Qatar Univ. Sci. Bull. (1984) 4: 23-31

ON THE BEHAVIOUR OF THE SOLUTIONS OF

$ff'' - f'^2 - 3(1+\in e^{2\alpha})f' + 2(1+2\in e^{2\alpha})f - 2(1+\in e^{2\alpha})^2 = 0;\ \in=0,\ \pm1$

By

M.K. GABR
University of Qatar
State of Qatar

SH. SALEM
Al-Azhar University
Egypt

ABSTRACT

The above equation plays an important role in the geometric theory of functions of a complex variable.

To determine interval estimations for its solutions, an inequality is obtained using the fixed point theorem. Then, it is computed for $\in=-1$ only*. Some selected formulae, of the solutions, are depicted graphically.

INTRODUCTION

The nonlinear differential equation

$$ff'' - f^{12} - 3(1+\in e^{2\alpha})f' + 2(1+2\in e^{2\alpha})f - 2(1+\in(1+\in e^{2\alpha})^2 = 0,\ \in=0;\ \pm1 \ \ldots\ldots\ 1)$$

was known at first for $\in = 0$ as a requirement to obtain a fundamental principle in complex analysis (6), (8). On extending this principle in non-euclidean geometry, equation (1) appeared for $\in=\pm1$ (1). To show, briefly, how the considered equation is obtained and what is its role consider the following:

1. Suppose that we are concerned with the hyperbolic metric in the z-plane and the $\in$-metric in the w-plane, where $\in = 0, +1, -1$ implies, respectively, euclidean-, elliptic- and hyperbolic metric.

2. Let w(z) be a regular function in the unit disk $|z|<1$ and is non-ramified ($w' = 0$). Hence we have the following differential invariants under the motions of the considered geometries (7):

$$\alpha = \log \frac{|w'|\ (1-z\bar{z})}{1+\in w\bar{w}} \quad \text{and} \quad \delta 1^{a})^*$$

*Since, for $\in=-1$, the same geometry is considered in both z- and w- planes.

3. Upon calculating $\delta_2\ 2\varnothing$, where $\varnothing=\delta_1 a - f(\propto)$ and $f(\propto)\in C_2$ is unknown so-far, we have

$$\delta 2\varnothing = \frac{1}{\delta_1 \propto}[\delta_1\varnothing + 2(2 + \in e^{2\propto} + f')Re(\propto,\varnothing)] - (2+4\in e^{2\propto}+f'')\ \varnothing + D_0 = 0,$$

where

$$D_0 = -ff'' + f'^2 + 3(1+\in e^{2\propto})f' - 2(1+\in e^{2\propto})f + 2(1+ \in e^{2\propto})^2.$$

One of Peschl's requirements, to deduce his fundamental principle, is the vanishing of the term D_0. Hence of interest to us are those functions $f(\propto)$ satisfying the Eq. $D_0=0$, which is a short form of (1).

These functions are given in (7) parametrically and corresponding to each positive one interval estimations for the differential $\propto$ and $|w(z)|$ have been obtained. Furthermore a set of families of functions w(z) are introduced and classified (see e.g. (3,4,8).

Our aim in this article is to obtain an inequality, using the fixed point theorem (2). Making use of that inequality, interval estimations for the solutions of (1) can be determined. We shall be concerned then for example with the hyperbolic geometry i.e. $\in = -1$. Interval estimation is computed and some selected formulae of the solution are depicted graphically.

(1) An inequality to determine interval estimations for the solutions of equation (1)

Using the transformation

$$f = h - 1 + \in e^{2\propto}\ , \in = 0, \pm 1$$

in equation (1), we have

$$(\in e^{2\propto} - h - 1)h'' + h'^2 + (3 - \in e^{2\propto})h' - 2h = 0,$$

which for $\frac{h'}{h} = u$, $\in e^{2\propto} - 1 = P$ and $e\ \int u d\propto = Q$ reduces to

$$(P - Q)u' + Pu^2 + (2 - P)u - 2 = 0 \qquad 2)$$

Next, we consider the operator A(u), where

$$A(u) = \int \frac{1}{P-Q}\{2 + (P-2)u - Pu^2\}\ d\propto \qquad 3)$$

satisfies the conditions of the fixed point theorem (2). For a function v with the same initial conditions as u Eq. (3) is satisfied and consequently we have:

$$p(A(u),A(v)) < 0\theta . p(u,v),$$

**) The first and the second Beltrami operators δ_1 and δ_2 are defined as follows:

(i) $\delta_1(u,v) = (1-z\bar{z})^2 u_z v_{\bar{z}}$; hence $\delta_1(u,v) = \overline{\delta_1(v,u)}$

(ii) $\delta_2\ u = (1-z\bar{z})^2 u_{z\bar{z}}$.

where p is the distance function and $\theta<1$.

Recalling that $P - \int u d\alpha$ satisfies Eq. (1) we obtain

$$p(A(u),A(v)) < \frac{|\ 2+2Pu-u\ |.}{Q-P}\ (u,v)$$

which implies

$$-1 < \frac{3-\in e^{2\alpha} -2u(1-\in e^{2\alpha})}{1 -\in e^{2\alpha} + e^{\int u d\alpha}} < 1 < 1 \quad \ldots\ldots (*)$$

This inequality introduces a useful tool to determine interval estimations for the solutions of Eq. (1). In the next section we are restricted only to one of the three values of $\in$, namely $\in = -1$.

(2) An interval estimation for the solutions of equation (1) in hyperbolic geometry

In hyperbolic geometry inequality (*) takes the form

$$-1 < \frac{3+e^{2\alpha} -2(1+e^{2\alpha})h'/h}{1+e^{2\alpha} + h} < 1$$

which implies

$$2-h < 2(1+e^{2\alpha})h'/h < 4 + 2e^{2\alpha} + h$$

We have therefore the following two differential inequalities

$$2 - h < 2(1+e^{2\alpha})h'/h \quad \ldots\ldots 4)$$

and

$$2(1+e^{2\alpha})h'/h < 4+2e^{2\alpha} + h \quad \ldots\ldots 5)$$

Upon using separation of variables inequality (4) takes the form

$$\frac{d\alpha}{2(1+e^{2\alpha})} < \frac{dh}{h(2-h)}.$$

Hence, by integration, considering some results obtained from the mean value theorem (5), we have

$$(1+e^{2}\alpha)^{-1/2} < h(2-h)^{-1} ,$$

from which we induce

$$2e^{\alpha}\{e^{\alpha} +(1+e^{2\alpha})^{-1/2}\} < h \qquad 6)$$

To deduce the upper bound for h we consider inequality (5) and rewrite it in the form

$$h' - \frac{2+e^{2\alpha}}{1+e^{2\alpha}} h < \frac{1}{2(1+e^{2\alpha})} h$$

Substituting $\Psi = 1/h$, we have

$$\frac{-1}{2(1+e^{2\alpha})} < \Psi \frac{2+e^{2\alpha}}{1+e^{2\alpha}} \Psi \qquad 7)$$

This represents a first order linear differential inequality with integrating factor

$$\exp \frac{(\int 2+e^{2\alpha} \, d\alpha)}{1+e^{2\alpha}} = \frac{e^{2\alpha}}{\sqrt{(1+e^{2\alpha})}}$$

Inequality (7), when solved, gives

$$\frac{1 \quad \Psi}{2 \sqrt{(1+e^{2\alpha})}} < \frac{e^{2\alpha}}{\sqrt{(1+e^{2\alpha})}}$$

It follows that

$$h < 2e^{2\alpha} \qquad 8)$$

From (6) and (8) we have

$$\frac{2e^{2\alpha}}{e^{\alpha}+ \sqrt{(1+e^{2\alpha})}} < h < e^{2\alpha}$$

Consequently,

$$1 - e^{2\alpha} + 2e^{\alpha} \sqrt{(1 + e^{2\alpha})} < f < 1 + 3e^{2\alpha} \qquad 9)$$

which is one of the main targets of this work.

(3) Some formulae for the solution of equation (1) for $\in = -1$ and their graph

The formulae of the solution of Eq. (1) are given in (7). The procedures for obtaining them, $\in = \pm 1$*), are outlined as follows:

---

*) The easier case $\in = 0$ has been studied in (8).

Since we are interested in the positive solutions only, we let $f(\alpha) = g^2(x)$ and $x = e^{\alpha}$. Thus for arbitrary $L(x) = -1 - \in e^2$, using the transformation $q = \frac{xgg' - L(x)}{g}$, the new form of Eq. (1) is

$$(q^2 - 1)\frac{dx}{dq} = xg, \ (q^2 - 1)\frac{dg}{dq} = qg + L(x), \text{ where } q^2 \neq 1.$$

We seek a transformation for q such that this system leads to an elementary solvable equation. To do so let $q = P(x,g,t)$ and consider then the linear transformation $q = x\varnothing_1(t) + g\varnothing_2(t)$, where $\varnothing_1$ and $\varnothing_2$ are to be selected to acquire an integrable system for particular L(x). It has been shown in (7) that for $\varnothing_2 = 1$, $\varnothing_1 = \sinh t$ for $\in = +1$ and $\pm\cosh t$, $\cos t$ for $\in = -1$ we have a system of equations which implies $(\dot{} = \frac{d}{dt})$:

For $\in = +1$

$$x\ddot{x} - \dot{x}^2 + \frac{1 + x^2}{\cosh^2 t} = 0 \text{ with } g = \dot{x}\cosh t.$$

For $\in = -1$

i) $x\ddot{x} - \dot{x}^2 + T(x,t) = 0$, where

$$T(x,t) = \frac{1 - x^2}{\sin^2 t}, \frac{1 - x^2}{\sin^2 t} \text{ with } g = \pm\dot{x}\sinh t, \ \dot{x}\sin t$$

ii) $xgg' + 1 - x^2 - g^2 = \pm xg.$

The solutions of these equations yield the parametric representation x(t), g(t) for the solutions of Eq. (1).

Now, we select the following formulae for the solution of Eq. (1) for $\in = -1$:

$$\alpha = \log\frac{\sinh(a(t+b))}{a\sinh t} \qquad a>0, = | 1 \ b, \ t \text{ arbitrary}$$

(I) $\quad f = \{\cosh(a(t+b)) \pm e^{\alpha}.\cosh t\}^2$

and $\quad \alpha = \frac{\log\sin(a(t+b))}{a\sin t}$ $\qquad$ $a>0$, b arbitrary, $0<t<\pi$, $\pi<t<2\pi$

(II) $\quad f = \{\cos(a(t+b)) \pm e^{\alpha}.\cos t\}^2$

One can see that (I), (II) satisfy the inequality (9). The above solutions are computed and depicted graphically in fig. (1) and (2), where $G=\sqrt{f}$ and $|X|=e^{\alpha}$. Limiting cases, $a \to 0$, of (I), (II) are also represented in fig. (3) and (4).

(4) Conclusion

We have obtained an inequality (9) for interval estimation of the solutions of the considered equation in the hyperbolic geometry, $\in = -1$, which is chosen upon symmetry grounds in the z- and w- plane. For space limitations, only some selected formulae are presented graphically.

The other two cases $\in = 0, +1$ corresponding to euclidean and elliptic geometry respectively can be dealt with in the same way.

REFERENCES

(1) **K.-W. Bauer.** 1960 Ueber die Abschaetzung von Loesungen gewisser partieller Differentialgleichungen vom elliptischen Typus, Bonner Math. Schr. Nr. 10.

(2) **E. Codington.** 1955 Theory of ordinary differential equations, Tata Mcgraw Hill Publishing Comp. Ltd., New Delhi.

(3) **M.-K. Gabr.** 1974 Ueber die Anwendung einer Abschaetzungsmethode von E. Peschl auf verschiedene Funktionenfamilien, Diss. Bonn.

(4) **M.-K. Gabr.** 1979 On Peschl's principle, The 4th Int. Congr. on St., Comp. Sc. and Dem. Res., Cairo.

(5) **A.S.-T. Lue.** 1974 Basic pure Mathematics II, VNR New Math. Libr. London.

(6) **E. Peschl.** 1955 Les invariant differentials non holomorphes et leur role dans les theorie de fonctions, Rand. Sem. Math. Messina.

(7) **E. Peschl, K.-W. Bauer.** 1963 Ueber eine nichtlineare Differentialgleichung 2. Ordnung, die bei einem gewissen Abschaetzundsverfahren eine besondere Rolle spielt, Forschungsber. Des Landes Nordrh. Westf. 1306.

(8) **E. Raupach.** 1960 Eine Abschaetzungsmethode fuer reellwertige Loesungen der Differentialgleichung $\Delta\alpha = \frac{-4}{1-|z|^2}$, Bonner Math. Schr. Nr. 9.

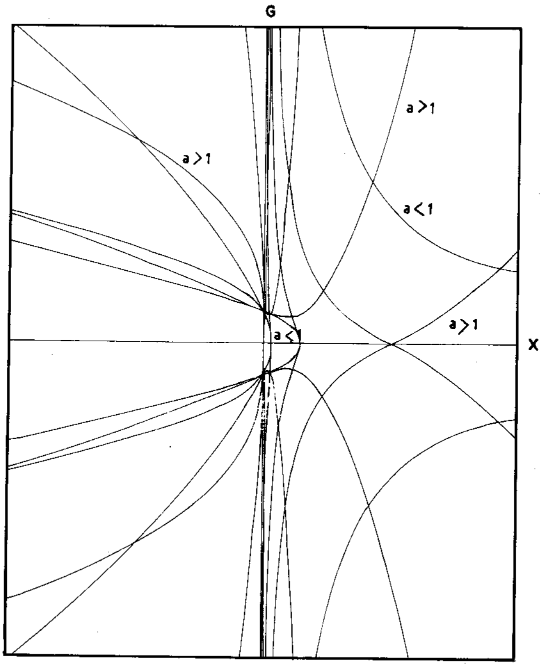

Fig. 1

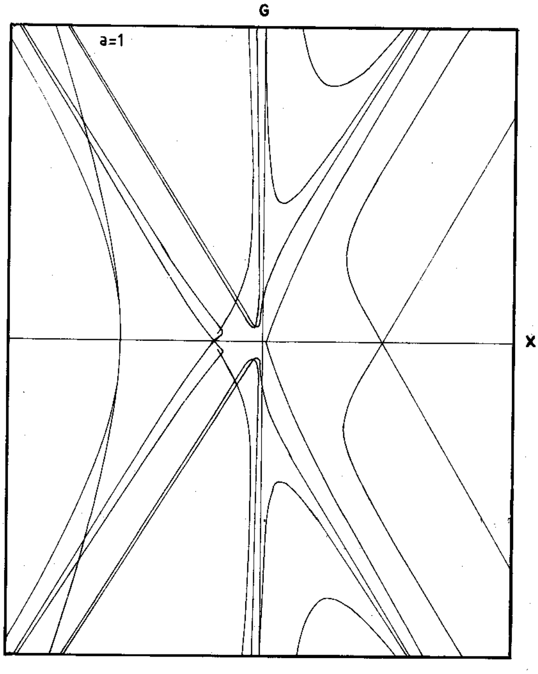

Fig. 2

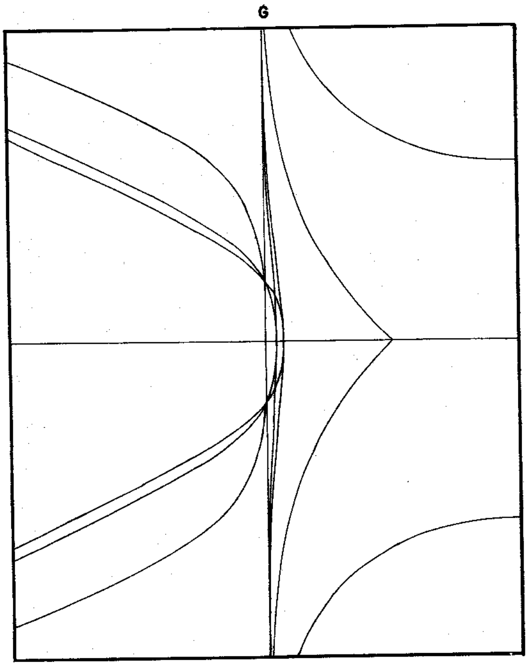

Fig. 3

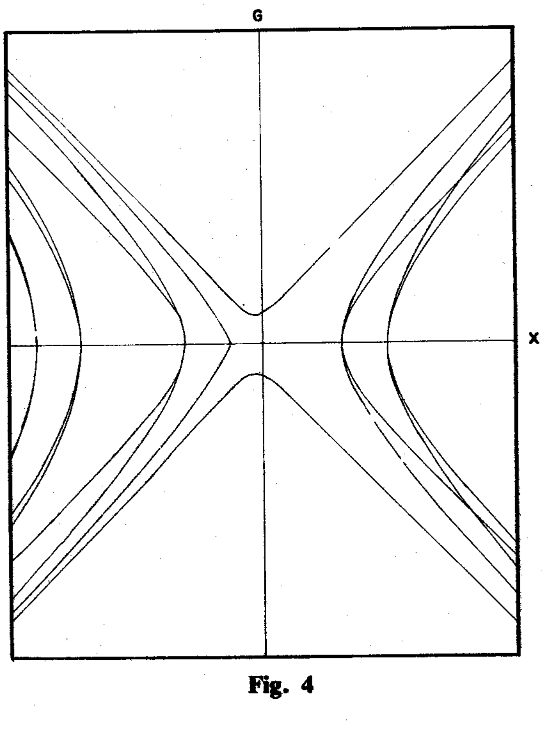

Fig. 4

حول سلوك حلول معادلة تفاضل غير خطية

د . محمد كامل جبر   شـعبان سـالم

يعنى هذا البحث بدراسة سلوك حلول معادلة تفاضلية غير خطية لها أهميتها في مجال النظريات الهندسية لدوال المتغير المركب . لقد تم التوصل الى تباينه تساعد في حساب فترة تقدير لحلولها ، هذه الفترة حسبت في احدى المجـالات المأخـوذة في الاعتبار حيث يوجد نفس النوع من الهندسة (الزائدية) في كل من المستـوى Z والمستوى W بالاضافة إلى ذلك تم رسم بعض الحلول المختارة لهذه المعادلة باستخدام الحاسب الآلى .