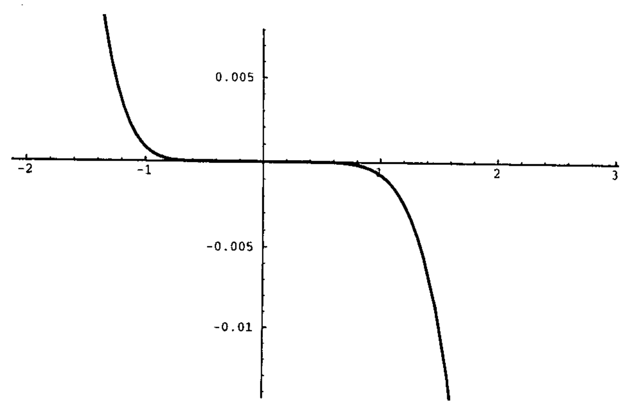

Fig. 2: Error of app. circle in (c) $\forall\, t \in (-2, 2.9)$

Fig. 3 shows the curvature of approximating the circle given in (c).

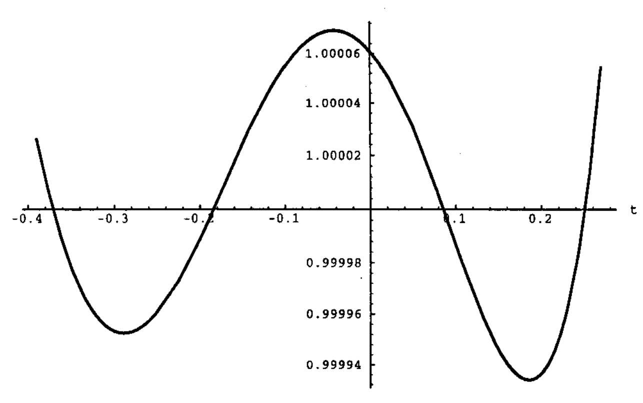

Fig. 3: The curvature of app. circle in (c)

Remark: The error of approximation depends on the choice of $X_1(0)$ and $X_1(1)$. This fact can be used to improve the order of approximation. It is possible to get better approximation order by choosing suitable values for $X_1(0)$, $X_1(1)$, for example by choosing $X_1(0) = X_1(1) = .9665$ in the case (c) one gets

$$e_8 = .4 \times 10^{-7}$$

which gives an approximation order

$$\alpha = -13.7$$

which means that the error is multiplied with

$$\left(\frac{1}{2}\right)^{13} = \frac{1}{8192}$$

ACKNOWLEDGEMENT

I gratefully acknowledge the support of Professor Klaus Höllig of the Department of Mathematics at University of Stuttgart who brought the conjecture discussion in this paper to my attention.

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