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COMPLETENESS OF DISTRIBUTIVE p-ALGEBRAS

By

S. EL-ASSAR and M. ATALLAH
Dept. of Mathematics, Faculty of Science,
Tanta University, Egypt

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ABSTRACT

According to the triple characterization of distributive p-algebras which is given by T. Katrinák, we characterize the completeness of distributive p-algebras.

INTRODUCTION

It is known that there can be assigned to every distributive lattice with pseudocomplementation (= distributive p-algebra) L a Boolean algebra B (L) and a distributive lattice D (L) with 1 (see [4], [6] and [7] A problem for distributive p-algebras is considered solved if it can be reduced to two problems: One for Boolean algebras and one for distributive Lattices with 1, (Gratzer [4]). Except for Stone algebras (see [1], [2]) characterization of completeness of various classes of p-algebras is still an open question. In the present paper we discuss the completeness of a distributive p-algebra L by means of the triple B (L), D (L) and φ (L).

PRELIMINARIES

A universal algebra $<L;\vee,\wedge,^*,0,1>$ of type $<2,2,1,0,0>$ is called a p-algebra iff $<L;\vee,\wedge,0,1>$ is a bounded lattice and the unary operation*(of pseudocomplementation) is defined by $x \wedge a = 0$ iff $x \leq a^*$ for each $a \in L$. The p-algebra is called distributive if the lattice $<L;\vee,\wedge,0,1>$ is distributive. The standard results on p-algebras may be found in [3], [4] and [5].

The following rules of computation (see, e.g. (3), (4). will be used frequently.

(1) $a \leq b$ implies $b^* \leq a^*$;
(2) $a \leq a^{**}$;
(3) $a^* = a^{***}$;
(4) $a^* \wedge a^{**} = 0$;
(5) $(a \vee b)^* = a^* \wedge b^*$,

(6) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,
(7) $(a \wedge b)^{*} \geqslant a^{*} \vee b^{*}$;
(8) $(a \vee b)^{**} = (a^{**} \vee b^{**})^{**}$;

The identities (1) - (8) hold in any p-algebra.

The Stone algebra is a distributive p-algebra which satisfies the additional identity $x^{*} \vee x x^{**} = 1$

In any p-algebra L we can define the set of closed elements $B(L) = \{x : x = x^{**}\}$. It is known that $< B(L); \nabla, \wedge, *, 0,1 >$ is a Boolean algebra, where $a \nabla b = (a^{*} \wedge b^{**})$. Then the set of dense elements D(L) where $D(L) = \{x : x^{*} = 0\}$ forms a filter, namely a dual ideal of L. Let F(D(L)) denote the poset of all filters of D(L) ordered by set inclusion. For a distributive p-algebra L, F(D)L)) is a distributive lattice. Finally define the mapping

$(L): B(L) \rightarrow F(D(L))$

where $\varphi(L)$: $a\{x \in D(L): x \geqslant a^{*}\}$

$= (a^{*}) \cap D(L)$, $A \in B(L)$, where (a^{*}) is the filter of L generated by a^{*}.

For a distributive p-algebra L, the sublattice of dense elements D(L) is distributive and contains 1. The mapping $\varphi(L)$ is a $\{0,1,\vee\}$ homomorphism of B (L) into F (D(L)), and for $a \in B(L)$ we have

$a^{*}(L) \quad \varphi(L) = (a \vee a^{*})$.

The following theorem shows the role played by B(L), D(L) and $\varphi(L)$.

THEOREM A (Katriňák (7))

A distributive p-algebra L is determined up to isomorphism by the triple < b (L), D (L), φ (L) >

According to Theorem A, any $x \in L$ is determined by the pair < a, d>, where $a \in B(L)$, $d \in D_a$, $d_a = \{d \in a\varphi(L): d \leqslant d_a\}$, and

$d_a = a v a^{*}$.

For $x, y \in L$, $x \simeq <a,d< \; y \simeq <b,e>$

Then $x \leqslant y$ iff $a \leqslant b$ and $d \leqslant e\varphi a$, where the map $\varphi_a: D(L) \rightarrow D(L)$ is given by $d_a = d v a$, $a \in B(L)$ $d \in D(L)$.

MAIN DESULTS

LEMMA 1 (Frink, (3))

For any complete p-algebra L, the Boolean algebra of closed elements B (L) is complete.

PROOF

Let HCB(L) and x be a lower bound of H in L. The fact $x \leqslant a \; \forall \; a \in H$ implies that $x^{**} \leqslant a^{**} = a$, so x^{**} is also a lower bound of H. Consequently, if $x = \inf_L H$, then $x = x^{**}$ and $x = \inf_{B(L)} H$. Hence, B(L) is a complete lattice.

Now, we can formulate.

THEOREM 1

A distributive p-algebra L is complete iff the following conditions are satisfied;

(i) B (L) is complete,
(ii) D (L) is conditionally complete,
(iii) for any subset $E \subset D(L)$ the set

$B_E = \{a: a \in B(L)$ and $\inf_{D(L)} E \varphi_a)$exists$\}$ has a greatest element in B(L).

PROOF

Let L be a complete distributive p-algebra. B (L) is complete by Lemma 1, and (ii) is clearly satisfied. For (iii), let $E \subset D(L)$.Put $s = \inf_L E$. We claim that $s^{**} = \max B_E$. Indeed, since $s \leqslant d$ for all $d \in E$, $s v s^* \leqslant d v s^* = d v s^{***} = d \varphi_s^{**}$ for all $d \in$. Since $s v s^* \in D(L)$, we infer that $E \varphi_s^*$ has a lower bound in D(L); hence $\inf_L E \varphi_s^{**} = \inf_D (L) E \; s^{**}$ and thus $s^{**} \in B_E$. Now, consider any $b \in B_E$. This means that $\inf_{D(L)} E \varphi_b = d_b$ exists. We have
$d_b \wedge b \leqslant (d \; v \; b^*) \wedge b = d \wedge b \leqslant d$ for all d E.

Thus $d_b \wedge b \leqslant s$ since $s = \inf_L E$.
Consequently, $b = b^{**} = (d_b \wedge b^{**}) \leqslant s^{**}$, that is, $b \leqslant s^{**}$ and indeed $s^{**} = \max B_E$

To prove the converse, assume the validity of the conditions (i) - (iii) and consider $M \subset L$. Put

$B_1 = \{m^{**}; m \in M\}$, $E: = \{m \; v \; m^*; m \in M\}$, $a: = \inf_{B(L)} B_1$ (using (i)), b = greatest element $c \in B(L)$ such that $\inf_{D(L)} E \;_c$ exists (using (iii) and finally $d_b := \inf_{D(L)} E \varphi_b$.

We claim that $a \wedge b \wedge d_b = \text{in}_L M$. For this end we have:
(1) we have, for all $m \in M$, $a \leqslant m^{**}$ and $d_b \leqslant m \; v \; m^* \; v \; b^*$. Hence $a \wedge b \wedge d_v \leqslant m^{**} \wedge b \wedge (m \; v \; m^* \; v \; b^*) = m \wedge b \leqslant m$,

thus $a \wedge b \wedge d_b \leqslant M$

(2) Let $z \leqslant M$. Hence $z \leqslant m \; v \; m^* \; v \; b^*$ for all $m \in M$, Consider $z \; v \; d_b$, $z \; v \; d_b \in D(L)$ and $z \; v \; d_b \leqslant m \; v \; m^* \; v \; b^*$ for all $m \in M$, thus $z \; v \; d_b \leqslant d_b = \inf_{b(L)} E \varphi_b$ and we conclude $z \leqslant d_b$.

Moreover $z \leqslant m$ implies $z^{**} \leqslant m^{**}$ for all $m \in M$, hence $z \leqslant z^{**} \leqslant a = \inf_{N(L)} B_1$ and we conclude $z \leqslant a$.

$$\text{Consider } E\varphi_z vb = \{m \vee m^* \vee (z^* \wedge b^*);\ m \in M\}$$
$$= \{(m \vee m^* \vee z^*) \wedge (m \vee m^* \vee b^*);\ m \in M\}$$

$(z \vee z^*) \wedge d_b \in D(L)$. Further, $z \vee z^* \leqslant m \vee m^* \vee z^*$ for all $m \in M$. So $(z \vee z^*) \wedge d_b \leqslant (m \vee m^* \vee z^*) \wedge (m \vee m^* \vee b^*)$ for all $m \in M$. Consequently, $E\varphi_{zvb}$ has a lower bound in D(L) and by (ii) $\inf_{D(L)} E\varphi_{zvb}$ exists. But $E\varphi_{zvb} = E\varphi_{(zvb)^{**}}$
So (iii) gives $b \geqslant (z \vee b)^{**} \geqslant z \vee b$ and we conclude $z \leqslant b$. Furthermore $z \leqslant a \wedge b \wedge d_b$.

COROLLARY 1

Let L be a distributive p-algebra. If B(L) and D(L) are complete then so is L.

In this case B_E = B(L) which has 1 as a greatest element.

COROLLARY 2

For the distributive p-algebra L, if B(L) is finite and D(L) is conditionally complete then L is complete.

In this case B_E is an ideal of a finite Boolean algebra, and by this B_E is necessarily principal.

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