Bull. Korean Math. Soc. **44** (2007), No. 3, pp. 385–394

AN UPPER BOUND OF THE BASIS NUMBER OF THE SEMI-STRONG PRODUCT OF CYCLES WITH BIPARTITE GRAPHS

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ABSTRACT. An upper bound of the basis number of the semi-strong product of cycles with bipartite graphs is given. Also, an example is presented where the bound is achieved.

1. Introduction

Throughout this paper, we assume that graphs are finite, undirected and simple. We adopt the standard notation $\Delta(G)$ for the maximum degree of the vertices of G. Our terminologies and notations will be as in [4]. Let G be a graph and $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then any subset S of $E(G)$ corresponds to a $(0,1)$-vector $(\zeta_1, \zeta_2, \cdots, \zeta_{|E(G)|}) \in (Z_2)^{|E(G)|}$ with $\zeta_i = 1$ if $e_i \in S$ and $\zeta_i = 0$ if $e_i \notin S$. Let $\mathcal{C}(G)$, called the cycle space, be the subspace of $(Z_2)^{|E(G)|}$ generated by the vectors corresponding to the cycles in G. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is well known that if r is the number of components of G, then $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r$ (see [5]).

A basis of $\mathcal{C}(G)$ is called *d-fold* if each edge of G occurs in at most d of the cycles in the basis. The *basis number* of G, $b(G)$, is the smallest non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The *required basis* of $\mathcal{C}(G)$ is a basis that is $b(G)$-fold. Let G and H be two graphs, $\varphi : G \longrightarrow H$ be an isomorphism and $\mathcal{B}$ be a (required) basis of $\mathcal{C}(G)$. Then $\{\varphi(c) | c \in \mathcal{B}\}$ is called the *corresponding* (*required*) *basis* of $\mathcal{B}$ in H. The first use of the basis number of a graph was the theorem of MacLane [13] when he proved that a graph G is planar if and only if $b(G) \leq 2$. Schmeichel proved that there are graphs with arbitrary large basis numbers. Moreover, Schmeichel proved that $b(K_n) \leq 3$.

Let G_1 and G_2 be two graphs. The direct product $G = G_1 \wedge G_2$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1) \text{ and } u_2v_2 \in E(G_2)\}$. The semi-strong product $G = G_1 \bullet G_2$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and

Received May 10, 2005; Revised May 10, 2007.
2000 *Mathematics Subject Classification.* 05C38, 05C75.
Key words and phrases. basis number, cycle space, fold, semi-strong product.

the edge set $E(G) = \{(u_1,u_2)(v_1,v_2)|u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$ or $u_1 = v_1$ and $u_2v_2 \in E(G_2)\}$. The cartesian product $G = G_1 \times G_2$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1,u_2)(v_1,v_2)|u_1v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_2v_2 \in E(G_2)$ and $u_1 = v_1\}$. Thus, by specializing G_1 and G_2 in the direct product by two edges, say $e = u_1v_1, e' = u_2v_2$, we have that $E(e \wedge e') = \{(u_1,u_2)(v_1,v_2),(u_1,v_2)(v_1,u_2)\}$. Also, by specializing G_1 and G_2 in the cartesian product by a vertex and an edge, say $u, e = u_1v_1$, we have that $E(u \times e) = \{(u,u_1)(u,v_1)\}$. It is clear that the semi-strong product is non-commutative.

The semi-strong product and the direct product was studied by Schmeichel [14], Ali [1] and Jaradat and Alzoubi [12]. They proved the following results.

Theorem 1.1. ([14]) *For each* $n \geq 5$, $b(K_n \bullet P_2) \leq 1 + b(K_n)$.

Theorem 1.2. ([1]) *For each* $n, m \geq 5$, $b(K_n \bullet K_m) \leq 3 + b(K_m) + b(K_n)$.

A tree T consisting of n equal order paths $\{P^{(1)}, P^{(2)}, \ldots, P^{(n)}\}$ is called an n-special star if there is a vertex, say v_1, such that v_1 is an end vertex for each path in $\{P^{(1)}, P^{(2)}, \ldots, P^{(n)}\}$ and $V(P^{(i)}) \cap V(P^{(j)}) = \{v_1\}$ for each $i \neq j$ (see [7]).

Theorem 1.3. ([12]) *Let G be a bipartite graph and C_n be a cycle. Then $b(G \bullet C_n) \leq 4 + b(G)$. Moreover, $b(G \bullet C_n) \leq 3 + b(G)$ if G has a spanning tree contains no subgraph isomorphic to a 3-special star of order 7.*

Many papers appeared to investigate the basis number of other graph products, we refer the reader to [2], [3], [8], [9], [10] and [11].

In view of the above results and since the semi-strong product is non commutative, one is naturally led to the following question:

Problem. *Can we obtain an upper bound of the basis number of the semi-strong product of cycles with bipartite graphs?*

This question will be solved in the affirmative. Moreover, we will give an example to show the upper bound is achieved. The method employed in this paper is based in part on the ideas of Ali [1], Jaradat [7] and Schmeichel [14]. Throughout this paper $f_B(e)$ stands for the number of cycles in B containing the edge e, $E(B) = \cup_{d \in B} E(d)$ where $B \subseteq \mathcal{C}(G)$ and $\mathcal{B}_G$ stands for a required basis of G.

2. Main results

In this section, we give an upper bound of the basis number of the semi-strong product of a cycle with a bipartite graph. Also, as a consequence we show that $b(C_n \bullet P_m) \leq 3$ and the equality holds under some conditions on their orders. Throughout this work $C_n = u_1u_2 \cdots u_nu_1$, $e_i = u_iu_{i+1}$ for each $1 \leq i \leq n-1$ and $e_n = u_nu_1$. Since trees have no uniform forms, we recall the following proposition which decompose trees into paths of order 3 and stars. This proposition will be used frequently in our work.

Proposition 2.1. ([7]) *For each tree T of order ≥ 3, there is a set of paths $S(T) = \left\{P_3^{(1)}, P_3^{(2)}, \ldots, P_3^{(m)}\right\}$, which called a path sequence, such that*

(i) *each $P_3^{(i)}$ is a path of length 2,*
(ii) $\bigcup_{i=1}^{m} E(P_3^{(i)}) = E(T)$,
(iii) *every edge $uv \in E(T)$ appears in at most three paths of $S(T)$,*
(iv) *each $P_3^{(j)}$ contains one edge which is not in $\bigcup_{i=1}^{j-1} P_3^{(i)}$,*
(v) *if uv appears in three paths of $S(T)$, then the paths have forms of either uva, uvd and cuv or auv, duv and uvc, where a, d, c are vertices of T,*
(vi) *every edge with an end vertex occurs in at most two paths of $S(T)$,*
(vii) $m = |V(T)| - 2 = |E(T)| - 1$.

Remark 2.2. The proof of Proposition 2.1 (see [7]) guarantees the existence of $S(T)$ which satisfies the conditions (i)-(vii) in the proposition in addition to the following condition: (viii) there exists at least two edges of T each of which occurs only in one path of $S(T)$ and incident with an end vertex. In fact, $P_3^{(|V(T)|-2)}$ contains one of those two edges.

Let $e = uv$ and e' be edges. We define $\mathcal{A}_{ee'}$ to be the cycle which consists of the edge set $E(e \wedge e') \cup E(u \times e') \cup E(v \times e')$. Let T be a tree with $S(T) = \{P_3^{(1)} = a_1b_1c_1, P_3^{(2)} = a_2b_2c_2, \ldots, P_3^{(|V(T)|-2)} = a_{|V(T)|-2}\, b_{|V(T)|-2}\, c_{|V(T)|-2}\}$ as in Proposition 2.1 and Remark 2.2. For each $j = 1, 2, \ldots, |V(T)| - 2$, we define

$$\mathcal{B}_{(uv)P_3^{(j)}} = \{(u, a_j)(u, b_j)(u, c_j)(v, b_j)(u, a_j)\},$$

and

$$\mathcal{B}_{(uv)T} = \cup_{j=1}^{|V(T)|-2} \mathcal{B}_{(uv)P_3^{(j)}}.$$

Lemma 2.3. *Let $e = uv$ and T be a tree with $e' \in E(T)$. Then $\mathcal{B}_{(uv)T}^{(e')} = \mathcal{B}_{(uv)T} \cup \mathcal{A}_{ee'}$ is linearly independent subset of $\mathcal{C}(e \bullet T)$.*

Proof. We use induction on $|S(T)|$ to show that $\mathcal{B}_{(uv)T}$ is linearly independent. If $|S(T)| = 1$, then $\mathcal{B}_{(uv)T}$ consists only of one cycle and so it is linearly independent. By induction step on $|S(T)|$ and noting that $\mathcal{B}_{(uv)P_3^{(|E(T)|-1)}}$ consists only of one cycle, we have that both of $\cup_{i=1}^{|E(T)|-2} \mathcal{B}_{(uv)P_3^{(i)}}$ and $\mathcal{B}_{(uv)P_3^{(|E(T)|-1)}}$ are linearly independent. By Remark 2.2, $P_3^{(|E(T)|-1)}$ contains an edge, say $b_{|E(T)|-1}c_{|E(T)|-1}$, which does not appear in any other path of $S(T)$. Thus, $(u, b_{|E(T)|-1})(u, c_{|E(T)|-1})$ occurs only in $\mathcal{B}_{(uv)P_3^{(|E(T)|-1)}}$. Therefore,

$$\mathcal{B}_{(uv)P_3^{(|E(T)|-1)}}$$

can not be written as a linear combination of cycles of $\cup_{i=1}^{|E(T)|-2} \mathcal{B}_{(uv)P_3^{(i)}}$. And so, $\mathcal{B}_{(uv)T}$ is linearly independent. Note that $\mathcal{A}_{ee'}$ contains the edge of $E(v \times e')$ which is not in any cycle of $\mathcal{B}_{(uv)T}$. Therefore, $\mathcal{B}_{(uv)T}^{(e')}$ is linearly independent. The proof is complete. □

Lemma 2.4. *Let C_n be a cycle and T be a tree with $e' \in E(T)$. Then $\mathcal{B}^{(e')}_{(C_n)T} = \left(\cup_{i=1}^{n-1}\mathcal{B}^{(e')}_{(u_iu_{i+1})T}\right) \cup \mathcal{B}^{(e')}_{(u_nu_1)T}$ is linearly independent subset of $\mathcal{C}(C_n \bullet T)$. Moreover, $\sum_{i=1}^{n} A_{e_ie'}$ (mod 2) is the only linear combination of cycles of $\mathcal{B}^{(e')}_{(C_n)T}$ which contains no edge of $\{u_i \times e \mid e \in E(T)\}$.*

Proof. By Lemma 2.3, each of $\mathcal{B}^{(e')}_{(u_iu_{i+1})T}$ and $\mathcal{B}^{(e')}_{(u_nu_1)T}$ is linearly independent. Since

$$E(\cup_{i=1}^{j-1}\mathcal{B}^{(e')}_{(u_iu_{i+1})T}) \cap E(\mathcal{B}^{(e')}_{(u_ju_{j+1})T}) = E(u_j \times e')$$

which is an edge, $\cup_{i=1}^{n-1}\mathcal{B}^{(e')}_{(u_iu_{i+1})T}$ is linearly independent. Similarly,

$$E(\cup_{i=1}^{n-1}\mathcal{B}^{(e')}_{(u_iu_{i+1})T}) \cap E(\mathcal{B}^{(e')}_{(u_nu_1)T}) = E(u_1 \times e') \cup E(u_n \times e')$$

which is a set of two edges. Thus, $\mathcal{B}^{(e')}_{(C_n)T}$ is linearly independent. We now show the second part. It is easy to see that $\sum_{i=1}^{n} A_{e_ie'}$ (mod 2) contains no edge of $\{u_i \times e \mid e \in E(T)\}$. Since any linear combination of cycles of $\mathcal{B}_{(u_iu_{i+1})T}$ (or $\mathcal{B}_{(u_nu_1)T}$) contains at least two edges of $E(u_i \times T)$ (or $E(u_n \times T)$) and $E(u_i \times T) \cap E(u_j \times T) = \varnothing$ for each $i \neq j$, as a result any linear combination of cycles of $\left(\cup_{i=1}^{n-1}\mathcal{B}_{(u_iu_{i+1})T}\right) \cup \mathcal{B}_{(u_nu_1)T}$ contains at least two edges of $E(u_i \times T)$ for some $1 \leq i \leq n$. Let O be any linear combination of cycles of S where S is a proper subset of $\{\mathcal{A}_{e_ie'}\}_{i=1}^{n}$. Then there is at least one cycle of $\{\mathcal{A}_{e_ie'}\}_{i=1}^{n}$ which does not belong to S, say $\mathcal{A}_{e_{i_O}e'} \notin S$ for some $1 \leq i_O \leq n$. Hence, O contains exactly one edge of $E(u_{i_O} \times T)$ and at most one edge of $(u_i \times T)$ for each $i = 1, 2, \ldots, i_O - 1, i_O + 1, \ldots, n$. Therefore, by the above arguments, any linear combination of cycles of $\mathcal{B}^{(e')}_{(C_n)T}$ other than $\sum_{i=1}^{n} A_{e_ie'}$ (mod 2) must contains at least one edge of $\{u_i \times e \mid e \in E(T)\}$. The proof is complete. □

Lemma 2.5. *For each tree T and cycle C_n,*

$$b(C_n \bullet T) \leq \begin{cases} 3, & \text{if } T \text{ is a path,} \\ 5, & \text{if } T \text{ is not a path.} \end{cases}$$

Proof. Let

$$S(T) = \{P_3^{(1)} = a_1b_1c_1, \ldots, P_3^{(|V(T)|-2)} = a_{|V(T)|-2}b_{|V(T)|-2}c_{|V(T)|-2}\}$$

as in Proposition 2.1 and Remark 2.2. By Remark 2.2, we may assume that $b_{|V(T)|-2}c_{|V(T)|-2}$ appears only in $P_3^{(|V(T)|-2)}$. Let $\mathcal{B}^{(e')}_{(C_n)T}$ be the linearly independent set as in Lemma 2.4 which is obtained by taking $e' = b_{|V(T)|-2}c_{|V(T)|-2}$. Now, for each $j = 1, 2, \ldots, |V(T)| - 2$, set

$$\begin{aligned}\mathcal{B}_{P_3^{(j)}} = \{&\mathcal{B}_{(i+1)P_3^{(j)}} \\ &= (u_{i+1}, a_j)(u_{i+2}, b_j)(u_{i+1}, c_j)(u_i, b_j)(u_{i+1}, a_j) | i = 1, 2, \ldots n-2\} \\ &\cup \{\mathcal{B}_{nP_3^{(j)}} = (u_n, a_j)(u_1, b_j)(u_n, c_j)(u_{n-1}, b_j)(u_n, a_j)\} \\ &\cup \{\mathcal{B}_{1P_3^{(j)}} = (u_1, a_j)(u_2, b_j)(u_1, c_j)(u_n, b_j)(u_1, a_j)\}.\end{aligned}$$

Let $\mathcal{B}_{S(T)} = \cup_{j=1}^{|V(T)|-2}\mathcal{B}_{P_3^{(j)}}$. Then, by Theorem 5.1 of Jaradat [7], $\mathcal{B}_{S(T)}$ is a linearly independent subset of $\mathcal{C}(C_n \wedge T)$ and so it is linearly independent subset of $\mathcal{C}(C_n \bullet T)$. Now we have two cases to consider:

Case 1. n is odd. We now show that $\mathcal{B}_{S(T)} \cup \mathcal{B}^{(e')}_{(C_n)T}$ is linearly independent. By Lemma 2.4, any linear combination of cycles of $\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'}$ contains an edge of $E(u_i \times T)$ for some $1 \le i \le n$ which is not in any cycle of $\mathcal{B}_{S(T)}$. Thus, $\mathcal{B}_{S(T)} \cup \left(\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'}\right)$ is linearly independent. Now, one can see that $\mathcal{A}_{e_n e'}$ is linearly independent of cycles of $\mathcal{B}_{S(T)} \cup \left(\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'}\right)$ if and only if F is linearly independent of cycles of $\mathcal{B}_{S(T)} \cup \left(\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'}\right)$ where

$$\begin{aligned} F &= (u_1, b_{|V(T)|-2})(u_2, c_{|V(T)|-2})(u_3, b_{|V(T)|-2}) \cdots (u_n, b_{|V(T)|-2}) \\ &\quad (u_1, c_{|V(T)|-2})(u_2, b_{|V(T)|-2}) \cdots (u_n, c_{|V(T)|-2})(u_1, b_{|V(T)|-2}). \end{aligned}$$

Therefore, to show that $\mathcal{B}_{S(T)} \cup \mathcal{B}^{(e')}_{(C_n)T}$ is linearly independent it is enough to prove that $\mathcal{B}_{S(T)} \cup \left(\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'}\right) \cup \{F\}$ is linearly independent. By Theorem 5.1 of Jaradat [7], $\mathcal{B}_{S(T)} \cup \{F\}$ is linearly independent. By Lemma 2.4, any linear combinations of cycles of $\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'}$ contains at least one edge of $E(u_i \times T)$ for some i. On the other hand no cycle of $\mathcal{B}_{S(T)} \cup F$ contains such edges. Hence $\mathcal{B}_{S(T)} \cup (\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'}) \cup \{F\}$ is linearly independent. And so $\mathcal{B}_{S(T)} \cup \mathcal{B}^{(e')}_{(C_n)T}$ is linearly independent. Let c_{j_0} be an end vertex such that $b_{j_0}c_{j_0}$ appears only in $P_3^{(j_0)}$ and $b_{j_0}c_{j_0} \neq b_{|V(T)|-2}c_{|V(T)|-2}$ (see Remark 2.2). Set

$$C^* = \{(u_1, c_{j_0})(u_1, b_{j_0})(u_2, c_{j_0})(u_2, b_{j_0}) \cdots (u_n, b_{j_0})(u_1, c_{j_0})\}.$$

Now, we prove that $\mathcal{B}(C_n \bullet T) = \{C^*\} \cup \mathcal{B}_{S(T)} \cup \mathcal{B}^{(e')}_{(C_n)T}$ is a linearly independent set. For simplicity, let $\mathcal{B}_{(u_n u_{n+1})P_3^{(k)}} = \mathcal{B}_{(u_n u_1)P_3^{(k)}}$. Assume that C^* can be written as a linear combination of cycles from $\mathcal{B}_{S(T)} \cup \mathcal{B}^{(e')}_{(C_n)T}$, say $R_1, R_2, \ldots, R_k$. Since $(u_i, c_{j_0})(u_i, b_{j_0}) \in E(C^*)$ and the only cycle contains such an edge is $\mathcal{B}_{(u_i u_{i+1})P_3^{(j_0)}}$ for each $i = 1, 2, \ldots n$, as a result $\left(\bigcup_{i=1}^{n} \mathcal{B}_{(u_i u_{i+1})P_3^{(j_0)}}\right)$ $\subseteq \{R_1, R_2, \ldots, R_k\}$. Since $\{(u_i, a_{j_0})(u_i, b_{j_0})\}_{i=1}^{n} \subseteq E\left(\oplus_{i=1}^{n}\mathcal{B}_{(u_i u_{i+1})P_3^{(j_0)}}\right)$ and $(u_i, a_{j_0})(u_i, b_{j_0}) \notin E(C^*)$ for each $i = 1, 2, \ldots, n$, and since $(u_i, a_{j_0})(u_i, b_{j_0})$ appears only in $P_3^{(j_0)}$ and in at most two other paths, say $P_3^{(1_1)}$ and/or $P_3^{(1_2)}$, as a result there is a set $A_1 = \{1_k\}_{k=1}^{s_1 \le 2}$ such that each member 1_k of A_1 associates with a set $A_1^{(1_k)} \subseteq \{1, 2, \ldots, n\}$. Furthermore, $\cup_{1_k \in A_1} A_1^{(1_k)} = \{1, 2, \ldots, n\}$ and $\left(\bigcup_{1_k \in A_1} \bigcup_{i \in A_1^{(1_k)}} \mathcal{B}_{(u_i u_{i+1})P_3^{(1_k)}}\right) \subseteq \{R_1, R_2, \ldots, R_k\}$. For each $1_k \in A_1$, let $a_{1_k}b_{1_k}$ be the edge of $P_3^{(1_k)}$ which is not $a_{j_0}b_{j_0}$. Let $B_1 = \{1_k | 1_k \in A_1$ and

$P_3^{(1_k)}$ contains an edge which appears in no other paths of $S(T)\}$. Since

$$\left\{(u_i,a_{1_k})(u_i,b_{1_k})|1_k\in A_1-B_1, i\in A_1^{(1_k)}\right\}$$
$$\subseteq E\left(\left(\oplus_{i=1}^{n}\mathcal{B}_{(u_iu_{i+1})P_3^{(j_0)}}\right)\oplus\left(\oplus_{1_k\in A_1}\oplus_{i\in A_1^{(1_k)}}\mathcal{B}_{(u_iu_{i+1})P_3^{(1_k)}}\right)\right)$$

and $(u_i,a_{1_k})(u_i,b_{1_k})\notin E(C^*)$ for each $1_k\in A_1-B_1, i\in A_1^{(1_k)}$, and since $(u_i,a_{1_k})(u_i,b_{1_k})$ appears only in $P_3^{(1_k)}$ and in at most two paths for each $1_k\in A_1-B_1$, as a result there is a set $A_2=\{2_k\}_{k=1}^{s_2\leq 4}$ such that each member 2_k of A_2 associates with a set $A_2^{(2_k)}$. Furthermore, $B_1\cup\left(\cup_{2_k\in A2}A_2^{(2_k)}\right)=\{1,2,\ldots,n\}$, $a_{1_k}b_{1_k}\in E(P_3^{(2_k)})$ and also $\left(\bigcup_{2_k\in A_2}\bigcup_{i\in A_2^{(2_k)}}\mathcal{B}_{(u_iu_{i+1})P_3^{(2_k)}}\right)\subseteq\{R_1,R_2,\ldots,R_k\}$. For each $2_k\in A_2$, let $a_{2_k}b_{2_k}$ be the edge of $P_3^{(2_k)}$ which is not $a_{1_k}b_{1_k}$. Let $B_2=\{2_k|2_k\in A_2$ and $P_3^{(2_k)}$ contains an edge which appears in no other paths of $S(T)\}$. Since

$$\left\{(u_i,a_{2_k})(u_i,b_{2_k})|2_k\in A_2-B_2, i\in A_2^{(2_k)}\right\}$$
$$\subseteq E\left(\left(\oplus_{i=1}^{n}\mathcal{B}_{(u_iu_{i+1})P_3^{(j_0)}}\right)\oplus\left(\oplus_{1_k\in A_1}\oplus_{i\in A_1^{(1_k)}}\mathcal{B}_{(u_iu_{i+1})P_3^{(1_k)}}\right)\right.$$
$$\left.\oplus\left(\oplus_{2_k\in A_2}\oplus_{i\in A_2^{(2_k)}}\mathcal{B}_{(u_iu_{i+1})P_3^{(2_k)}}\right)\right)$$

and $(u_i,a_{2_k})(u_i,b_{2_k})\notin E(C^*)$ for each $2_k\in A_2-B_2, i\in A_2^{(2_k)}$, and since $(u_i,a_{2_k})(u_i,b_{2_k})$ appears only in $P_3^{(2_k)}$ for each $2_k\in A_2-B_2$, as a result there is a set $A_3=\{3_k\}_{k=1}^{s_3\leq 8}$ such that each member 3_k of A_2 associates with a set $A_3^{(3_k)}$. Further, $B_1\cup B_2\cup\left(\cup_{3_k\in A3}A_3^{(3_k)}\right)=\{1,2,\ldots,n\}$, $a_{2_k}b_{2_k}\in E(P_3^{(3_k)})$ and also $\left(\bigcup_{3_k\in A_3}\bigcup_{i\in A_3^{(3_k)}}\mathcal{B}_{(u_iu_{i+1})P_3^{(3_k)}}\right)\subseteq\{R_1,R_2,\ldots,R_k\}$. For each $3_k\in A_3$, let $a_{3_k}b_{3_k}$ be the edge of $P_3^{(3_k)}$ which is not $a_{2_k}b_{2_k}$. Let $B_3=\{3_k|3_k\in A_3$ and $P_3^{(3_k)}$ contains edge which appears in no other paths of $S(T)\}$. By continuing in this process and since $|S(T)|$ is finite, there is an integer l and a set $A_l=\{l_k\}_{k=1}^{s_l\leq 2^l}$ such that each member l_k of A_l associates with a set $A_l^{(l_k)}$. Furthermore, $\left(\cup_{i=1}^{l-1}B_i\right)\cup\left(\cup_{l_k\in A_l}A_l^{(l_k)}\right)=\{1,2,\ldots,n\}$ and also $\left(\bigcup_{l_k\in A_l}\bigcup_{i\in A_l^{(l_k)}}\mathcal{B}_{(u_iu_{i+1})P_l^{(l_k)}}\right)\subseteq\{R_1,R_2,\ldots,R_k\}$. Moreover $B_l=\{l_k|l_k\in A_l$ and $P_3^{(l_k)}$ contains an edge which appears in no other paths of $S(T)\}=A_l$. Thus, for every $t_k\in B_t$, $P_3^{(t_k)}$ contains an edge, $a_{t_k}b_{t_k}$, which appears in no other path of $S(T)$. Therefore,

$$\cup_{t=1}^{l}\left\{(u_i,b_{t_k})(u_i,c_{t_k})|t_k\in B_t, i\in A_t^{t_k}\right\}$$
$$\subseteq E\left(\left(\oplus_{i=1}^{n}\mathcal{B}_{(u_iu_{i+1})P_3^{(j_0)}}\right)\oplus\left(\oplus_{1_k\in A_1}\oplus_{i\in A_1^{(1_k)}}\mathcal{B}_{(u_iu_{i+1})P_3^{(1_k)}}\right)\right.$$

$$\oplus \cdots \oplus \left(\oplus_{l_k \in A_l} \oplus_{i \in A_l^{(l_k)}} \mathcal{B}_{(u_i u_{i+1}) P_3^{(l_k)}}\right)\bigg) .$$

To this end, we consider the following subcases:

Subcase 1a. At least one edge of $\cup_{t=1}^{l} \{b_{t_k} c_{t_k} | t_k \in B_t\}$ differs from

$$b_{|V(T)|-2} c_{|V(T)|-2},$$

say $b_{t_{k_0}} c_{t_{k_0}}$ (Clearly $b_{t_{k_o}} c_{t_{k_o}} \neq b_{j_0} c_{j_0}$). Then C^* contains at least one edge of the form $(u_i, b_{t_{k_o}})(u_i, c_{t_{k_o}})$ for some $1 \leq i \leq n$. This is a contradiction.

Subcase 1b. All the edges of $\cup_{t=1}^{l} \{b_{t_k} c_{t_k} | t_k \in B_t\}$ have the form

$$b_{|V(T)|-2} c_{|V(T)|-2}.$$

Then either $\mathcal{A}_{e_1 e'} \in \{R_1, R_2, \ldots, R_k\}$ or $\mathcal{A}_{e_n e'} \in \{R_1, R_2, \ldots, R_k\}$. Therefore, we consider the following two subsubcases:

Subsubcase 1b1. $\mathcal{A}_{e_1 e'} \in \{R_1, R_2, \ldots, R_k\}$. Then $\mathcal{A}_{e_2 e'} \notin \{R_1, R_2, \ldots, R_k\}$ and so $\mathcal{A}_{e_3 e'} \in \{R_1, R_2, \ldots, R_k\}$. Continuing in this way, we get $\mathcal{A}_{e_n e'} \in \{R_1, R_2, \ldots, R_k\}$. Therefore, $(u_1, b_{|V(T)|-2})(u_1, c_{|V(T)|-2}) \in E(C^*)$. This is a contradiction.

Subsubcase 1b2. $\mathcal{A}_{e_n e'} \in \{R_1, R_2, \ldots, R_k\}$. Then by using the same argument as in Case 1b1 we get $(u_n, b_{|V(T)|-2})(u_n, c_{|V(T)|-2}) \in E(C^*)$. This is a contradiction.

Therefore, $\mathcal{B}(C_n \bullet T)$ is linearly independent. Since

$$\begin{aligned} |\mathcal{B}(C_n \bullet T)| &= |\mathcal{B}_{S(T)}| + |\mathcal{B}_{(C_n)T}^{(e')}| + |C^*| \\ &= n(|E(T)| - 1) + \sum_{i=1}^{n}((\sum_{j=1}^{|V(T)|-2} 1) + 1) + 1 \\ &= n(|E(T)| - 1) + n(|V(T)| - 1) + 1 \\ &= 2n|E(T)| - n + 1 = \dim \mathcal{C}(C_n \bullet T), \end{aligned}$$

$\mathcal{B}$ is a basis for $\mathcal{C}(C_n \bullet T)$. To complete the proof of the lemma, it suffices to show that $\mathcal{B}$ satisfies the required fold. Let $e \in C_n \bullet T$. Then

(1) if $e = (u_i, b_{j_0})(u_{i+1}, c_{j_0})$ or $(u_i, c_{j_0})(u_{i+1}, b_{j_0})$, then

$$f_{\mathcal{B}_{S(T)}}(e) \leq 1, f_{\mathcal{B}_{(C_n)T}^{(e')}}(e) \leq 1, \text{ and } f_{\{C^*\}}(e) \leq 1.$$

(2) If $e = (u_i, b_{|V(T)|_2})(u_{i+1}, c_{|V(T)|_2})$ or $(u_i, c_{|V(T)|_2})(u_{i+1}, b_{|V(T)|_2})$, then

$$f_{\mathcal{B}_{S(T)}}(e) \leq 1, f_{\mathcal{B}_{(C_n)T}^{(e')}}(e) \leq 2, \text{ and } f_{\{C^*\}}(e) = 0.$$

(3) If $e = (u_i, a_j)(u_{i+1}, b_j)$ or $(u_i, a_j)(u_{i-1}, b_j)$ or $e = (u_i, b_j)(u_{i+1}, c_j)$ or $(u_i, b_j)(u_{i-1}, c_j)$ and is not one of the edges as in (1) and (2), then

$$f_{\mathcal{B}_{S(T)}}(e) \leq \begin{cases} 2, & \text{if } T \text{ is a path,} \\ 3, & \text{if } T \text{ is not a path,} \end{cases}$$

$$f_{\mathcal{B}_{(C_n)T}^{(e')}}(e) \leq \begin{cases} 1, & \text{if } T \text{ is a path,} \\ 2, & \text{if } T \text{ is not a path,} \end{cases} \quad \text{and } f_{\{C^*\}}(e) = 0.$$

(4) If $e = (u_i, b_j)(u_i, c_j)$ where $j \neq j_0$ or $|V(T)| - 2$, then

$$f_{\mathcal{B}_{S(T)}}(e) = 0, f_{\mathcal{B}^{(e')}_{(C_n)T}}(e) \leq \begin{cases} 2, & \text{if } T \text{ is a path,} \\ 3, & \text{if } T \text{ is not a path,} \end{cases} \quad \text{and } f_{\{C^*\}}(e) = 0.$$

(5) If $e = (u_i, b_{j_0})(u_i, c_{j_0})$, then

$$f_{\mathcal{B}_{S(T)}}(e) = 0, f_{\mathcal{B}^{(e')}_{(C_n)T}}(e) \leq 2, \text{ and } f_{\{C^*\}}(e) \leq 1.$$

(6) If $e = (u_i, b_{|V(T)|-2})(u_i, c_{|V(T)|-2})$, then

$$f_{\mathcal{B}_{S(T)}}(e) = 0, f_{\mathcal{B}^{(e')}_{(C_n)T}}(e) \leq 2, \text{ and } f_{\{C^*\}}(e) = 0.$$

Case 2. n is even. Then choose $\mathcal{B}(C_n \bullet T) = \mathcal{B}_{S(T)} \cup (\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'}) \cup \{F_1\}$ $\cup \{F_2\}$, where $\mathcal{B}_{S(T)}$ and $(\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'})$ are as in Case 1,

$$F_1 = (u_1, c_{j_0})(u_2, b_{j_0})(u_3, c_{j_0}) \cdots (u_n, b_{j_0})(u_1, c_{j_0}).$$

and

$$F_2 = (u_1, b_{j_0})(u_2, c_{j_0})(u_3, b_{j_0}) \cdots (u_n, c_{j_0})(u_1, b_{j_0})$$

where $b_{j_0} c_{j_0}$ is as in Case 1. By Theorem 5.1 of Jaradat [7], $\mathcal{B}_{S(T)} \cup \{F_1\} \cup \{F_2\}$ is linearly independent. By Lemma 2.4, every linear combination of cycles of $\mathcal{B}^{(e')}_{(C_n)T} - \mathcal{A}_{e_n e'}$ contains an edge of $E(u_i \times T)$ for some $1 \leq i \leq n$, on the other hand no cycle of $\mathcal{B}_{S(T)} \cup \{F_1\} \cup \{F_2\}$ contains such an edge. Thus, $\mathcal{B}(C_n \bullet T)$ is linearly independent. By using the same arguments as in Case 1 and counting the cycles of $\mathcal{B}(C_n \bullet T)$, we have that $\mathcal{B}(C_n \bullet T)$ is a basis for $\mathcal{C}(C_n \bullet T)$ and satisfies the required fold. The proof is complete. □

The following proposition (See [6] and [7]) will be needed in proving the next result.

Proposition 2.6. *Let G be a bipartite graph and P_2 be a path of order 2. Then $G \wedge P_2$ consists of two components G_1 and G_2 each of which is isomorphic to G.*

Let G be a graph. Then T_G stand for a spanning tree of G such that $\Delta(T_G) = \min\{\Delta(T) \mid T \text{ is a spanning tree of } G\}$ (See [2]).

Theorem 2.7. *For any bipartite graph H and cycle C_n, we have*

$$b(C_n \bullet H) \leq b(H) + \begin{cases} 3, & \text{if } T_H \text{ is a path,} \\ 5, & \text{if } T_H \text{ is not a path.} \end{cases}$$

Proof. Let $\mathcal{B}(C_n \bullet T_H)$ be the basis of $\mathcal{C}(C_n \bullet T_H)$ as in Lemma 2.5. Let $E(C_n) = \{e_1, e_2, \ldots, e_n\}$. By Proposition 2.6, for each $i = 1, 2, \ldots, n$, $e_i \wedge H$ consists of two components each of which is isomorphic to H. Thus, we set $\mathcal{B}_{e_i} = \mathcal{B}^{(1)}_{e_i} \cup \mathcal{B}^{(2)}_{e_i}$ where $\mathcal{B}^{(1)}_{e_i}$ and $\mathcal{B}^{(2)}_{e_i}$ are the corresponding required basis of $\mathcal{B}_H$ of the two copies of H in $e_i \wedge H$. Set $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{B}_{e_i}$. Since $E(\mathcal{B}^{(1)}_{e_i}) \cap E(\mathcal{B}^{(2)}_{e_i}) = \varnothing$ for each $i = 1, 2, \ldots, n$ and $E(\mathcal{B}_{e_i}) \cap E(\mathcal{B}_{e_j}) = \varnothing$ for each $i \neq j$, we obtain that $\mathcal{T}$ is linearly independent. We now show that the cycles of $\mathcal{T}$ are linearly independent of the cycles of $\mathcal{B}(C_n \bullet T_H)$. Let $O = \sum_{i\in A\subseteq\{1,2,\ldots,n\}} \sum_{j=1}^{\alpha_i} c_{e_i}^{(j)} \pmod 2$ where $c_{e_i}^{(j)} \in \mathcal{B}_{e_i}$. By Proposition 2.6, $e_i \wedge T_H$ is a forest for each i. Thus, the ring sum $c_{e_i}^{(1)} \oplus c_{e_i}^{(2)} \oplus \cdots \oplus c_{e_i}^{(\alpha_i)}$ contains at least one edge of $E(e_i \wedge (H - T_H))$ where $H - T_H$ is the complement of T_H in H. Since $E(e_i \wedge H) \cap E(e_j \wedge H) = \varnothing$ for each $i \neq j$, the ring sum $O = \oplus_{i\in A\subseteq\{1,2,\ldots,n\}} \oplus_{j=1}^{\alpha_i} c_{e_j}^{(j)}$ contains at least one edge of $E(C_n \wedge (H - E(T)))$. On the other hand, no cycle of $\mathcal{B}(C_n \bullet T_H)$ contains such kind of edges. Thus, $\mathcal{B}(C_n \bullet T_H) \cup \mathcal{T}$ is linearly independent. Now, for each $i = 1, 2, \ldots, n$, let $\mathcal{B}_{u_i}$ be the corresponding required basis of $\mathcal{B}_H$ in $u_i \times H$. Let $\mathcal{V} = \bigcup_{i=1}^{n} \mathcal{B}_{u_i}$, and $\mathcal{B}(C_n \bullet H) = \mathcal{B}(C_n \bullet T_H) \bigcup \mathcal{T} \bigcup \mathcal{V}$. Since $E(u_i \times H) \cap E(u_j \times H) = \varnothing$ whenever $i \neq j$, we conclude that $\mathcal{V}$ is linearly independent. Note that each linear combination of cycles of $\mathcal{V}$ contains at least one edge of $E(u_i \times (H - T_H))$ for some $1 \le i \le n$ where $H - T_H$ is the complement of T_H in H, on the other hand no cycle of $\mathcal{B}(C_n \bullet T_H) \cup \mathcal{T}$ contains such an edge. Therefore, $\mathcal{B}(C_n \bullet H)$ is linearly independent. To this end, we have

$$\begin{aligned}
|\mathcal{B}(C_n \bullet H)| &= |\mathcal{B}(C_n \bullet T)| + |\mathcal{T}| + |\mathcal{V}| \\
&= 2n|E(T_H)| - n + 1 + \sum_{i=1}^{n} |\mathcal{B}_{e_i}| + \sum_{i=1}^{n} |\mathcal{B}_{u_i}| \\
&= 2n|E(T_H)| - n + 1 + 2n\dim \mathcal{C}(H) + n\dim \mathcal{C}(H) \\
&= 2n(|E(T_H)| + \dim \mathcal{C}(H)) - n + 1 + n(|E(H)| - |V(H)| + 1) \\
&= 2n|E(H)| + n|E(H)| - n|V(H)| + 1 = \dim \mathcal{C}(C_n \bullet H).
\end{aligned}$$

Thus, $\mathcal{B}(C_n \bullet H)$ is a basis for $\mathcal{C}(C_n \bullet H)$. Now, one can easily see that $\mathcal{B}(C_n \bullet H)$ satisfy the required fold. The proof is complete. □

The following result gives an example where the above upper bound is achieved.

Corollary 2.8. *$b(C_n \bullet P_m) = 3$ if one of the following holds:*
(i) *n is even and $m \ge 3$;*
(ii) *n is odd and $m \ge 2$.*

Proof. To prove the corollary it suffices to show that $C_n \bullet P_m$ is non-planar. Let $P_m = v_1 v_2 \cdots v_m$. To prove (i), consider the subgraph H_1 whose vertex set $\{(u_1, v_1), (u_2, v_1), (u_4, v_1), (u_5, v_1), \ldots, (u_n, v_1), (u_1, v_2), (u_2, v_2), (u_3, v_2), \ldots, (u_n, v_2), (u_2, v_3)\}$ and whose edge set consists of the following nine paths: $P_1 = (u_1, v_1)(u_1, v_2)$, $P_2 = (u_1, v_1)(u_2, v_2)$, $P_3 = (u_1, v_2)(u_2, v_1)$, $P_4 = (u_2, v_1)(u_2, v_2)$, $P_5 = (u_1, v_2)(u_2, v_3)$, $P_6 = (u_2, v_2)(u_2, v_3)$, $P_7 = (u_3, v_2)(u_2, v_3)$, $P_8 = (u_2, v_1)(u_3, v_2)$, $P_9 = (u_1, v_1)(u_n, v_2)(u_n, v_1)(u_{n-1}, v_2) \cdots (u_4, v_1)(u_3, v_2)$. Then H_1 is homeomorphic to $K_{3,3}$ and so $C_n \bullet P_m$ is non planar. To prove (ii), consider the subgraph H_2 whose vertex set $\{(u_1, v_1), (u_2, v_1), (u_3, v_1), \ldots, (u_n, v_1), (u_1, v_2), (u_2, v_2), (u_3, v_2), \ldots, (u_n, v_2)\}$ and whose edge set consists of the following nine paths: $P_1 = (u_1, v_1)(u_1, v_2)$, $P_2 = (u_1, v_1)(u_2, v_2)$, $P_3 = (u_1, v_2)(u_2, v_1)$, $P_4 = (u_n, v_1)(u_n, v_2)$, $P_5 = (u_1, v_2)(u_n, v_1)$, $P_6 = (u_1, v_1)(u_n, v_2)$, $P_7 = (u_2, v_1)(u_2, v_2)$, $P_8 = (u_2, v_1)(u_3, v_2) \cdots (u_{n-1}, v_1)(u_n, v_2)$, $P_9 = (u_2, v_2)(u_3, v_1) \cdots (u_{n-1}, v_2)(u_n, v_1)$. Then H_2 is homeomorphic to $K_{3,3}$ and so $C_n \bullet P_m$ is non planar. The proof is complete. □

Acknowledgment. The author wish to thank Prof. C. Y. Chao and the referee for their valuable comments.

References

[1] A. A. Ali, *The basis number of complete multipartite graphs*, Ars Combin. **28** (1989), 41–49.
[2] A. A. Ali and G. T. Marougi, *The basis number of the Cartesian product of some graphs*, J. Indian Math. Soc. (N.S.) **58** (1992), no. 1-4, 123–134.
[3] S. Y. Alsardary and J. Wojciechowski, *The basis number of the powers of the complete graph*, Discrete Math. **188** (1998), no. 1-3, 13–25.
[4] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, America Elsevier Publishing Co., Inc., New York, 1976.
[5] W.-K. Chen, *On vector spaces associated with a graph*, SIAM J. Appl. Math. **20** (1971), 525–529.
[6] W. Imrich and S. Klavžar, *Product Graphs*, Structure and recognition. With a foreword by Peter Winkler. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
[7] M. M. M. Jaradat, *On the basis number of the direct product of graphs*, Australas. J. Combin. **27** (2003), 293–306.
[8] ———, *The basis number of the strong product of trees and cycles with some graphs*, J. Combin. Math. Combin. Comput. **58** (2006), 195–209.
[9] ———, *An upper bound of the basis number of the strong product of graphs*, Discuss. Math. Graph Theory **25** (2005), no. 3, 391–406.
[10] ———, *The basis number of the direct product of a theta graph and a path*, Ars Combin. **75** (2005), 105–111.
[11] M. M. M. Jaradat and M. Y. Alzoubi, *An upper bound of the basis number of the lexicographic product of graphs*, Australas. J. Combin. **32** (2005), 305–312.
[12] ———, *On the basis number of the semi-strong product of bipartite graphs with cycles*, Kyungpook Math. J. **45** (2005), no. 1, 45–53.
[13] S. MacLane, *A combinatorial condition for planar graphs*, Fundamenta Math. **28** (1937), 22–32.
[14] E. F. Schmeichel, *The basis number of a graph*, J. Combin. Theory Ser. B **30** (1981), no. 2, 123–129.

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