Qatar Univ. Sci. J. (1993), 13(2): 187 - 189

KREIN'S METHOD FOR SINGULAR MIXED PROBLEMS

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طريقة كرين في المسائل الشاذة المختلطة

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استكمالاً للطرق السابقة المختلفة في حل المعادلة التكاملية لفريد هولم من النوع الأول ذو النواة الشاذة والناتجة في العلوم التطبيقية (علم المرونة عند وضع جسم على لوح ذو عمق لا نهائي أو علم الموائع اللزجة عند غمر جسم في سائل لزج لا نهائي) استطاع الباحثان استخدام طريقة كرين للحصول على العلاقات الطيفية المختلفة للمعادلة السابقة تحت شروط معينة وأثبت الباحثان أن هذه الطريقة أسهل استخداماً من الطرق السابقة لتجنبها النقطة الشاذة وتحويل المعادلة التكاملية الشاذة إلى تكاملات عادية يمكن تطبيقها في مجالات أوسع انتشاراً .

ABSTRACT

In this paper, the spectral relations of the Fredholm integral equation of the first kind for the mixed problems with singular kernel are obtained.

INTRODUCTION

In the displacement problems (in the theory of elasticity) of the anti-plane deformation of an infinite rigid strip with width 2a, putting on an elastic layer of thickness h; or (in hydro-dynamic) immersed in a viscous fluid layer of deep h, we obtain the integral equation

$$-\int_{-a}^{a} \tau(\xi) \ln \left| \operatorname{th} \frac{\pi(\xi - x)}{4h} \right| d\xi = G \qquad (1.1)$$

where G is the displacement magnitude, $\tau(\zeta)$ and is the unknown displacement stress. In the mixed problem G is a variable function, and we have the integral equation

$$\int_{-a}^{a} k(|x-y|)\, \phi(y)\, dy = f(x) \qquad (1.2)$$

which represents the Fredholm integral equation of the first kind with singular kernel at x = y.

In paper [1] the different methods, Fourier transformation method, Chebeshev polynomials, Potential methods, singular method (Cauchy method) and Krein's method, for solving (1.2) are given. Also the authers proved that Krein's method is the best and has a large field of applications in the theory of elasticity and visco-dynamic problems.

In this paper Krein's method is used for obtaining the spectral relation of the Fredholm integral equation of the first kind under certain boundary conditions.

FORMULATION OF THE PROBLEM

Consider the integral equation

$$\int_{-a}^{a} k\left(\frac{x-y}{\lambda}\right) P(y)\, dy = \pi f(x) \quad (\lambda = \frac{h}{a} \in (o, \infty),\ |x| \le a) \qquad (2.1)$$

under the conditions

$$(1) \int_{-a}^{a} P(y)\, dy = P < \infty; \quad (2)\ k(t) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{th\, s}{s} e^{ist} ds \qquad (2.2)$$

where λ is a parameter, P is a constant, f (x) (displacement magnitude) is a known function belongs to the class of continuous function C [-a, a] with continuous first derivatives in [-a, a], while P (x) (displacement stress) is unknown function. When λ is very large (i.e. $\lambda \to \infty$) the second condition of (2.2) takes the form

$$k\left(\frac{x-y}{\lambda}\right) = -\ln|y-x| + d, \quad d = \ln \frac{4\lambda}{\pi} \qquad (2.3)$$

which has a singularity at $y = x$

As the same way of [1], we can establish the general solution of (2.1) under (2.2) and (2.3), using Krein's method in the form

$$P_+(x)=\frac{J(a)}{\pi\left[ln\frac{2}{a}+d\right]}\frac{1}{\sqrt{a^2-x^2}}-\frac{2}{\pi}\int_x^a\frac{du}{\sqrt{u^2-x^2}}\frac{d}{du}\left[u\frac{d}{du}\int_o^u\frac{f_+(y)dy}{\sqrt{u^2-y^2}}\right] \quad (2.4)$$

and

$$P_-(x)=\frac{-2}{\pi}\frac{d}{dx}\int_x^a\frac{u\,du}{\sqrt{u^2-x^2}}\int_o^a\frac{df_-(y)}{\sqrt{u^2-y^2}}\bullet \quad (2.5)$$

where

$$J(u) = 2\left[\int_o^u\frac{f_+(y)\,dy}{\sqrt{u^2-y^2}} + u\,ln\left(\frac{2}{u}+d\right)\frac{d}{du}\int_o^u\frac{f_+(y)\,dy}{\sqrt{u^2-y^2}}\right] \quad (2.6)$$

$$f(x) = f_+(x) + f_-(x)\ ,\quad P(x) = P_+(x) + P_-(x)$$

$$(f_\pm(-x) = \pm f_\pm(x)\quad,\quad P_\pm(-x) = \pm P_\pm(x),\ x\in\quad(-a,a)).$$

Now, we are going to obtain the spectral relations of (2.1) using (2.4) and (2.5) when the known function takes the form of Chebeshev polynomials.

METHOD OF SOLUTION

The solution of the problem can be derived in the following theorem.

Theorem 1.

The spectral relations for the Fredholm integral equation of the first type with the kernel defined by (2.3) take the form

$$\int_{-1}^{1}\left[-ln|x-s|+d\right]\frac{T_{2n}(s)}{\sqrt{1-s^2}}ds=\begin{cases}\frac{\pi}{2n}T_{2n}(x)\ , & n=1,2,3\ldots\\ \pi(ln\,2+d)^{-1}\ , & n=o\end{cases} \quad (3.1)$$

when T_n (x) are Chebeshev polynomials of order n. the proof of (3.1) depends on the following way:

For all positive integers n, the values of J_n (u) have the form

$$J_n(u) = 2\pi\left[\frac{1}{2}P_n^{(-1,o)}(2u^2-1)+n\,u^2\,ln\left(\frac{2}{u}+d\right).\ P_{n-1}^{(o,1)}(2u^2-1)\right] \quad (3.2)$$

where $P_n^{(\propto,\beta)}$ (x) are Jacobi polynomials of order n (n >, o; $P_n^{(\propto,\beta)}$ > o).

To prove (3.2), let f_+ (y) = T_{2n} (Y) in (2.6), where it may take the form

$$J_n(u) = 2\left[I_n(u) + u\,ln\left(\frac{2}{u}+d\right)\frac{d}{du}I_n(u)\right]\ , \quad (3.3)$$

where

$$I_n = \int_o^u\frac{T_{2n}(s)\,ds}{\sqrt{u^2-s^2}} \quad (3.4)$$

Using the famaus relations [2] between Chebyshev polynomials, Legender polynomials and Jacobe polynomials,

$$\int_{-1}^{1}(1-t^2)^{-1/2}\,T_n(1-t^2y)dt=\frac{\pi}{2}\left[P_n(1-y)+P_{n-1}(1-y)\right],$$

$$2P_n(-1,o)\,(x)=P_n(x)-P_{n-1}(x)\bullet \quad (3.5)$$

where P_n (x) Legender polynomials of order n. Using the substitution s = ut together with the relation T_{2n} (x) = T_n $(2x^2 - 1)$, the formula (3.4) takes the form

$$I_{n.} = \int_o^1 (1-t^2)^{-1/2}\,T_n\,(2t^2-1)\,dt \quad (3.6)$$

In the veid of (3.5) the previous formula and its first derivative take the form

$$I_n\,(u) = \frac{\pi}{2}P_n^{(-1,o)}\,(2u^2-1). \quad (3.7)$$

and

$$\frac{dI_n}{du} = n\pi u\,P_{n-1}^{(o,1)}(2u^2-1)\ (n=1,2\ldots) \quad (3.8)$$

The required result of (3.2) is obtained, after introducing (3.7) and (3.8) in (3.3).

Put u = 1 in (3.2), we have

$$J_n(1) = 2\pi\left[\frac{1}{2}P_n^{(-1,o)}(1) + n\,ln(2+d).\ P_{n-1}^{(o,1)}(1)\right], \quad (3.9)$$

Using the famous relation [3]

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(1+a)}, \quad (3.10)$$

equation (3.9) takes the form

$$J_n(1) = 2n\,ln\,(2+d).\qquad (n=1,2\ldots) \quad (3.11)$$

where Γ (x) is the Gamma function.

Corollary:

The second derivative of $I_n(u)$ $\left(I_n^{(2)}(u) = \frac{d}{du}\left(u\frac{dI_n}{du}\right)\right)$ can be obtained in the form

$$I_n^{(2)}(u) = 2n\pi u\left[P_{n-1}^{(o,1)}(2u^2-1) + (n+1)u^2\,P_{n-2}^{(1,2)}\,(2u^2-1)\right] \quad (3.12)$$

Now, we are in a position to evaluate the integral

$$A_n(x) = \int_x^1\frac{du}{\sqrt{u^2-x^2}}\frac{d}{du}\left[u\frac{d}{du}\int_o^u\frac{T_{2n}(s)\,ds}{\sqrt{u^2-s^2}}\right]. \quad (4.1)$$

Introducing (3.12) in (4.1), we have

$$A_n(x) = 2n\pi\left[\int_x^1 \frac{u\,P_{n-1}^{(o,1)}(2u^2-1)}{\sqrt{u^2-x^2}}\,du + (n+1)\int_x^1 \frac{u^3\,P_{n-2}^{(1,2)}(2u^2-1)du}{\sqrt{u^2-x^2}}\right] \tag{4.2}$$

Assuming $2u^2$-1 = y, $2x^2$-1 = t; we can write (4.2) in the form

$$A_n(t) = \frac{\pi n}{\sqrt{2}}\int_t^1 \frac{P_{n-1}^{(o,1)}(y)\,dy}{\sqrt{y-t}} + \frac{n(n+1)}{2\sqrt{2}}\int_t^1 \frac{y\,P_{n-2}^{(1,2)}(y)\,dy}{\sqrt{y-t}}$$

$$+\frac{n(n+1)}{2\sqrt{2}}\int_t^1 \frac{P_{n-2}^{(1,2)}(y)\,dy}{\sqrt{y-t}}. \tag{4.3}$$

Putting y = 1 - (1-t) τ, then (4.3) becomes

$$A_n(t) = \frac{\pi n}{\sqrt{2}}\sqrt{1-t}\int_o^1 (1-\tau)^{-1/2} P_{n-1}^{(o,1)}[1-(1-t)\,\tau]d\tau +$$

$$+\frac{\pi n(n+1)}{2\sqrt{2}}\cdot\sqrt{1-t}\,(1+t)\int_o^1 (1-\tau)^{-1/2} P_{n-2}^{(1,2)}[1-(1-t)\,\tau]\,d\tau +$$

$$+\frac{n(n+1)}{2\sqrt{2}}(1-t)^{3/2}\int_o^1 (1-\tau)^{-1/2} P_{n-2}^{(1,2)}[1-(1-t)\,\tau]\,d\tau \tag{4.4}$$

Using the famous formulae [2]

$$\int_o^1 t^{\lambda-1}(1-t)^{\mu-1} P_n^{(\alpha,\beta)}(1-\gamma t)dt =$$

$$=\frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(\xi\ )}{n!\Gamma(1+\alpha)\Gamma(\lambda+\ \ )}\,{}_3F_2(-n,n+\alpha+\beta+1;\lambda,\alpha+1,\lambda+\xi;\frac{\gamma}{2})$$

$$(\mathrm{Re}\lambda > o;\ Re\xi > o) \tag{4.5}$$

and

$$P_n^{(\alpha,\beta)}(y) = \binom{n+\alpha}{n}F(-n,n+\alpha+\beta+1;\alpha+1;\ \frac{1-y}{2}), \tag{4.6}$$

where $_3F_2$ ($\propto_1$, $\propto_2$, $\propto_3$; β_1, β_2; z) is the generalized hypergeometric series, and F (∝, β; τ; z) is the hypergeometric Gauss function, the first term of (4.4) can be written in the form

$$\int_o^1 (1-\tau)^{-1/2} P_{n-1}^{(o,1)}[1-(1-t)\tau]d\tau = \frac{\sqrt{\pi}(n-1)!}{\Gamma(n+1/2)}P_{n-1}^{(1/2,1/2)}(y). \tag{4.7}$$

By using the same way, it is easy to prove that

$$\int_0^1 (1-\tau)^{-1/2} P_{n-2}^{(1,2)}[1-(1-t)\tau]d\tau =$$

$$= \frac{-\sqrt{\pi}\Gamma(n-1)}{2\Gamma(n-1/2).(n+1)}P_{n-1}^{(-1/2,3/2)}(1-2t) + \frac{1}{2(n^2-1)} \tag{4.8}$$

and

$$\int_0^1 (1-\tau)^{1/2} P_{n-2}^{(1,2)}[1-(1-t)\tau]d\tau =$$

$$= \frac{-\sqrt{\pi}(n-1)!}{\Gamma(n\times 1/2).(n+1)(1-t)}P_{n-1}^{(1/2,1/2)}(t) + \frac{2}{(n+1)(1-t)} \tag{4.9}$$

Introducing the last three formulae (4.7), (4.8), and (4.9) in (4.4), after some algebra, we obtain

$$A_n(x) = \frac{\pi^{3/2}n!}{\sqrt{2}\Gamma(n-1/2)}\cdot\frac{1}{(1-y)}\left[\frac{1-y}{2n-1}P_{n-1}^{(1/2,1/2)}(y)_-\right.$$

$$\left.-(1+y)P_{n-1}^{(-1/2,3/2)}(y)\ \right] + \frac{\sqrt{2}n\pi}{\sqrt{1-y}}(y = 2x^2-1,\ n = 1,2,\ldots) \tag{4.10}$$

For writing (4.10) in Chebyshev polynomials form, we must use the famous formulae [2,3] relations:

$$P_n^{(-1/2,-1/2)}(2x^2-1) = \frac{\Gamma(n+1/2)\Gamma(\lambda)}{\sqrt{\pi}(n+\lambda)}C_{2n}^{\lambda}(x), \tag{5.1}$$

$$P_n^{(\lambda-1/2,1/2)}(2x^2-1) = \frac{\Gamma(n+3/2)\Gamma(\lambda)}{\sqrt{\pi}x\Gamma(n+\lambda+1)}C_{2n+1}^{\lambda}(x), \tag{5.2}$$

and

$$\lim_{\lambda\to o}\Gamma(\lambda)C_n^{\lambda}(x) = \frac{2}{n}T_n(x)\ (n = 1,2,\ldots) \tag{5.3}$$

Substituting these formulae in (4.10) we have

$$A_n(x) = \frac{n\pi(1-T_{2n}(x))}{\sqrt{1-x^2}}(n = 1,2,\ldots) \tag{5.4}$$

Introducing (5.4) and (3.11) in (2.4) then the theorem may be proved.

By using the same way we can try to prove the following theorem.

Theorem 2.

The spectral relation for the Fredholm integral equation with the kernel defined by (2.2) and the known function is odd is given by

$$\int_{-1}^1 \left[-ln|x-s|+d\right]T_{2n-1}(s)\,ds = \frac{\pi}{2n-1}T_{2n-1}(x).$$

$$(n = 1,2,\ldots$$

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