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A SPECIAL SYSTEM OF BOUNDARY VALUE PROBLEMS

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ABSTRACT

In this paper we introduce a special system of boundary value problems and give a method for solving it. Then we give a detailed application of this method to a system of boundary value problems of Hilbert type.

Let G_1 and G_2 be simply connected bounded regions with simple closed contours c_1 and c_2 in the z − plane and let $\overline{G}_1 \cap \overline{G}_2 = \Phi$.

The aim is to find two sectionally holomorphic functions $\varnothing_1(z)$ and $\varnothing_2(z)$ whose boundary values $\varnothing_1^{\pm}(t)$ and $\varnothing_2^{\pm}(t)$ satisfy the following conditions:

On c_1

$$A_1[\varnothing_1^{+}(t), \varnothing_1^{-}(t)] = F_1[\varnothing_2(t), \overline{\varnothing_2(t)}] \cdots\cdots\cdots\cdots (1)$$

and on c_2

$$A_2[\varnothing_2^{+}(t), \varnothing_2^{-}(t)] = F_2[\varnothing_1(t), \overline{\varnothing_1(t)}] \cdots\cdots\cdots\cdots (2)$$

where A_1 and A_2 are linear with respect to their arguments.

First we suppose that both of right hand sides of (1) and (2) are given, then from (1) and (2) we obtain $\varnothing_1$ and $\varnothing_2$ respectively.

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Consider that

$$\varnothing_1(z) = \frac{1}{2\pi i} \int_{c_1} \frac{\varphi_1(\tau)}{\tau - z} dt \quad \cdots\cdots\cdots\cdots (3)$$

where the density function $\varphi_1(t) \in H$ (Hölder class).

The boundary values of the function $\varnothing_1(z)$ take the form

$$\left.\begin{aligned} \varnothing_1^{+}(t) &= \frac{\varphi_1(t)}{2} + \frac{1}{2\pi i} \int_{c_1} \frac{\varphi_1(t)}{\tau - t} d\tau \\ \varnothing_1^{-}(t) &= -\frac{\varphi_1(t)}{2} + \frac{1}{2\pi i} \int_{c1} \frac{\varphi_1(t)}{\tau - t} d\tau \end{aligned}\right\} \cdots\cdots\cdots\cdots (4)$$

Substituting from (4) into (1), we have

$$A_1 \left[\frac{\varphi_1(t)}{2} + \frac{1}{2\pi i} \int_{c_1} \frac{\varphi_1(\tau)}{2} d\tau , -\frac{\varphi_1(t)}{2} + \right.$$

$$\left. + \frac{1}{2\pi i} \int_{c_1} \frac{\varphi_1(t)}{\tau - t} d\tau \right] = F[\varnothing_2(t), \overline{\varnothing_2(t)}] \cdots\cdots\cdots (5)$$

and therefore we have the singular integral equation (5) with respect to the unknown function $\varphi_1(t)$ for which we can apply Noether's theorems. Under certain conditions we obtain $\varphi_1(t)$ and consequently $\varnothing_1(z)$.

It is known that the function $\varnothing_1(z)$ can be written in the form

$$\varnothing_1(z) = \frac{1}{2\pi i} \int_{c_1} \frac{\in_1 [\varnothing_2(t), \overline{\varnothing_2(t)}]}{t - z} dt , \cdots\cdots\cdots\cdots (6)$$

Thus, on c_2

$$\varnothing_1(t) = \frac{1}{2\pi i} \int_{c_1} \frac{\in_1[\varnothing_2(\tau), \overline{\varnothing_2(\tau)}]}{\tau - t} d\tau \quad \cdots\cdots\cdots\cdots (7)$$

Subtituting from (7) into (2), we have

$$A_2[\varnothing_2^+(t), \varnothing_2^-(t)] = F_2\Big[\frac{1}{2\pi i} \int_{c_1} \frac{\in_1[\varnothing_2(\tau), \overline{\varnothing_2(\tau)}]}{\tau - t} d\tau$$

$$- \frac{1}{2\pi i} \int_{c_1} \frac{\overline{\in_1[\varnothing_2(t), \overline{\varnothing_2(t)}]}}{\overline{\tau} - \overline{t}} \overline{d\tau}\Big] \quad \cdots\cdots\cdots\cdots (8)$$

Let A_2 have properties such that $\varnothing_2(z)$ can be obtained from (8) and assume that whenever $z = t$, on c_1, we define $\varnothing_2(t)$.

Thereby from (8), we immediately obtain an integral equation with respect to $\varnothing_2(t)$.

By obtaining the solution $\varnothing_2(t)$ of such integral equation the function $\varnothing_1(z)$ follows directly from (6). Similarly, we find $\varnothing_2(z)$.

The method is complete.

We now give an application of this method:

Consider the following system of boundary value problems of Hilbert type [1].

On c_1

$$\varnothing_1^+(t) - A_1(t)\varnothing_1^-(t) = f_1(t) + \alpha_1(t)\varnothing_2(t) + \alpha_2(t)\overline{\varnothing_2(t)} \quad \cdots\cdots\cdots (9),$$

and on c_2

$$\varnothing_2^+(t) - A_2(t)\varnothing_2^-(t) = f_2(t) + \beta_1(t)\varnothing_1(t) + \beta_2(t)\overline{\varnothing_1(t)} \quad \cdots\cdots\cdots (10)$$

and let the index ae_1 of $A_1(t)$ be not negative.

From (9), we have

$$\varnothing_1(z) = \frac{X(z)}{2\pi i} \int_{c_1} \frac{f_1(t) + \propto_1(t)\,\varnothing_2(t) + \propto_2(t)\,\overline{\varnothing_2(t)}}{X^+(t)} \frac{dt}{t-z} +$$

$$+ p_{ae_1}(z)\, X(z) \qquad (11)$$

where $X(z)$ is the canonical function of the associated homogeneous equation with respect to (9) and $p_{ae_1}(z)$ is a polynomial of degree ae_1 with arbitrary coefficients.

Whenever $z = t$, on c_2, then

$$\varnothing_1(t) = \frac{X(t)}{2\pi i} \int_{c_1} \frac{f_1(\tau) + \propto_1(\tau)\,\varnothing_2(\tau) + \propto_2(\tau)\,\overline{\varnothing_2(\tau)}}{X^+(\tau)} \frac{d\tau}{\tau-t}$$

$$+ p_{ae_1}(t)\, X(t) \qquad (12)$$

Thus, the right hand side of (10) can be written in the form

$$F(t),\ \frac{1}{2\pi i} \int_{c_1} \frac{\propto_1(\tau)\,\varnothing_2(\tau) + \propto_2(\tau)\,\overline{\varnothing_2(\tau)}}{X^+(\tau)} \frac{d\tau}{\tau - t,}$$

$$\frac{1}{2\pi i} \int_{c_1} \frac{\overline{\propto_1(\tau)}\,\overline{\varnothing_2(\tau)} + \overline{\propto_2(\tau)}\,\varnothing_2(\tau)}{X^+(\tau)} \frac{d\bar{\tau}}{\tau - \bar{t},}$$

$$= F\,\varnothing_2 \qquad (13)$$

From (13) the boundary condition (10)

$$\varnothing_2^+(t) - A_2(t)\,\varnothing_2^-(t) = F\varnothing_2 \qquad (14)$$

Let the index ae_2 of $A_2(t)$ be not negative. Then we have

$$\varnothing_2(t) = \frac{y(t)}{2\pi i} \int_{c_2} \frac{F\varnothing_2}{y^+(t)} \frac{d\tau}{\tau - z} + Q_{ae_2}(z)\, y(z) \qquad (15)$$

where y (z) is the cononical function of the associated homogeneous equation with respect to (14) and Q_{ae_2} with arbitrary coefficients.

Whenever z = t, on c_1, we have

$$\varnothing_2(t) = \frac{y(t)}{2\pi i} \int_{c_2} \frac{F\varnothing_2}{y^{+}(\tau)} \frac{d\tau}{\tau - t} + Q_{ae_2}(t)\, y(t) \quad \cdots\cdots (16)$$

and therefore from (13) and (16), we immediately obtain an integral equation with respect to $\varnothing_2(t)$. By applying Fredholm's Integral Equation Theory we obtain $\varnothing_2(t)$ and consequently from (11) we find $\varnothing_1(z)$. Similarly we find $\varnothing_2(z)$.

REFERENCES

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نظام خاص من مسائل القيم الحدية

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في هذا البحث درس نظام خاص من مسائل القيم الحدية وأعطيت طريقة حل هذا النظام ، ثم طبقت الطريقة بالتفصيل لنظام من مسائل القيم الحدية من نوع هلبرت .