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Ordered S_p -metric spaces and some fixed point theorems for contractive mappings with application to periodic boundary value problems

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Abstract

In this paper, we introduce the structure of S_p -metric spaces as a generalization of both S -metric and S_b -metric spaces. Also, we present the notions of \tilde{S} -contractive mappings in the setup of ordered S_p -metric spaces and investigate the existence of a fixed point for such mappings under various contractive conditions. We provide examples to illustrate the results presented herein. An application to periodic boundary value problems is presented.

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1 Introduction and preliminaries

There is a large number of generalizations of Banach contraction principle via using different forms of contractive conditions in various generalized metric spaces. Some of such generalizations are obtained via contractive conditions expressed by rational terms (see, e.g., [7, 9, 17, 19]).

Ran and Reurings initiated the studying of fixed point results on partially ordered sets in [14]. Further, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order. For more details, we refer the reader to [11, 12].

Parvaneh introduced in [13] the concept of extended b -metric spaces as follows.

Definition 1.1 ([13]) Let \mathcal{E} be a (nonempty) set. A function $\tilde{d} : \mathcal{E} \times \mathcal{E} \rightarrow \mathbf{R}^+$ is a p -metric if there exists a strictly increasing continuous function $\Omega : [0, \infty) \rightarrow [0, \infty)$ with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ and $\Omega^{-1}(0) = 0 = \Omega(0)$ such that, for all $\zeta, \eta, \omega \in \mathcal{E}$, the following conditions hold:

- (\tilde{d}_1) $\tilde{d}(\zeta, \eta) = 0$ iff $\zeta = \eta$,
- (\tilde{d}_2) $\tilde{d}(\zeta, \eta) = \tilde{d}(\eta, \zeta)$,
- (\tilde{d}_3) $\tilde{d}(\zeta, \omega) \leq \Omega(\tilde{d}(\zeta, \eta) + \tilde{d}(\eta, \omega))$.

In this case, the pair (\mathcal{E}, \tilde{d}) is called a p -metric space or an extended b -metric space.

A b -metric [2] is a p -metric with $\Omega(t) = st$ for some fixed $s \geq 1$, while a metric is a p -metric when $\Omega(t) = t$. We have the following proposition.

Proposition 1.2 ([13]) *Let (\mathcal{E}, d) be a metric space, and let $\tilde{d}(\zeta, \eta) = \xi(d(\zeta, \eta))$, where $\xi : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing continuous function with $t \leq \xi(t)$ and $0 = \xi(0)$. In this case, \tilde{d} is a p -metric with $\Omega(t) = \xi(t)$.*

The above proposition provides several examples of p -metric spaces.

Example 1.3 Let (\mathcal{E}, d) be a metric space, and let $\tilde{d}(\zeta, \eta) = e^{d(\zeta, \eta)} \sec^{-1}(e^{d(\zeta, \eta)})$. Then \tilde{d} is a p -metric with $\Omega(t) = e^t \sec^{-1}(e^t)$.

In [16], Sedghi et al. introduced the notion of an S -metric space as follows.

Definition 1.4 ([16]) Let \mathcal{E} be a nonempty set and $S : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \rightarrow \mathbf{R}^+$ be a function satisfying the following properties:

- (S1) $S(\zeta, \eta, \omega) = 0$ iff $\zeta = \eta = \omega$;
- (S2) $S(\zeta, \eta, \omega) \leq S(\zeta, \zeta, a) + S(\eta, \eta, a) + S(\omega, \omega, a)$ for all $\zeta, \eta, \omega, a \in \mathcal{E}$ (rectangle inequality).

Then the function S is called an S -metric on \mathcal{E} and the pair (\mathcal{E}, S) is called an S -metric space.

Example 1.5 ([16]) Let \mathbf{R} be the real line. Then $S(\zeta, \eta, \omega) = |\zeta - \eta| + |\zeta - \omega|$ for all $\zeta, \eta, \omega \in \mathbf{R}$ is an S -metric on \mathbf{R} . This S -metric on \mathbf{R} is called the usual S -metric on \mathbf{R} .

Souayahya and Mlaiki in [18], motivated by the concepts of b -metric and S -metric, introduced the concept of S_b -metric spaces, and then they presented some basic properties of such spaces.

The following is the definition of modified S -metric spaces, which is a proper generalization of the notions of S -metric spaces and S_b -metric spaces.

Definition 1.6 Let \mathcal{E} be a nonempty set and $\Omega : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing continuous function such that $t \leq \Omega(t)$ for all $t > 0$ and $\Omega(0) = 0$. Suppose that a mapping $\tilde{S} : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \rightarrow \mathbf{R}^+$ satisfies:

- (\tilde{S} 1) $\tilde{S}(\zeta, \eta, \omega) = 0$ iff $\zeta = \eta = \omega$,
- (\tilde{S} 2) $\tilde{S}(\zeta, \eta, \omega) \leq \Omega[\tilde{S}(\zeta, \zeta, \alpha) + \tilde{S}(\eta, \eta, \alpha) + \tilde{S}(\omega, \omega, \alpha)]$ for all $\zeta, \eta, \omega, \alpha \in \mathcal{E}$ (rectangle inequality).

Then \tilde{S} is called an S_p -metric and the pair (\mathcal{E}, \tilde{S}) is called an S_p -metric space.

Remark 1.7 In an S_p -metric space, we have $\tilde{S}(\zeta, \zeta, \eta) \leq \Omega[\tilde{S}(\eta, \eta, \zeta)]$ for all $\zeta, \eta \in \mathcal{E}$. Indeed, putting $(\zeta, \zeta, \eta, \zeta)$ instead of $(\zeta, \eta, \omega, \alpha)$ in (\tilde{S} 2) and using (\tilde{S} 1), we obtain the previous inequality.

Each S -metric space is an S_p -metric space with $\Omega(t) = t$ and every S_b -metric space with parameter $s \geq 1$ is an S_p -metric space with $\Omega(t) = st$.

Proposition 1.8 *Let (\mathcal{E}, S) be an S_b -metric space with coefficient $s \geq 1$, and let $\tilde{S}(\zeta, \eta, \omega) = \xi(S(\zeta, \eta, \omega))$, where $\xi : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing continuous function with $t \leq \xi(t)$ for all $t > 0$ and $\xi(0) = 0$. Then \tilde{S} is an S_p -metric with $\Omega(t) = \xi(st)$.*

Proof For all $\zeta, \eta, \omega, a \in \mathcal{E}$,

$$\begin{aligned} \tilde{S}(\zeta, \eta, \omega) &= \xi(S(\zeta, \eta, \omega)) \leq \xi(sS(\zeta, \zeta, a) + sS(\eta, \eta, a) + sS(\omega, \omega, a)) \\ &\leq \xi(s\xi(S(\zeta, \zeta, a)) + s\xi(S(\eta, \eta, a)) + s\xi(S(\omega, \omega, a))) \\ &= \xi(s\tilde{S}(\zeta, \zeta, a) + s\tilde{S}(\eta, \eta, a) + s\tilde{S}(\omega, \omega, a)) \\ &= \Omega(\tilde{S}(\zeta, \zeta, a) + \tilde{S}(\eta, \eta, a) + \tilde{S}(\omega, \omega, a)). \end{aligned}$$

So, \tilde{S} is an S_p -metric with $\Omega(t) = \xi(st)$. □

The above proposition provides several examples of S_p -metric spaces.

Example 1.9 Let (\mathcal{E}, S) be an S_b -metric space with coefficient $s \geq 1$. Then:

1. $\tilde{S}(\zeta, \eta, \omega) = e^{S(\zeta, \eta, \omega)} \sec^{-1}(e^{S(\zeta, \eta, \omega)})$ is an S_p -metric with $\Omega(t) = e^{st} \sec^{-1}(e^{st})$.
2. $\tilde{S}(\zeta, \eta, \omega) = [S(\zeta, \eta, \omega) + 1] \sec^{-1}([S(\zeta, \eta, \omega) + 1])$ is an S_p -metric with $\Omega(t) = [st + 1] \sec^{-1}([st + 1])$.
3. $\tilde{S}(\zeta, \eta, \omega) = e^{S(\zeta, \eta, \omega)} \tan^{-1}(e^{S(\zeta, \eta, \omega)} - 1)$ is an S_p -metric with $\Omega(t) = e^{st} \tan^{-1}(e^{st} - 1)$.
4. $\tilde{S}(\zeta, \eta, \omega) = S(\zeta, \eta, \omega) \cosh(S(\zeta, \eta, \omega))$ is an S_p -metric with $\Omega(t) = st \cosh(st)$.
5. $\tilde{S}(\zeta, \eta, \omega) = e^{S(\zeta, \eta, \omega)} \ln(1 + S(\zeta, \eta, \omega))$ is an S_p -metric with $\Omega(t) = e^{st} \ln(1 + st)$.
6. $\tilde{S}(\zeta, \eta, \omega) = S(\zeta, \eta, \omega) + \ln(1 + S(\zeta, \eta, \omega))$ is an S_p -metric with $\Omega(t) = st + \ln(1 + st)$.

In all the given examples 1–6, it can be checked by routine calculation that the respective function ξ satisfies all the requirements given in Proposition 1.8, i.e., it is continuous, strictly increasing, $\xi(0) = 0$, and $\xi(t) > t$ for $t > 0$.

Definition 1.10 Let \mathcal{E} be an S_p -metric space. A sequence $\{\zeta_n\}$ in \mathcal{E} is said to be:

- (1) S_p -Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geq n_0$, $\tilde{S}(\zeta_m, \zeta_n, \zeta_n) < \varepsilon$.
- (2) S_p -convergent to a point $\zeta \in \mathcal{E}$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$, $\tilde{S}(\zeta_n, \zeta_n, \zeta) < \varepsilon$.
- (3) An S_p -metric space \mathcal{E} is called S_p -complete if every S_p -Cauchy sequence is S_p -convergent in \mathcal{E} .

In general, an S_p -metric mapping $\tilde{S}(\zeta, \eta, \omega)$ with a nontrivial function Ω need not be jointly continuous in all its variables (see [10]). Thus, in some proofs we will need the following simple lemma about the S_p -convergent sequences.

Lemma 1.11 Let (\mathcal{E}, \tilde{S}) be an S_p -metric space.

1. Suppose that $\{\zeta_n\}$ and $\{\eta_n\}$ are S_p -convergent to ζ and η , respectively. Then we have

$$\begin{aligned} \frac{\Omega^{-1}[\frac{1}{2}\Omega^{-1}[\tilde{S}(\zeta, \eta, \eta)]]}{2} &\leq \liminf_{n \rightarrow \infty} \tilde{S}(\zeta_n, \eta_n, \eta_n) \leq \limsup_{n \rightarrow \infty} \tilde{S}(\zeta_n, \eta_n, \eta_n) \\ &\leq \Omega[2\Omega[\tilde{S}(\zeta, \eta, \eta)]]. \end{aligned}$$

In particular, if $\zeta = \eta$, then we have $\lim_{n \rightarrow \infty} \tilde{S}(\zeta_n, \eta_n, \eta_n) = 0$.

2. Suppose that $\{\zeta_n\}$ is S_p -convergent to ζ and $\omega \in \mathcal{E}$ is arbitrary. Then we have

$$\frac{\Omega^{-1}[\tilde{S}(\zeta, \omega, \omega)]}{2} \leq \liminf_{n \rightarrow \infty} \tilde{S}(\zeta_n, \omega, \omega) \leq \limsup_{n \rightarrow \infty} \tilde{S}(\zeta_n, \omega, \omega) \leq \Omega[2\tilde{S}(\zeta, \omega, \omega)].$$

Proof 1. Using the rectangle inequality in the S_p -metric space, it is easy to see that

$$\begin{aligned} \tilde{S}(\zeta, \eta, \eta) &\leq \Omega[\tilde{S}(\zeta, \zeta, \zeta_n) + 2\tilde{S}(\eta, \eta, \zeta_n)] \\ &\leq \Omega[\tilde{S}(\zeta, \zeta, \zeta_n) + 2\Omega[2\tilde{S}(\eta, \eta, \eta_n) + \tilde{S}(\eta_n, \zeta_n, \zeta_n)]] \end{aligned}$$

and

$$\begin{aligned} \tilde{S}(\zeta_n, \eta_n, \eta_n) &\leq \Omega[\tilde{S}(\zeta_n, \zeta_n, \zeta) + 2\tilde{S}(\eta_n, \eta_n, \zeta)] \\ &\leq \Omega[\tilde{S}(\zeta_n, \zeta_n, \zeta) + 2\Omega[2\tilde{S}(\eta_n, \eta_n, \eta) + \tilde{S}(\eta, \zeta, \zeta)]] \end{aligned}$$

Taking the lower limit as $n \rightarrow \infty$ in the first inequality and the upper limit as $n \rightarrow \infty$ in the second inequality, we obtain the desired result.

2. Using the rectangle inequality, we see that

$$\tilde{S}(\zeta, \omega, \omega) \leq \Omega[\tilde{S}(\zeta, \zeta, \zeta_n) + 2\tilde{S}(\omega, \omega, \zeta_n)]$$

and

$$\tilde{S}(\zeta_n, \omega, \omega) \leq \Omega[\tilde{S}(\zeta_n, \zeta_n, \zeta) + 2\tilde{S}(\omega, \omega, \zeta)]. \quad \square$$

Let \mathfrak{B} denote the class of all real functions $\beta : [0, \infty) \rightarrow [0, 1)$ satisfying the condition

$$\beta(t_n) \rightarrow 1 \text{ implies that } t_n \rightarrow 0, \text{ as } n \rightarrow \infty.$$

In order to generalize the Banach contraction principle, Geraghty proved in 1973 the following result.

Theorem 1.12 ([5]) *Let (\mathfrak{E}, d) be a complete metric space, and let $f : \mathfrak{E} \rightarrow \mathfrak{E}$ be a self-map. Suppose that there exists $\beta \in \mathfrak{B}$ such that*

$$d(f\zeta, f\eta) \leq \beta(d(\zeta, \eta))d(\zeta, \eta)$$

holds for all $\zeta, \eta \in \mathfrak{E}$. Then f has a unique fixed point $\omega \in \mathfrak{E}$ and for each $\zeta \in \mathfrak{E}$ the Picard sequence $\{f^n \zeta\}$ converges to ω .

In 2010, Amini-Harandi and Emami [1] characterized the result of Geraghty in the setting of a partially ordered complete metric space. In [4], some fixed point theorems for mappings satisfying Geraghty-type contractive conditions were proved in various generalized metric spaces. Also, Zabihi and Razani [19] and Shahkoobi and Razani [17] obtained some fixed point results for rational Geraghty contractions in b -metric spaces.

Motivated by [9], in this paper we present some fixed point theorems for various contractive mappings in tripled partially ordered modified S -metric spaces. Our results extend some existing results in the literature. Examples are provided to show the usefulness of the results. In the last section, an application is given to a first-order boundary value problem for differential equations.

2 Main results

2.1 Fixed point results using Geraghty contractions

Let (\mathcal{E}, \tilde{S}) be an S_p -metric space with function Ω , and let \mathfrak{B}_Ω denote the class of all functions $\beta : [0, \infty) \rightarrow [0, \Omega^{-1}(1))$ satisfying the following condition:

$$\limsup_{n \rightarrow \infty} \beta(t_n) = \Omega^{-1}(1) \text{ implies that } t_n \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Example 2.1

- (1) Let $\mathcal{E} = \mathbb{R}$ and $\tilde{S}(\zeta, \eta, \omega) = e^{|\zeta - \eta| + |\eta - \omega|} - 1$ for all $\zeta, \eta, \omega \in \mathbb{R}$, with $\Omega(t) = e^t - 1$. Then, by $\beta(t) = (\ln 2)e^{-t}$ for $t > 0$ and $\beta(0) \in [0, \ln 2)$, a function β belonging to \mathfrak{B}_Ω is given.
- (2) Another example of a function in \mathfrak{B}_Ω may be given by $\beta(t) = W(1)e^{-t}$ for $t > 0$ and $\beta(0) \in [0, W(1))$, where $\tilde{S}(\zeta, \eta, \omega) = (|\zeta - \eta| + |\eta - \omega|)e^{|\zeta - \eta| + |\eta - \omega|}$ for all $\zeta, \eta, \omega \in \mathbb{R}$. Here, W is the Lambert W -function (see, e.g., [3]).

Definition 2.2 Let $(\mathcal{E}, \leq, \tilde{S})$ be an ordered S_p -metric space. A mapping $f : \mathcal{E} \rightarrow \mathcal{E}$ is called an S_p -Geraghty contraction if there exists $\beta \in \mathfrak{B}_\Omega$ such that

$$\Omega^2(2\tilde{S}(f\zeta, f\eta, f\omega)) \leq \beta(M(\zeta, \eta, \omega))M(\zeta, \eta, \omega) \tag{2.1}$$

for all mutually comparable elements $\zeta, \eta, \omega \in \mathcal{E}$, where

$$M(\zeta, \eta, \omega) = \max \left\{ \tilde{S}(\zeta, \eta, \omega), \frac{\Omega^{-1}[\tilde{S}(\zeta, f\zeta, f\eta)]}{2}, \tilde{S}(\eta, f\eta, f\omega) \right\}.$$

An ordered S_p -metric space $(\mathcal{E}, \leq, \tilde{S})$ is said to have the s.l.c. property if, whenever $\{\zeta_n\}$ is an increasing sequence in \mathcal{E} such that $\zeta_n \rightarrow u \in \mathcal{E}$, one has $\zeta_n \leq u$ for all $n \in \mathbb{N}$.

Theorem 2.3 Let $(\mathcal{E}, \leq, \tilde{S})$ be an ordered S_p -complete S_p -metric space. Let $f : \mathcal{E} \rightarrow \mathcal{E}$ be an increasing mapping with respect to \leq such that there exists an element $\zeta_0 \in \mathcal{E}$ with $\zeta_0 \leq f\zeta_0$. Suppose that f is an S_p -Geraghty contraction. If

- (I) f is continuous, or
- (II) $(\mathcal{E}, \leq, \tilde{S})$ has the s.l.c. property,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Put $\zeta_n = f^n \zeta_0$. Since $\zeta_0 \leq \zeta_1$ and f is increasing, we obtain by induction that the sequence $\{\zeta_n\}$ is increasing w.r.t. \leq . We will show that $\lim_{n \rightarrow \infty} \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0$. Without loss of generality, we may assume that $\zeta_n \neq \zeta_{n+1}$ for all $n \in \mathbb{N}$. Since ζ_n and ζ_{n+1} are comparable, then by (2.1) we have

$$\tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = \tilde{S}(f\zeta_{n-1}, f\zeta_n, f\zeta_n) \leq \beta(M(\zeta_{n-1}, \zeta_n, \zeta_n))M(\zeta_{n-1}, \zeta_n, \zeta_n), \tag{2.2}$$

where

$$\begin{aligned} &M(\zeta_{n-1}, \zeta_n, \zeta_n) \\ &= \max \left\{ \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n), \frac{1}{2} \Omega^{-1}[\tilde{S}(\zeta_{n-1}, f\zeta_{n-1}, f\zeta_n)], \tilde{S}(\zeta_n, f\zeta_n, f\zeta_n) \right\} \end{aligned}$$

$$\begin{aligned}
 &= \max \left\{ \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n), \frac{1}{2} \Omega^{-1} [\tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_{n+1})], \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \right\} \\
 &\leq \max \left\{ \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n), \frac{\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + \tilde{S}(\zeta_{n+1}, \zeta_{n+1}, \zeta_n)}{2}, \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \right\} \\
 &= \max \{ \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n), \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \}.
 \end{aligned}$$

If $\max \{ \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n), \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \} = \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})$, then from (2.2) we have

$$\begin{aligned}
 \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) &\leq \beta(M(\zeta_{n-1}, \zeta_n, \zeta_n)) \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \\
 &< \Omega^{-1}(1) \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \\
 &\leq \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}),
 \end{aligned}$$

which is a contradiction.

Hence, $\max \{ \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n), \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \} = \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n)$. So, from (2.2),

$$\tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leq \beta(M(\zeta_{n-1}, \zeta_n, \zeta_n)) \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n) < \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n). \tag{2.3}$$

That is, $\{ \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \}$ is a decreasing sequence. Then there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = r$. We will prove that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, letting $n \rightarrow \infty$, from (2.3) we have

$$r \leq \lim_{n \rightarrow \infty} \beta(M(\zeta_{n-1}, \zeta_n, \zeta_n)) r \leq \Omega^{-1}(1)r,$$

which implies that $\Omega^{-1}(1) \leq 1 \leq \lim_{n \rightarrow \infty} \beta(M(\zeta_{n-1}, \zeta_n, \zeta_n)) \leq \Omega^{-1}(1)$. Now, as $\beta \in \mathfrak{B}_\Omega$, we conclude that $M(\zeta_{n-1}, \zeta_n, \zeta_n) \rightarrow 0$, which yields that $r = 0$, a contradiction. Hence, the assumption that $r > 0$ is false. That is,

$$\lim_{n \rightarrow \infty} \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = \lim_{n \rightarrow \infty} \tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1}) = 0. \tag{2.4}$$

Now, we prove that the sequence $\{\zeta_n\}$ is an S_p -Cauchy sequence. Suppose the contrary, *i.e.*, there exists $\varepsilon > 0$ for which we can find two subsequences $\{\zeta_{m_i}\}$ and $\{\zeta_{n_i}\}$ of $\{\zeta_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad \tilde{S}(\zeta_{m_i}, \zeta_{n_i}, \zeta_{n_i}) \geq \varepsilon. \tag{2.5}$$

This means that

$$\tilde{S}(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}) < \varepsilon. \tag{2.6}$$

From the rectangular inequality, we get

$$\begin{aligned}
 &\tilde{S}(\zeta_{m_i}, \zeta_{m_i+1}, \zeta_{n_i}) \\
 &\leq \Omega [\tilde{S}(\zeta_{m_i}, \zeta_{m_i}, \zeta_{n_i-1}) + \tilde{S}(\zeta_{m_i+1}, \zeta_{m_i+1}, \zeta_{n_i-1}) + \tilde{S}(\zeta_{n_i}, \zeta_{n_i}, \zeta_{n_i-1})] \\
 &\leq \Omega [\tilde{S}(\zeta_{m_i}, \zeta_{m_i}, \zeta_{n_i-1}) + \Omega [2\tilde{S}(\zeta_{m_i+1}, \zeta_{m_i+1}, \zeta_{m_i}) + \tilde{S}(\zeta_{n_i-1}, \zeta_{n_i-1}, \zeta_{m_i})] \\
 &\quad + \tilde{S}(\zeta_{n_i}, \zeta_{n_i}, \zeta_{n_i-1})].
 \end{aligned}$$

Taking the upper limit as $i \rightarrow \infty$ and by (2.4), we get

$$\limsup_{i \rightarrow \infty} \tilde{S}(\zeta_{m_i}, \zeta_{m_i+1}, \zeta_{n_i}) \leq \Omega(\varepsilon + \Omega(\varepsilon)).$$

From the rectangular inequality, we get

$$\varepsilon \leq \tilde{S}(\zeta_{m_i}, \zeta_{n_i}, \zeta_{n_i}) \leq \Omega[\tilde{S}(\zeta_{m_i}, \zeta_{m_i}, \zeta_{m_i+1}) + \tilde{S}(\zeta_{n_i}, \zeta_{n_i}, \zeta_{m_i+1}) + \tilde{S}(\zeta_{n_i}, \zeta_{n_i}, \zeta_{m_i+1})].$$

Taking the upper limit as $i \rightarrow \infty$ and by (2.4), we get

$$\frac{1}{2}\Omega^{-1}(\varepsilon) \leq \limsup_{i \rightarrow \infty} \tilde{S}(\zeta_{m_i+1}, \zeta_{n_i}, \zeta_{n_i}).$$

From the definition of $M(\zeta, \eta, \omega)$ and the above limits,

$$\begin{aligned} & \limsup_{i \rightarrow \infty} M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}) \\ &= \limsup_{i \rightarrow \infty} \max \left\{ \tilde{S}(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}), \frac{1}{2}\Omega^{-1}[\tilde{S}(\zeta_{m_i}, f\zeta_{m_i}, f\zeta_{n_i-1})], \right. \\ & \quad \left. \tilde{S}(\zeta_{n_i-1}, f\zeta_{n_i-1}, f\zeta_{n_i-1}) \right\} \\ &= \limsup_{i \rightarrow \infty} \max \left\{ \tilde{S}(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}), \frac{1}{2}\Omega^{-1}[\tilde{S}(\zeta_{m_i}, \zeta_{m_i+1}, \zeta_{n_i})], \right. \\ & \quad \left. \tilde{S}(\zeta_{n_i-1}, \zeta_{n_i}, \zeta_{n_i}) \right\} \\ &\leq \Omega(\varepsilon). \end{aligned}$$

Now, since the sequence $\{\zeta_n\}$ is increasing, we can apply (2.1) and the above inequalities to get

$$\begin{aligned} \Omega(\varepsilon) &= \Omega^2\left(2 \cdot \frac{1}{2}\Omega^{-1}(\varepsilon)\right) \\ &\leq \Omega^2\left[\limsup_{i \rightarrow \infty} 2\tilde{S}(\zeta_{m_i+1}, \zeta_{n_i}, \zeta_{n_i})\right] \\ &\leq \limsup_{i \rightarrow \infty} \beta(M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1})) \limsup_{i \rightarrow \infty} M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}) \\ &\leq \Omega(\varepsilon) \cdot \limsup_{i \rightarrow \infty} \beta(M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1})), \end{aligned}$$

which implies that $\Omega^{-1}(1) \leq 1 \leq \lim_{n \rightarrow \infty} \beta(M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1})) \leq \Omega^{-1}(1)$. Now, as $\beta \in \mathfrak{B}_\Omega$ we conclude that $M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}) \rightarrow 0$, which yields that $\tilde{S}(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}) \rightarrow 0$. Consequently,

$$\tilde{S}(\zeta_{m_i}, \zeta_{n_i}, \zeta_{n_i}) \leq \Omega[\tilde{S}(\zeta_{m_i}, \zeta_{m_i}, \zeta_{n_i-1}) + 2\tilde{S}(\zeta_{n_i}, \zeta_{n_i}, \zeta_{n_i-1})] \rightarrow 0,$$

a contradiction with (2.5). Therefore, $\{\zeta_n\}$ is an S_p -Cauchy sequence. S_p -completeness of \mathcal{E} yields that $\{\zeta_n\}$ S_p -converges to a point $u \in \mathcal{E}$.

We will prove that u is a fixed point of f . First, let f be continuous. Then we have

$$u = \lim_{n \rightarrow \infty} \zeta_{n+1} = \lim_{n \rightarrow \infty} f \zeta_n = fu.$$

Now, let (II) hold. Using the assumption on \mathcal{E} , we have that $\zeta_n \preceq u$ for all $n \in \mathbb{N}$. Now, by Lemma 1.11,

$$\begin{aligned} \Omega^2 \left[2 \frac{\Omega^{-1}[\tilde{S}(u, fu, fu)]}{2} \right] &\leq \Omega^2 \left[2 \limsup_{n \rightarrow \infty} \tilde{S}(\zeta_{n+1}, fu, fu) \right] \\ &\leq \limsup_{n \rightarrow \infty} \beta(M(\zeta_n, u, u)) \limsup_{n \rightarrow \infty} M(\zeta_n, u, u), \end{aligned}$$

where

$$\lim_{n \rightarrow \infty} M(\zeta_n, u, u) = \lim_{n \rightarrow \infty} \max \{ \tilde{S}(\zeta_n, u, u), \tilde{S}(\zeta_n, \zeta_{n+1}, fu), \tilde{S}(u, fu, fu) \} = \tilde{S}(u, fu, fu).$$

Therefore, we deduce that $\tilde{S}(u, fu, fu) = 0$, so $u = fu$.

Finally, suppose that the set of fixed points of f is well ordered. Assume on the contrary, that u and v are two fixed points of f such that $u \neq v$. Then, by (2.1), we have

$$\begin{aligned} \tilde{S}(u, v, v) = \tilde{S}(fu, fv, fv) &\leq \beta(M(u, v, v))M(u, v, v) = \beta(\tilde{S}(u, v, v))\tilde{S}(u, v, v) \\ &< \Omega^{-1}(1)\tilde{S}(u, v, v) \end{aligned}$$

because $M(u, v, v) = \tilde{S}(u, v, v)$. So, we get $\tilde{S}(u, v, v) < \Omega^{-1}(1)\tilde{S}(u, v, v)$, a contradiction. Hence, $u = v$, and f has a unique fixed point. Conversely, if f has a unique fixed point, then the set of fixed points of f is trivially well ordered. \square

2.2 Fixed point results via comparison functions

Let Ψ be the family of all nondecreasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that

$$\lim_{n \rightarrow \infty} \psi^n(t) = 0$$

for all $t > 0$. It is easy to see that the following holds.

Lemma 2.4 (see, e.g., [15]) *If $\psi \in \Psi$, then the following are satisfied:*

- (a) $\psi(t) < t$ for all $t > 0$;
- (b) $\psi(0) = 0$.

Definition 2.5 Let $(\mathcal{E}, \preceq, \tilde{S})$ be an ordered S_p -metric space. A mapping $f : \mathcal{E} \rightarrow \mathcal{E}$ is called an S_p - ψ -contraction if there exists $\psi \in \Psi$ such that

$$\Omega^2(2\tilde{S}(f\zeta, f\eta, f\omega)) \leq \psi(M(\zeta, \eta, \omega)) \tag{2.7}$$

for all mutually comparable elements $\zeta, \eta, \omega \in \mathcal{E}$, where

$$M(\zeta, \eta, \omega) = \max \left\{ \tilde{S}(\zeta, \eta, \omega), \frac{1}{2}\Omega^{-1}[\tilde{S}(\zeta, f\zeta, f\eta)], \tilde{S}(\eta, f\eta, f\omega) \right\}.$$

Theorem 2.6 *Let $(\mathcal{E}, \preceq, \tilde{\mathcal{S}})$ be an ordered S_p -complete S_p -metric space. Let $f : \mathcal{E} \rightarrow \mathcal{E}$ be an increasing mapping with respect to \preceq such that there exists an element $\zeta_0 \in \mathcal{E}$ with $\zeta_0 \preceq f\zeta_0$. Suppose that f is an S_p - ψ -contractive mapping. If*

- (I) f is continuous, or
- (II) $(\mathcal{E}, \preceq, \tilde{\mathcal{S}})$ has the s.l.c. property,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Put $\zeta_n = f^n \zeta_0$.

Step 1. We will show that $\lim_{n \rightarrow \infty} \tilde{\mathcal{S}}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0$. We assume that $\zeta_n \neq \zeta_{n+1}$ for all $n \in \mathbb{N}$ (otherwise there is nothing to prove). As in the proof of Theorem 2.3, we have that the sequence $\{\zeta_n\}$ is increasing. Hence, by (2.7) we have

$$\begin{aligned} \tilde{\mathcal{S}}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) &= \tilde{\mathcal{S}}(f\zeta_{n-1}, f\zeta_n, f\zeta_n) \leq \Omega(2\tilde{\mathcal{S}}(f(\zeta_{n-1}), f\zeta_n, f\zeta_n)) \\ &\leq \psi(M(\zeta_{n-1}, \zeta_n, \zeta_n)) \leq \psi(\tilde{\mathcal{S}}(\zeta_{n-1}, \zeta_n, \zeta_n)) \\ &< \tilde{\mathcal{S}}(\zeta_{n-1}, \zeta_n, \zeta_n) \end{aligned} \tag{2.8}$$

because

$$M(\zeta_{n-1}, \zeta_n, \zeta_{n+1}) \leq \max\{\tilde{\mathcal{S}}(\zeta_{n-1}, \zeta_n, \zeta_n), \tilde{\mathcal{S}}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})\},$$

and it is easy to see that $\max\{\tilde{\mathcal{S}}(\zeta_{n-1}, \zeta_n, \zeta_n), \tilde{\mathcal{S}}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})\} = \tilde{\mathcal{S}}(\zeta_{n-1}, \zeta_n, \zeta_n)$. So, from (2.8) we conclude that $\{\tilde{\mathcal{S}}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})\}$ is decreasing. Then there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \tilde{\mathcal{S}}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = r$. It is easy to verify that

$$r = \lim_{n \rightarrow \infty} \tilde{\mathcal{S}}(\zeta_{n-1}, \zeta_n, \zeta_n) = 0.$$

Step 2. Similarly as in the proof of Theorem 2.3, if $\{\zeta_n\}$ were not an S_p -Cauchy sequence, then there would exist $\varepsilon > 0$ for which we can find two subsequences $\{\zeta_{m_i}\}$ and $\{\zeta_{n_i}\}$ of $\{\zeta_n\}$ such that (2.5) and (2.6) hold. Then we would have

$$\limsup_{i \rightarrow \infty} M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}) \leq \Omega(\varepsilon).$$

Now, from (2.7) and the mentioned inequalities, we have

$$\begin{aligned} \Omega(\varepsilon) &= \Omega^2\left(2 \cdot \frac{1}{2} \Omega^{-1} \varepsilon\right) \\ &\leq \Omega^2\left[\limsup_{i \rightarrow \infty} (2\tilde{\mathcal{S}}(\zeta_{m_i+1}, \zeta_{n_i}, \zeta_{n_i}))\right] \\ &\leq \limsup_{i \rightarrow \infty} \psi(M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1})) < \Omega(\varepsilon), \end{aligned}$$

which is a contradiction. Hence, $\{\zeta_n\}$ is a S_p -Cauchy sequence and S_p -completeness of ζ yields that $\{\zeta_n\}$ S_p -converges to a point $u \in \mathcal{E}$.

Step 3. u is a fixed point of f . This step is proved as in the proof of Theorem 2.3 with some elementary changes. □

If in the above theorem we take $\psi(t) = \sinh t$ and $\tilde{S}(\zeta, \eta, \omega) = \sinh(S_b(\zeta, \eta, \omega))$, then we have the following corollary in the framework of S_b metric spaces.

Corollary 2.7 *Let $(\mathcal{E}, \preceq, S_b)$ be an ordered S_b -complete S_b -metric space with coefficient $s > 1$. Let $f : \mathcal{E} \rightarrow \mathcal{E}$ be an increasing mapping with respect to \preceq such that there exists an element $\zeta_0 \in \mathcal{E}$ with $\zeta_0 \preceq f\zeta_0$. Suppose that*

$$\sinh(s \cdot \sinh(s \cdot 2 \sinh(S_b(f\zeta, f\eta, f\omega)))) \leq \sinh(M(\zeta, \eta, \omega))$$

for all mutually comparable elements $\zeta, \eta, \omega \in \mathcal{E}$, where

$$M(\zeta, \eta, \omega) = \max \left\{ \sinh(S_b(\zeta, \eta, \omega)), \frac{S_b(\zeta, f\zeta, f\eta)}{2s}, \sinh(S_b(\eta, f\eta, f\omega)) \right\}.$$

If

- (I) f is continuous, or
- (II) $(\mathcal{E}, \preceq, S_b)$ enjoys the s.l.c. property,

then f has a fixed point.

2.3 Fixed point results related to JS-contractions

Jleli et al. [8] introduced the class Θ_0 consisting of all functions $\theta : (0, \infty) \rightarrow (1, \infty)$ satisfying the following conditions:

- (θ_1) θ is nondecreasing;
- (θ_2) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n \rightarrow \infty} \theta(t_n) = 1$ if and only if $\lim_{n \rightarrow \infty} t_n = 0$;
- (θ_3) there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t \rightarrow 0^+} \frac{\theta(t)-1}{t^r} = \ell$;
- (θ_4) θ is continuous.

They proved the following result.

Theorem 2.8 ([8, Corollary 2.1]) *Let (\mathcal{E}, d) be a complete metric space, and let $T : \mathcal{E} \rightarrow \mathcal{E}$ be a given mapping. Suppose that there exist $\theta \in \Theta_0$ and $k \in (0, 1)$ such that*

$$\zeta, \eta \in \mathcal{E}, \quad d(Tx, Ty) \neq 0 \implies \theta(d(Tx, T\eta)) \leq \theta(d(\zeta, \eta))^k.$$

Then T has a unique fixed point.

From now on, we denote by Θ the set of all functions $\theta : [0, \infty) \rightarrow [1, \infty)$ satisfying the following conditions:

- θ_1 . θ is a continuous strictly increasing function;
- θ_2 . for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n \rightarrow \infty} \theta(t_n) = 1$ if and only if $\lim_{n \rightarrow \infty} t_n = 0$.

Remark 2.9 ([6]) It is clear that $f(t) = e^t$ does not belong to Θ_0 , but it belongs to Θ . Other examples are $f(t) = \cosh t$, $f(t) = \frac{2 \cosh t}{1 + \cosh t}$, $f(t) = 1 + \ln(1 + t)$, $f(t) = \frac{2 + 2 \ln(1 + t)}{2 + \ln(1 + t)}$, $f(t) = e^{te^t}$, and $f(t) = \frac{2e^{te^t}}{1 + e^{te^t}}$ for all $t > 0$.

Definition 2.10 Let $(\mathcal{E}, \preceq, \tilde{S})$ be an ordered S_p -metric space. A mapping $f : \mathcal{E} \rightarrow \mathcal{E}$ is called an S_p rational-JS-contraction if

$$\theta(\mathcal{S}[2\tilde{S}(f\zeta, f\eta, f\omega)]) \leq \theta(M(\zeta, \eta, \omega))^k \tag{2.9}$$

for all mutually comparable elements $\zeta, \eta, \omega \in \zeta$, where $\theta \in \Theta, k \in [0, 1]$ and

$$M(\zeta, \eta, \omega) = \max \left\{ \tilde{S}(\zeta, \eta, \omega), \frac{\tilde{S}(\zeta, \zeta, f\zeta)\tilde{S}(\eta, \eta, f\eta)}{1 + \tilde{S}(\zeta, \eta, \eta) + \tilde{S}(\zeta, \omega, \omega)}, \frac{\tilde{S}(\eta, \eta, f\eta)\tilde{S}(\omega, \omega, f\omega)}{1 + \tilde{S}(\eta, f\omega, f\omega) + \tilde{S}(\eta, \zeta, \zeta)} \right\}.$$

Theorem 2.11 *Let $(\mathcal{E}, \preceq, \tilde{S})$ be an ordered S_p -complete S_p -metric space. Let $f : \mathcal{E} \rightarrow \mathcal{E}$ be an increasing mapping with respect to \preceq such that there exists an element $\zeta_0 \in \mathcal{E}$ with $\zeta_0 \preceq f\zeta_0$. Suppose that f is an S_p -rational JS-contractive mapping. If*

- (I) f is continuous, or
- (II) $(\mathcal{E}, \preceq, \tilde{S})$ enjoys the s.l.c. property,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Put $\zeta_n = f^n \zeta_0$.

Step 1. We will show that $\lim_{n \rightarrow \infty} \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0$. Without loss of generality, we may assume that $\zeta_n \neq \zeta_{n+1}$ for all $n \in \mathbb{N}$. Since $\zeta_{n-1} \preceq \zeta_n$ for each $n \in \mathbb{N}$, then by (2.9) we have

$$\begin{aligned} \theta(\tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})) &\leq \theta(\Omega[2\tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})]) = \theta(\tilde{S}(f\zeta_{n-1}, f\zeta_n, f\zeta_n)) \\ &\leq \theta(M(\zeta_{n-1}, \zeta_n, \zeta_n))^k \leq \theta(\tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n))^k \end{aligned} \tag{2.10}$$

because

$$\begin{aligned} M(\zeta_{n-1}, \zeta_n, \zeta_n) &= \max \left\{ \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n), \frac{\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, f\zeta_{n-1})\tilde{S}(\zeta_n, \zeta_n, f\zeta_n)}{1 + \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n) + \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n)} \right. \\ &\quad \left. \frac{\tilde{S}(\zeta_n, \zeta_n, f\zeta_n)\tilde{S}(\zeta_n, \zeta_n, f\zeta_n)}{1 + \tilde{S}(\zeta_n, f\zeta_n, f\zeta_n) + \tilde{S}(\zeta_n, \zeta_{n-1}, \zeta_{n-1})} \right\} \\ &= \max \left\{ \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n), \frac{\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)\tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})}{1 + \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n) + \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n)} \right. \\ &\quad \left. \frac{\tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})\tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})}{1 + \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) + \tilde{S}(\zeta_n, \zeta_{n-1}, \zeta_{n-1})} \right\} \\ &\leq \max \{ \tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n), \tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1}) \}. \end{aligned}$$

From (2.10) we deduce that

$$\theta(\tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})) \leq \theta(\tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n))^k.$$

Therefore,

$$1 \leq \theta(\tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})) \leq \theta(\tilde{S}(\zeta_{n-1}, \zeta_n, \zeta_n))^k \leq \dots \leq \theta(\tilde{S}(\zeta_0, \zeta_1, \zeta_1))^{k^n}. \tag{2.11}$$

Taking the limit as $n \rightarrow \infty$ in (2.11), we have

$$\lim_{n \rightarrow \infty} \theta(\tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})) = 1,$$

and since $\theta \in \Theta$, we obtain $\lim_{n \rightarrow \infty} \tilde{S}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0$. Therefore, we have

$$\lim_{n \rightarrow \infty} \tilde{S}(\zeta_{n+1}, \zeta_n, \zeta_n) = 0.$$

Step 2. Similarly as in the proof of Theorem 2.3, if $\{\zeta_n\}$ were not an S_p -Cauchy sequence, then there would exist $\varepsilon > 0$ for which we can find two subsequences $\{\zeta_{m_i}\}$ and $\{\zeta_{n_i}\}$ of $\{\zeta_n\}$ such that (2.5) and (2.6) hold. Hence,

$$\tilde{S}(\zeta_{m_i}, \zeta_{m_i}, \zeta_{n_i-1}) < \varepsilon.$$

From the rectangular inequality, we get

$$\varepsilon \leq \tilde{S}(\zeta_{m_i}, \zeta_{n_i}, \zeta_{n_i}) \leq \Omega [\tilde{S}(\zeta_{m_i}, \zeta_{m_i}, \zeta_{m_i+1}) + 2\tilde{S}(\zeta_{m_i+1}, \zeta_{n_i}, \zeta_{n_i})].$$

By taking the upper limit as $i \rightarrow \infty$, we get

$$\frac{1}{2}\Omega^{-1}(\varepsilon) \leq \limsup_{i \rightarrow \infty} \tilde{S}(\zeta_{m_i+1}, \zeta_{n_i}, \zeta_{n_i}).$$

From the definition of $M(\zeta, \eta, \omega)$ and the above limits,

$$\begin{aligned} \limsup_{i \rightarrow \infty} M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}) &= \limsup_{i \rightarrow \infty} \max \left\{ \tilde{S}(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}), \right. \\ &\quad \frac{\tilde{S}(\zeta_{m_i}, \zeta_{m_i}, f \zeta_{m_i}) \tilde{S}(\zeta_{n_i-1}, \zeta_{n_i-1}, f \zeta_{n_i-1})}{1 + \tilde{S}(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}) + \tilde{S}(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1})} \\ &\quad \left. \frac{\tilde{S}(\zeta_{n_i-1}, \zeta_{n_i-1}, f \zeta_{n_i-1}) \tilde{S}(\zeta_{n_i-1}, \zeta_{n_i-1}, f \zeta_{n_i-1})}{1 + \tilde{S}(\zeta_{n_i-1}, f \zeta_{n_i-1}, f \zeta_{n_i-1}) + \tilde{S}(\zeta_{n_i-1}, \zeta_{m_i}, \zeta_{m_i})} \right\} \\ &\leq \varepsilon. \end{aligned}$$

Now, from (2.9) and the above inequalities, we have

$$\begin{aligned} \theta \left(\Omega \left[2 \cdot \frac{1}{2} \Omega^{-1}(\varepsilon) \right] \right) &\leq \limsup_{i \rightarrow \infty} \theta(\Omega [2\tilde{S}(\zeta_{m_i+1}, \zeta_{n_i}, \zeta_{n_i})]) \\ &\leq \limsup_{i \rightarrow \infty} \theta(M(\zeta_{m_i}, \zeta_{n_i-1}, \zeta_{n_i-1}))^k \\ &\leq \theta(\varepsilon)^k, \end{aligned}$$

which implies that $\varepsilon = 0$, a contradiction. So, we conclude that $\{\zeta_n\}$ is an S_p -Cauchy sequence. By S_p -completeness of \mathcal{E} , it follows that $\{\zeta_n\}$ S_p -converges to a point $u \in \mathcal{E}$.

Step 3. u is a fixed point of f .

When f is continuous, the proof is straightforward.

Now, let (II) hold. Using the assumption on \mathcal{E} , we have $\zeta_n \preceq u$. By Lemma 1.11,

$$\begin{aligned} \theta \left(\Omega \left[2 \cdot \frac{\Omega^{-1}[\tilde{S}(u, u, fu)]}{2} \right] \right) &\leq \limsup_{n \rightarrow \infty} \theta(\Omega [2\tilde{S}(\zeta_{n+1}, \zeta_{n+1}, fu)]) \\ &\leq \limsup_{n \rightarrow \infty} \theta(M(\zeta_n, \zeta_n, u))^k, \end{aligned}$$

where

$$\lim_{n \rightarrow \infty} M(\zeta_n, \zeta_n, u) = \lim_{n \rightarrow \infty} \max \left\{ \tilde{S}(\zeta_n, \zeta_n, u), \frac{\tilde{S}(\zeta_n, \zeta_n, f \zeta_n) \tilde{S}(\zeta_n, \zeta_n, f \zeta_n)}{1 + \tilde{S}(\zeta_n, \zeta_n, \zeta_n) + \tilde{S}(\zeta_n, u, u)}, \frac{\tilde{S}(\zeta_n, \zeta_n, f \zeta_n) \tilde{S}(u, u, fu)}{1 + \tilde{S}(\zeta_n, fu, fu) + \tilde{S}(\zeta_n, \zeta_n, \zeta_n)} \right\} = 0.$$

Therefore, we deduce that $\tilde{S}(u, fu, fu) = 0$, so $u = fu$.

Finally, suppose that the set of fixed points of f is well ordered. Assume, on the contrary, that u and v are two fixed points of f such that $u \neq v$. Then, by (2.9), we have

$$\theta[\tilde{S}(u, v, v)] = \theta[\tilde{S}(fu, fv, fv)] \leq \theta(M(u, v, v))^k = \theta(\tilde{S}(u, v, v))^k.$$

So, we get $\tilde{S}(u, v, v) = 0$, a contradiction. Hence $u = v$, and f has a unique fixed point. \square

If in the above theorem we take $\theta(t) = \frac{2e^{t\theta}}{1+e^{t\theta}}$ and $\tilde{S}(\zeta, \eta, \omega) = e^{S_b(\zeta, \eta, \omega)} - 1$, then we have the following corollary in the framework of S_b metric spaces.

Corollary 2.12 *Let (\mathcal{E}, \leq, S_b) be an ordered S_b -complete S_b -metric space with coefficient $s > 1$. Let $f : \mathcal{E} \rightarrow \mathcal{E}$ be an increasing mapping with respect to \leq such that there exists an element $\zeta_0 \in \mathcal{E}$ with $\zeta_0 \leq f \zeta_0$. Suppose that*

$$\frac{2e^{[e^{s \cdot [2e^{S_b(f \zeta, f \eta, f \omega)} - 1] - 1]} e^{s \cdot [2e^{S_b(f \zeta, f \eta, f \omega)} - 1] - 1}}}{1 + e^{[e^{s \cdot [2e^{S_b(f \zeta, f \eta, f \omega)} - 1] - 1]} e^{s \cdot [2e^{S_b(f \zeta, f \eta, f \omega)} - 1] - 1}}} \leq \sqrt{\frac{2e^{M(\zeta, \eta, \omega)} e^{M(\zeta, \eta, \omega)}}{1 + e^{M(\zeta, \eta, \omega)} e^{M(\zeta, \eta, \omega)}}}$$

for all mutually comparable elements $\zeta, \eta, \omega \in \mathcal{E}$, where

$$M(\zeta, \eta, \omega) = \max \left\{ e^{S_b(\zeta, \eta, \omega)} - 1, \frac{[e^{S_b(\zeta, \zeta, f \zeta)} - 1][e^{S_b(\eta, \eta, f \eta)} - 1]}{1 + e^{S_b(\zeta, \eta, \eta)} - 1 + e^{S_b(\zeta, \zeta, f \zeta)} - 1}, \frac{[e^{S_b(\eta, \eta, f \eta)} - 1][e^{S_b(\omega, \omega, f \omega)} - 1]}{1 + e^{S_b(\eta, f \omega, f \omega)} - 1 + e^{S_b(\eta, \zeta, \zeta)} - 1} \right\}.$$

If

- (I) f is continuous, or
- (II) (\mathcal{E}, \leq, S_b) enjoys the s.l.c. property,

then f has a fixed point.

3 Examples

Example 3.1 Let $\mathcal{E} = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}$ be equipped with the partial order \leq given as

$$\leq := \left\{ (0, 0), \left(0, \frac{1}{2}\right), \left(0, \frac{1}{3}\right), \left(0, \frac{1}{5}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{5}, \frac{1}{5}\right) \right\}.$$

Define a metric d on \mathcal{E} by

$$d(\zeta, \eta) = \begin{cases} 0, & \text{if } \zeta = \eta, \\ \zeta + \eta, & \text{if } \zeta \neq \eta, \end{cases}$$

and let $\tilde{S}(\zeta, \eta, \omega) = \sinh[d(\zeta, \omega) + d(\eta, \omega)]$. It is easy to see that (\mathcal{E}, \tilde{S}) is an S_p -complete S_p -metric space. Indeed, in the same way as the usual S -metric is formed in Example 1.5, an S -metric $S(\zeta, \eta, \omega) = d(\zeta, \omega) + d(\eta, \omega)$ is formed, and using the function $\xi(t) = \sinh t$, as in Proposition 1.8, one obtains the S_p -metric \tilde{S} .

Define a self-map f by

$$f = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 & 0 \end{pmatrix}.$$

We see that f is an increasing mapping w.r.t. \leq and $(\mathcal{E}, \leq, \tilde{S})$ enjoys the s.l.c. property. Also, $0 \leq f0$.

Define $\theta : [0, \infty) \rightarrow [1, \infty)$ by $\theta(t) = \cosh(t)$ and take $k = \frac{2}{3}$. One can easily check that f is an S_p -rational JS-contractive mapping. Indeed, as a sample, we check some cases as follows:

1. $(\zeta, \eta, \omega) = (0, 0, \frac{1}{2})$. Then

$$\begin{aligned} \theta[\Omega(2\tilde{S}(f\zeta, f\eta, f\omega))] &= \cosh\left[\sinh 2 \sinh 2\left(f0 + f\frac{1}{2}\right)\right] \\ &= \cosh\left[\sinh 2 \sinh 2\left(0 + \frac{1}{5}\right)\right] \\ &\approx 1.451 \leq 1.465 \approx \sqrt[3]{\cosh\left[\sinh 2\left(0 + \frac{1}{2}\right)\right]^2} \\ &= \sqrt[3]{\theta(M(\zeta, \eta))^2}. \end{aligned}$$

2. $(\zeta, \eta, \omega) = (0, \frac{1}{2}, \frac{1}{2})$. Then

$$\begin{aligned} \theta[\Omega(2\tilde{S}(f\zeta, f\eta, f\omega))] &= \cosh\left[\sinh \sinh 2\left(f0 + f\frac{1}{2}\right)\right] \\ &= \cosh\left[\sinh \sinh 2\left(0 + \frac{1}{5}\right)\right] \\ &\approx 1.087 \leq 1.091 \approx \sqrt[3]{\cosh\left[\sinh\left(0 + \frac{1}{2}\right)\right]^2} \\ &= \sqrt[3]{\theta(M(\zeta, \eta))^2}. \end{aligned}$$

3. $(\zeta, \eta, \omega) = (0, 0, \frac{1}{3})$. Then

$$\begin{aligned} \theta[\Omega(2\tilde{S}(f\zeta, f\eta, f\omega))] &= \cosh\left[\sinh \sinh 2\left(f0 + f\frac{1}{3}\right)\right] \\ &= \cosh[\sinh \sinh 2(0 + 0)] \\ &= 0 \leq 1.172 \approx \sqrt[3]{\cosh\left[\sinh 2\left(0 + \frac{1}{3}\right)\right]^2} \\ &= \sqrt[3]{\theta(M(\zeta, \eta))^2}. \end{aligned}$$

4. $(\zeta, \eta, \omega) = (0, \frac{1}{5}, \frac{1}{5})$. Then

$$\begin{aligned} \theta[\Omega(2\tilde{S}(f\zeta, f\eta, f\omega))] &= \cosh\left[\sinh 2 \sinh\left(f0 + f\frac{1}{5}\right)\right] \\ &= \cosh[\sinh \sinh 2(0 + 0)] \\ &= 0 \leq 1.0135 \approx \sqrt[3]{\cosh\left[\sinh\left(0 + \frac{1}{5}\right)\right]^2} \\ &= \sqrt[3]{\theta(M(\zeta, \eta))^2}. \end{aligned}$$

The rest of the cases can be checked similarly. Thus, all the conditions of Theorem 2.11 are satisfied and hence f has a fixed point. Indeed, 0 is the fixed point of f .

Example 3.2 Let $\mathcal{E} = [0, 1.5]$ be equipped with the S_p -metric

$$\tilde{S}(\zeta, \eta, \omega) = e^{|\zeta - \omega| + |\eta - \omega|} - 1$$

for all $\zeta, \eta, \omega \in \mathcal{E}$, where $\Omega(t) = e^t - 1$ (the mapping \tilde{S} is obtained from the usual S -metric (Example 1.5) using the function $\xi(t) = e^t - 1$). Define a relation \leq on \mathcal{E} by $\zeta \leq \eta$ iff $\eta \leq \zeta$, a mapping $f : \mathcal{E} \rightarrow \mathcal{E}$ by

$$f\zeta = \frac{\zeta}{8} e^{-\frac{\zeta}{2}}$$

and a function β by $\beta(t) = \frac{1}{2} < 0.882 \approx \Omega^{-1}(1)$. For all mutually comparable elements $\zeta, \eta, \omega \in \mathcal{E}$, we have

$$\begin{aligned} \tilde{S}(f\zeta, f\eta, f\omega) &= e^{|\frac{\zeta}{8}e^{-\frac{\zeta}{2}} - \frac{\omega}{8}e^{-\frac{\omega}{2}}| + |\frac{\eta}{8}e^{-\frac{\eta}{2}} - \frac{\omega}{8}e^{-\frac{\omega}{2}}|} - 1 \\ &\leq e^{\frac{1}{8}|\zeta - \omega| + \frac{1}{8}|\eta - \omega|} - 1 \leq \frac{1}{8}(e^{|\zeta - \omega| + |\eta - \omega|} - 1) \\ &= \frac{1}{8}\tilde{S}(\zeta, \eta, \omega). \end{aligned}$$

So,

$$\begin{aligned} \Omega^2[2\tilde{S}sp(f\zeta, f\eta, f\omega)] &\leq \Omega^2\left(\frac{1}{4}\tilde{S}(\zeta, \eta, \omega)\right) = e^{(e^{\frac{1}{4}\tilde{S}(\zeta, \eta, \omega)} - 1)} - 1 \leq e^{\frac{1}{2}(e^{\frac{1}{2}\tilde{S}(\zeta, \eta, \omega)} - 1)} - 1 \\ &\leq \frac{1}{2}\tilde{S}(\zeta, \eta, \omega) = \beta(M(\zeta, \eta, \omega))\tilde{S}(\zeta, \eta, \omega). \end{aligned}$$

Therefore, it follows from Theorem 2.3 that f has a fixed point (which is $u = 0$).

Example 3.3 Let $\mathcal{E} = [0, 3]$ be equipped with the S_p -metric

$$\tilde{S}(\zeta, \eta, \omega) = e^{|\zeta - \omega| + |\eta - \omega|} - 1$$

as in the previous example. Define a relation \leq on \mathcal{E} by $\zeta \leq \eta$ iff $\eta \leq \zeta$, a mapping $f : \mathcal{E} \rightarrow \mathcal{E}$ by

$$f\zeta = \ln(\zeta + 12),$$

and a function ψ by $\psi(t) = \frac{1}{4}(e^t - 1)$. For all mutually comparable elements $\zeta, \eta, \omega \in \mathcal{E}$, by the mean value theorem, we have

$$\begin{aligned} \tilde{S}(f\zeta, f\eta, f\omega) &= e^{|\ln(\zeta+12) - \ln(\omega+12)| + |\ln(\eta+12) - \ln(\omega+12)|} - 1 \\ &= e^{|\ln \frac{\zeta+12}{\omega+12}| + |\ln \frac{\eta+12}{\omega+12}|} - 1 = \left| \frac{\zeta + 12}{\omega + 12} \cdot \frac{\eta - \omega}{\omega + 12} \right| - 1 \\ &\leq \frac{1}{12} (|\zeta - \omega| \cdot |\eta - \omega| - 1) \leq \frac{1}{12} (e^{|\zeta - \omega| + |\eta - \omega|} - 1) \\ &= \frac{1}{12} \tilde{S}(\zeta, \eta, \omega). \end{aligned}$$

So,

$$\begin{aligned} \Omega^2[2\tilde{S}(f\zeta, f\eta, f\omega)] &\leq \Omega^2\left(\frac{1}{6}\tilde{S}(\zeta, \eta, \omega)\right) = e^{(e^{\frac{1}{6}\tilde{S}(\zeta, \eta, \omega)} - 1)} - 1 \\ &\leq e^{\frac{1}{4}\tilde{S}(\zeta, \eta, \omega)} - 1 = \psi(\tilde{S}(\zeta, \eta, \omega)). \end{aligned}$$

Therefore, from Theorem 2.6, f has a fixed point.

Example 3.4 Let $\tilde{S} : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}^+$ be defined on $\mathcal{E} = [0, 20]$ by

$$\tilde{S}(\zeta, \eta, \omega) = e^{\frac{1}{3}(|\zeta - \eta| + |\eta - \omega| + |\omega - \zeta|)} - 1$$

for all $\zeta, \eta, \omega \in \mathcal{E}$ (the mapping \tilde{S} is obtained from the S -metric

$$S(\zeta, \eta, \omega) = \frac{1}{3}(|\zeta - \eta| + |\eta - \omega| + |\omega - \zeta|)$$

using the function $\xi(t) = e^t - 1$). Then (\mathcal{E}, \tilde{S}) is an S_p -complete S_p -metric space with $\Omega(t) = e^t - 1$. Define $\theta \in \Theta$ by $\theta(t) = e^{te^t}$. Let \mathcal{E} be endowed with the standard order \leq , and let $f : \mathcal{E} \rightarrow \mathcal{E}$ be defined by $f\zeta = \arctan(\frac{\zeta}{32})$. It is easy to see that f is an ordered increasing and continuous self-map on ζ and $0 \leq f0$. For any mutually comparable $\zeta, \eta, \omega \in \mathcal{E}$, we have

$$\begin{aligned} \tilde{S}(f\zeta, f\eta, f\omega) &= e^{\frac{1}{3}(|f\zeta - f\eta| + |f\eta - f\omega| + |f\omega - f\zeta|)} - 1 \\ &= e^{\frac{1}{3}(|\arctan \frac{\zeta}{32} - \arctan \frac{\eta}{32}| + |\arctan \frac{\eta}{32} - \arctan \frac{\omega}{32}| + |\arctan \frac{\omega}{32} - \arctan \frac{\zeta}{32}|)} - 1 \\ &\leq e^{\frac{1}{3}(|\frac{\zeta}{32} - \frac{\eta}{32}| + |\frac{\eta}{32} - \frac{\omega}{32}| + |\frac{\omega}{32} - \frac{\zeta}{32}|)} - 1 \leq \frac{1}{32} (e^{\frac{1}{3}(|\zeta - \eta| + |\eta - \omega| + |\omega - \zeta|)} - 1) \\ &= \frac{1}{32} \tilde{S}(\zeta, \eta, \omega). \end{aligned}$$

So,

$$\begin{aligned} \Omega [2\tilde{S}(f\zeta, f\eta, f\omega)] &= e^{2\tilde{S}(f\zeta, f\eta, f\omega)} - 1 \leq e^{\frac{1}{16}\tilde{S}(\zeta, \eta, \omega)} - 1 \\ &\leq \frac{1}{8}\tilde{S}(\zeta, \eta, \omega). \end{aligned}$$

Therefore,

$$\begin{aligned} \theta(\Omega [2\tilde{S}(f\zeta, f\eta, f\omega)]) &= e^{\Omega [2\tilde{S}(f\zeta, f\eta, f\omega)]} e^{\Omega [2\tilde{S}(f\zeta, f\eta, f\omega)]} \leq e^{\frac{1}{8}\tilde{S}(\zeta, \eta, \omega)} e^{\frac{1}{8}\tilde{S}(\zeta, \eta, \omega)} \\ &\leq [e^{\tilde{S}(\zeta, \eta, \omega)} e^{\tilde{S}(\zeta, \eta, \omega)}]^{\frac{1}{2}} = [\theta(\tilde{S}(\zeta, \eta, \omega))]^{\frac{1}{2}}. \end{aligned}$$

Thus, (2.9) is satisfied with $k = \frac{1}{\sqrt{2}}$. Hence, all the conditions of Theorem 2.11 are satisfied. We see that 0 is the unique fixed point of f .

4 Application

In this section we present an application of Theorem 2.6. This application is inspired by [12].

Let $\mathcal{E} = C(I)$ be the set of all real continuous functions on $I = [0, T]$. Obviously, this set with the p -metric given by

$$d(\zeta, \eta) = \sinh\left(\max_{t \in I} |\zeta(t) - \eta(t)|\right)$$

for all $\zeta, \eta \in \mathcal{E}$ is a p -complete p -metric space with $\Omega(t) = \sinh t$ (in the sense of [13]). Then $\tilde{S}(\zeta, \eta, \omega) = d(\zeta, \omega) + d(\eta, \omega)$ is an S_p -metric on \mathcal{E} as $\sinh t$ is super-additive. (\mathcal{E}, \tilde{S}) can also be equipped with the order given by

$$\zeta \leq \eta \quad \text{iff} \quad \zeta(t) \leq \eta(t) \quad \text{for all } t \in I.$$

Moreover, as in [12], it can be proved that $(C(I), \leq, \tilde{S})$ has the s.l.c. property.

Consider the first-order periodic boundary value problem

$$\begin{cases} \zeta'(t) = f(t, \zeta(t)), \\ \zeta(0) = \zeta(T), \end{cases} \tag{4.1}$$

where $t \in I$ and $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. A lower solution for (4.1) is a function $\alpha \in C^1(I)$ such that

$$\begin{cases} \alpha'(t) \leq f(t, \alpha(t)), \\ \alpha(0) \leq \alpha(T), \end{cases}$$

where $t \in I$. Assume that there exists $\lambda > 0$ such that, for all $\zeta, \eta \in \mathcal{E}$ and $t \in I$, we have

$$|f(t, \zeta(t)) + \lambda\zeta(t) - f(t, \eta(t)) - \lambda\eta(t)| \leq \frac{\lambda}{8} |\zeta(t) - \eta(t)|. \tag{4.2}$$

We will show that the existence of a lower solution for (4.1) provides the existence of a unique solution of (4.1). Problem (4.1) can be rewritten as

$$\begin{cases} \zeta'(t) + \lambda\zeta(t) = f(t, \zeta(t)) + \lambda\zeta(t) \equiv F(t, \zeta(t)), \\ \zeta(0) = \zeta(T). \end{cases}$$

It is well known that this problem is equivalent to the integral equation

$$\zeta(t) = \int_0^T G(t,s)F(s, \zeta(s)) ds,$$

where G is the Green's function given as

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s \leq t \leq T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t \leq s \leq T. \end{cases}$$

Easy calculation gives

$$\int_0^T G(t,s) ds = \frac{1}{\lambda}.$$

Now define an operator $H : C(I) \rightarrow C(I)$ as

$$H\zeta(t) = \int_0^T G(t,s)F(s, \zeta(s)) ds. \tag{4.3}$$

It is easy to see that the mapping H is increasing w.r.t. \leq . Note that if $u \in C^1(I)$ is a fixed point of H , then u is a solution of (4.1).

Let $\zeta, \omega \in \mathcal{E}$ be comparable. Then we have

$$\begin{aligned} d(H\zeta, H\omega) &= \sinh\left(\max_{t \in I} |H\zeta(t) - H\omega(t)|\right) \\ &= \sinh\left(\max_{t \in I} \left| \int_0^T G(t,s)F(s, \zeta(s)) ds - \int_0^T G(t,s)F(s, \omega(s)) ds \right|\right) \\ &\leq \sinh\left(\max_{t \in I} \int_0^T G(t,s) \cdot |F(s, \zeta(s)) - F(s, \omega(s))| ds\right) \\ &\leq \sinh\left(\frac{1}{\lambda} \max_{s \in I} \frac{\lambda}{8} |\zeta(s) - \omega(s)|\right) = \sinh\left(\frac{1}{8} \sinh^{-1} d(\zeta, \omega)\right) \\ &\leq \frac{1}{8} d(\zeta, \omega). \end{aligned}$$

Therefore,

$$\tilde{S}(H\zeta, H\eta, H\omega) = d(H\zeta, H\eta) + d(H\eta, H\omega) \leq \frac{1}{8} (d(\zeta, \eta) + d(\eta, \omega)) = \frac{1}{8} \tilde{S}(\zeta, \eta, \omega).$$

So,

$$\begin{aligned}\Omega^2(2\tilde{S}(Hx, Hy, Hz)) &\leq \Omega^2\left(\frac{1}{4}\tilde{S}(\zeta, \eta, \omega)\right) = \sinh\left(\sinh\left(\frac{1}{4}\tilde{S}(\zeta, \eta, \omega)\right)\right) \\ &= \psi(\tilde{S}(\zeta, \eta, \omega)) \leq \psi(M(\zeta, \eta, \omega)),\end{aligned}$$

where

$$M(\zeta, \eta) = \max\left\{\tilde{S}(\zeta, \eta, \omega), \frac{\sinh^{-1}(\tilde{S}(\zeta, f\zeta, f\eta))}{2}, \tilde{S}(\eta, f\eta, f\omega)\right\},$$

and $\psi(t) = \sinh(\sinh(t/4))$. Finally, if there exists a lower solution α of 4.1, the hypotheses of Theorem 2.6 are satisfied. Therefore, there exists a fixed point $\hat{\zeta} \in C(I)$ of H , which is a solution of the given boundary value problem.

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Authors' contributions

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References

1. Amini-Harandi, A., Emami, H.: A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations. *Nonlinear Anal. TMA* **72**(5), 2238–2242 (2010)
2. Czerwik, S.: Contraction mappings in b -metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1**, 5–11 (1993)
3. Dence, T.P.: A brief look into Lambert W function. *Appl. Math.* **4**, 887–892 (2013)
4. Đukić, D., Kadelburg, Z., Radenović, S.: Fixed points of Geraghty-type mappings in various generalized metric spaces. *Abstr. Appl. Anal.* **2011**, Article ID 561245 (2011)
5. Geraghty, M.: On contractive mappings. *Proc. Am. Math. Soc.* **40**, 604–608 (1973)
6. Hussain, N., Kutbi, M.A., Salimi, P.: Fixed point theory in α -complete metric spaces with applications. *Abstr. Appl. Anal.* **2014**, Article ID 280817 (2014)
7. Jaggi, D.S.: Some unique fixed point theorems. *Indian J. Pure Appl. Math.* **8**(2), 223–230 (1977)
8. Jleli, M., Karapinar, E., Samet, B.: Further generalizations of the Banach contraction principle. *J. Inequal. Appl.* **2014**, 439 (2014)
9. Latif, A., Kadelburg, Z., Parvaneh, V., Roshan, J.R.: Some fixed point theorems for G -rational Geraghty contractive mappings in ordered generalized b -metric spaces. *J. Nonlinear Sci. Appl.* **8**, 1212–1227 (2015)
10. Mustafa, Z., Roshan, J.R., Parvaneh, V.: Coupled coincidence point results for (ψ, φ) -weakly contractive mappings in partially ordered G_b -metric spaces. *Fixed Point Theory Appl.* **2013**, 206 (2013)
11. Nieto, J.J., Rodríguez-López, R.: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **22**(3), 223–239 (2005)

12. Nieto, J.J., Rodríguez-López, R.: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. *Acta Math. Sin.* **23**(12), 2205–2212 (2007)
13. Parvaneh, V.: Fixed points of $(\psi, \varphi)_\Omega$ -contractive mappings in ordered p -metric spaces (submitted)
14. Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **132**(5), 1435–1443 (2004)
15. Rus, I.A.: *Generalized Contractions and Applications*. Cluj University Press, Cluj-Napoca (2001)
16. Sedghi, S., Shobe, N., Aliouche, A.: A generalization of fixed point theorem in S -metric spaces. *Mat. Vesn.* **64**, 258–266 (2012)
17. Shahkoobi, R.J., Razani, A.: Some fixed point theorems for rational Geraghty contractive mappings in ordered b -metric spaces. *J. Inequal. Appl.* **2014**, 373 (2014)
18. Souayah, N., Mlaiki, N.: A fixed point theorem in S_b -metric spaces. *J. Math. Comput. Sci.* **16**, 131–139 (2016)
19. Zabihi, F., Razani, A.: Fixed point theorems for hybrid rational Geraghty contractive mappings in ordered b -metric spaces. *J. Appl. Math.* **2014**, Article ID 929821 (2014)

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