Journal of Physics: Conference Series **96** (2008) 012045 doi:10.1088/1742-6596/96/1/012045

# Numerical Bifurcation of Predator-Prey Fractional Differential Equations with a Constant Rate Harvesting

**G. H. Erjaee**[1]

Mathematics & Physics Department, Qatar University, Doha, Qatar

E-mail: erjaee@qu.edu.qa

**Abstract.** In this article saddle and Hopf bifurcation points of predator-prey fractional differential equations system with a constant rate harvesting are investigated. The numerical results based on Grunwald-Letnikov discretization for fractional differential equations together with the Mickens' non-standard discretization method agree with those found by the corresponding ordinary differential equation system.

## 1. Introduction

Study of the Fractional Differential Equations (FDEs) as a dynamical system is a novel and appealing subject which has motivated the leading research literatures in recent years. For example see [1-7]. The non-local nature property of the fractional differential equations is a distinguish property. Indeed, a local operator, such as an integer order differential equation, has the property that only the present state of a system can determine its coming state, so this operator is oblivious to the history of the system. On the other hand, the so-called non-local property means that the next state of a system not only depends upon its current state but also upon its historical states starting from the initial time. This, of course, is closer to reality and is therefore the main reason that FDEs have become more and more popular and have been applied to dynamical systems.

In this article we consider the predator-prey system with constant harvesting rate in the form of FDEs and investigate its saddle and Hopf bifurcation points. Although this famous biological model, in the form of ordinary differential equations, has been widely used in forestry, fishery and wildlife management [7 and 8], however in the form of FDEs it is a new subject. Indeed we expect such biological model in the form of FDEs, by the above described non-local property, to be more consistent with the real natural phenomenon than its classic one. Here we first discretize the system by using the Grunwald-Letnikov discretization method [9 and 10] for FDEs, then in order to obtain more accurate numerical results we use the Mickens' method [11] in the discretization process. The numerical results for different orders of FDEs are illustrated in different figures. Finally, we summarized with some comments.

## 2. Discretization of Predator-Prey FDEs

We consider the special case of predator-prey FDEs with the constant rate harvesting as:

[1] To whom any correspondence should be addressed.

 

$$\begin{cases} Dx^{\alpha} = rx\left(1-\dfrac{x}{k}\right)-\dfrac{yx}{a+x} \\ Dy^{\alpha} = y\left(-d+\dfrac{x}{a+x}\right)-e. \end{cases} \tag{1}$$

In this system $x$ and $y$ are functions of time representing the population densities of prey and predator respectively, while $k$ is the capacity of the prey population, $d$ is the death rate of the predator, $r$ is the natural growth rate of the prey population, and $e$ is the harvesting rate. For the $\alpha$ fractional order of $D^{\alpha}$ we have used Caputo's definition [12] which is stated as $D^{\alpha}\mathbf{x}(t) = J^{n-\alpha}D^{n}\mathbf{x}(t)$ . We consider the case where $0<\alpha\le 1$ and $J^{n}$ is the $n^{\text{th}}$ -order Riemann–Liouville integral operator defined by

$$J^{n}\mathbf{x}(t) = \frac{1}{\Gamma(n)}\int_{0}^{t}(t-\tau)^{n-1}\mathbf{x}(\tau)d\tau,$$

for $t>0$. Using Grunwald-Letnikov method [9 and 10] we approximate $D^{\alpha}\mathbf{x}(t)$ as:

$$D^{\alpha}\mathbf{x}(t) = \lim_{h\to 0} h^{-\alpha}\sum_{i=0}^{[t/h]} c_i^{\alpha}\mathbf{x}(t-ih)). \tag{2}$$

Here, $h$ is the step size and $[t]$ denotes the integer part of $t$ and $c_i^{\alpha}$ are the Grunwald-Letnikov coefficients defined by

$$c_i^{\alpha} = h^{-\alpha}(-1)i\binom{\alpha}{i}, \qquad i = 0,1,2,\ldots .$$

They can also be evaluated recursively as

$$c_0^{\alpha} = h^{-\alpha} \text{ and } c_i^{\alpha} = \left(1-\frac{1+\alpha}{i}\right)c_{i-1}^{\alpha}, \quad i = 1,2,3,\ldots .$$

Under this discretization method system (1) is discretized as follows;

$$\begin{cases} x(t_{n+1}) = (\varphi_1(t))^{\alpha}x(t_n)\left(1-\dfrac{x(t_n)}{2}\right)-\dfrac{y(t_{n+1})x(t_n)}{1+x(t_n)}(\varphi_1(t))^{\alpha}-\displaystyle\sum_{i=1}^{N}\left(1-\frac{1+\alpha}{i}\right)x(t_{n-i}) \\ y(t_{n+1}) = (\varphi_2(t))^{\alpha}y(t_n)\left(-d+\dfrac{x(t_{n+1})}{1+x(t_{n+1})}\right)-e(\varphi_2(t))^{\alpha}-\displaystyle\sum_{i=1}^{N}\left(1-\frac{1+\alpha}{i}\right)y(t_{n-i}). \end{cases} \tag{3}$$

Note that here in discretized system (3) we have adapted non-standard Mickens' method. As discussed elsewhere [7 and 8], in the great number of the dynamical systems the difference equations found by non-standard Mickens' discretization method demonstrate significant qualitative improvements in the behaviour of the numerical solutions and the results are dynamically consistent with their differential equations; i.e., both possess the same dynamics in neighbourhoods of the bifurcation points. Here to use this non-standard discretization method for system (1) we replace $t$ in Euler's method by a function of $t$, $\varphi(t)$, and the term $yx$ by $y_n x_{n+1}$.

### 3. Numerical Results

In this section we illustrate the numerical results found by the solution of discretized system (3). For the results to be consistence with those found in classical order differential equations of system (1) [7 and 8], we have taken $\varphi_1(t)=1-e^{-t}$ and $\varphi_2(\tau)=e^{t}-1$ with the bifurcation parameters $d=0.5$ and $e=0.0625$. With these chooses, system (3) has a saddle point for $\alpha=1$ which is illustrated in Figure 1-a. This result is analogous to that found by classical differential equation form of system (1), [7]. This saddle point exists for the values $0.9 \le \alpha \le 1$. Figure 1-b shows the saddle point for the differential order $\alpha=0.9$. This saddle point for the values $\alpha \prec 0.9$ will disappear. Another interesting bifurcation point that we investigated here is the Hopf bifurcation point which can be found by choosing the pair of parameters $(d, e)=(0.3702,\ 0.05)$. This Hopf bifurcation point is illustrated in Figure 2 for different values of differential order $\alpha$. Figure 2-a shows the Hopf bifurcation point for $\alpha=1$ which is again completely similar to that found by classical differential equations of system (1) in [7]. This Hopf bifurcation point exists for different values of the bifurcation parameters $d$, $e$ and the differential order $\alpha$. The Hopf bifurcation for $\alpha=0.9$ is illustrated in Figure 2-b with the same values of the bifurcation parameters in Figure 2-a. This Hopf bifurcation is very sensitive for the differential order $\alpha \prec 0.9$ and needs a more considerable effort in the numerical method.

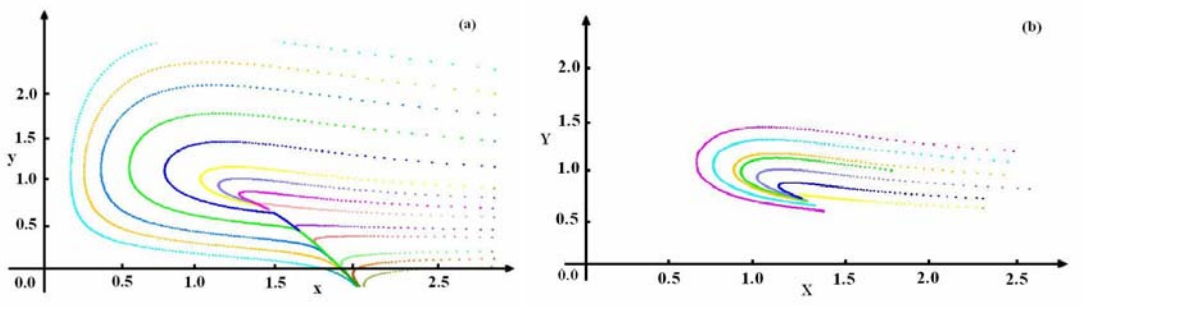


**Figure 1**. Saddle node for pair of bifurcation parameters $(d, e)=(0.5,\ 0.0625)$, (a) for differential order $\alpha=1$ and (b) for $\alpha=0.9$.

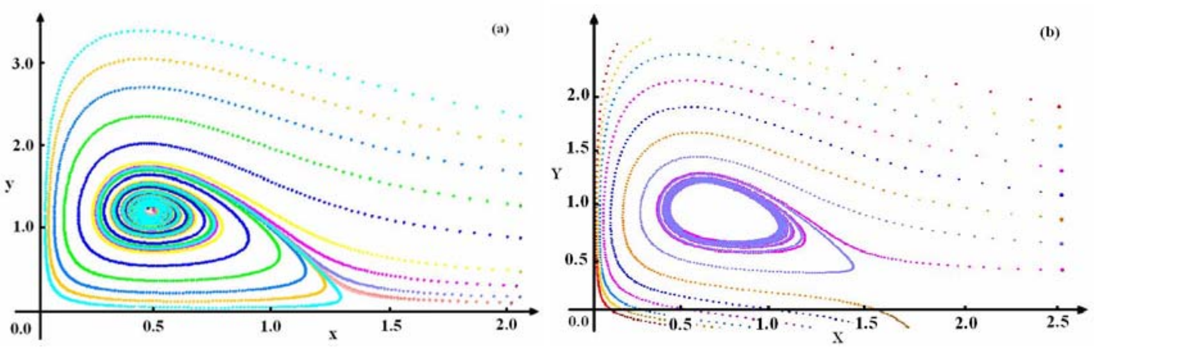


**Figure 2**. Hopf bifurcation for pair of bifurcation parameters $(d, e)=(0.3702,\ 0.05)$, (a) for differential order $\alpha=1$ and (b) for $\alpha=0.9$.

### 4. Consequence

Here we studied the saddle and Hopf bifurcation points of predator-prey FDEs system with the constant rate harvesting for different values of the bifurcation parameters and differential order $\alpha$ . The numerical results agreed with those are found by the same system with the classical order $\alpha = 1$. Yet, there are many other important bifurcation points of this system, such as Bogdanov-Takens bifurcation points, which have to be investigated. In addition, the stability analysis of these bifurcation points need more concrete machinery which yet is an untouched subject in the leading literatures. Although, there are some fascinating attempts which have been done by some researchers (for example by Matignon [4]), however the stability analysis of FDEs is relatively a new and attractive subject which needs more endeavour.

**References**

[1] Yan JP and Li CP 2007 *Chaos. Soliton. Fract.* **32** 725.
[2] Lu JG 2006 *Chaos. Soliton. Fract.***27** 519.
[3] Li CG and Chen G 2004 *Chaos. Soliton. Fract.***22** 549.
[4] Matignon D 1996 *Stability results of fractional differential equations with applications to control processing. In: IMACS, IEEE-SMC* (Lille, France) p.963
[5] Podlubny I 1999 *Fractional differential equations* (New York: Academic Press)
[6] Erjaee GH 2007 Analytical Justification of Phase Synchronization in Chaotic Systems *Chaos. Soliton. Fract.* accepted for publication
[7] Erjaee GH and Dannan FM 2004 *Int. J. Bifurcat. Chaos* **14** 4301
[8] Liu P and Elaydi S 2001 *J. Comput. Anal. Appl.* **3** 53.
[9] Podlubny I 1999 *Fractional differential equations* (New York: Academic Press)
[10] Oldham KB and Spanier J 1974 *The New fractional calculus* (NewYork: Academic Press)
[11] Mickens RE 1994 *Nonstandard Finite Difference Equations of Differential Equations* (Singapore :World Scientific)
[12] Caputo M 1967 *Geophys. J. Roy. Astron. Soc.***13**