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Fixed Point Results Using F_t-Contractions in Ordered Metric Spaces Having t-Property

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Abstract: In this paper, we prove the existence of fixed points of F_t-contraction mappings in partially ordered metric spaces not necessarily complete. We require that the ordered metric space has the t-property, which is a new concept introduced recently by Rashid et.al. We also give some examples to illustrate the new concepts and obtained results.

Keywords: t-property; ordered metric space; Ft-Contraction.

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1. Introduction and Preliminaries

Banach Contraction Principle is the most important result in metric fixed point theory. This result was due to Banach [1] in 1922. Banach contraction principle has been generalized by many researchers. Among the first generalizations in the setting of ordered metric spaces was proved by Ran-Reurings [2] in 2004. Many papers have been reported in ordered metric spaces (see [3–15]).

In 2012, Wardowski [16] generalized the Banach Contraction Principle by introducing a new type of contractions, called F-contractions. This concept attracted many researchers to contribute in this field. Many papers are reported on the existence of fixed points using F-contractions in different spaces (see [17–26]). For instance, let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is strictly increasing, i.e., for all $a, b \in (0, \infty)$ with $a < b$, then $F(a) < F(b)$.

(F_2) For each sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to\infty} \alpha_n = 0, \text{ iff } \lim_{n\to\infty} F(\alpha_n) = -\infty.$$

(F_3) There exists $k \in (0, 1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$.

Consider $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \ln \alpha + \alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln(\alpha^2 + \alpha)$. We have that $F_n \in \mathcal{F}$ for all $n = 1, 2, 3, 4$.

Definition 1. *[16]. Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. Then, T is said to be an F -contraction if for $F \in \mathcal{F}$, there exists $\tau > 0$ such that*

$$\forall x, y \in X \text{ with } [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))].$$

If we take $F(\alpha) = \ln \alpha$, the previous inequality becomes

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \quad \text{for all } x, y \in X, Tx \neq Ty.$$

In addition, for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ holds. Hence, T is a contraction mapping where the Lipshitiz constant is $\lambda = e^{-\tau}$. Thus, every contraction is also an F-contraction, but the converse is not true in general as it is proved in Example 2.5 of [16].

Definition 2. *[27]. A sequence $\{x_n\}$ in a partially ordered set $(X, \preceq)$ is said to be increasing or ascending if for $m < n$, $x_m \preceq x_n$. It is said strictly increasing if $x_m \preceq x_n$ and $x_m \neq x_n$. We denote it as $x_m \prec x_n$.*

In most fixed point results (including the ones dealing with F-contractions of Wardowski [16]), the completeness hypothesis is essential to ensure the existence of a fixed point. Note that this hypothesis is strong and it would be interesting to obtain fixed point results without the set being complete. The aim of this paper goes in this direction, that is, we have strong results for weaker hypotheses. More precisely, our motivation is based on a very recent paper [28], where the authors introduced the concept of t-property (for partially ordered metric spaces) to ovoid the completeness hypothesis, that is, the metric space may be incomplete.

Definition 3. *[28]. Let $(X, d, \preceq)$ be any ordered metric space. X has the t-property if every strictly increasing Cauchy sequence $\{x_n\}$ in X has a strict upper bound in X, i.e., there exists $u \in X$ such that $x_n \prec u$.*

We present the following examples illustrating Definition 3.

Example 1. *[28]. Let $X = \mathbb{R}, \mathbb{Q}, (a, b], a, b \in \mathbb{R}$ be equipped with the natural ordering $\leq$ and the usual metric. Then, X has the t-property.*

Example 2. *[28]. Let $X = \{(x, y) : x, y \in \mathbb{Q}\}$. We define $\preceq$ in X by $(x_1, x_2) \preceq (y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$. Let d be the Euclidean metric on X. Then, $(X, d, \preceq)$ has the t-property.*

Example 3. *[28]. Let $X = C[a, b]$ be equipped with the metric d defined as $d(f, g) = \int_a^b \mid f - g \mid dx$. Then, (X, d) is not a complete metric space. For $f, g \in X$, $f \preceq g$ iff $f(x) \leq g(x)$ for each $x \in [a, b]$. Obviously, $(C[a, b], d, \preceq)$ has t-property.*

In the following example, the increasing Cauchy sequence does not have any strict upper bound.

Example 4. *[28]. Let us consider $X = \{(x, y, z) : x, y, z \in \mathbb{Q} \text{ with } \max\{x, y, z\} < \sqrt{2}\}$. Endow X with the Euclidean metric on $\mathbb{R}^3$. Define $\preceq$ in X by $(x_1, y_1, z_1) \preceq (x_2, y_2, z_2)$ if $x_1 \leq x_2$, $y_1 \leq y_2$ and $z_1 \leq z_2$. Consider $x_n = (q_n, q_n, q_n)$ in X such that $q_0 = 1$ and $\{q_n\}$ is strictly increasing in $\mathbb{Q}$. We have that $q_n < \sqrt{2}$ for all $n \geq 0$. In addition, $\{x_n\}$ is a strictly increasing Cauchy sequence in X, but it does not have any strict upper bound in X.*

In this paper, we prove some fixed point results for F_t-contraction mappings (introduced in Definition 4) without requiring that the metric space is complete, but using the concept of the t-property. We give some examples to illustrate our obtained results.

2. Main Results

Definition 4. *Let $(X, d, \preceq)$ be an ordered metric space and $T : X \to X$ be a self-mapping. T is said an F_t-contraction if for $F \in \mathcal{F}$, there exists $\tau > 0$ such that for all $x, y \in X$ with $x \neq Tx$, $y \neq Ty$ and $x \prec y$, we have*

$$\tau + F(d(y, T(y))) \leq F(d(x, T(x))). \tag{1}$$

In the following example, the considered mapping T is not an F-contraction, but it is an F_t-contraction.

Example 5. *Let $X = \mathbb{Z}$ be endowed by the usual metric of $\mathbb{R}$ and the natural ordering $\leq$. Define $T : X \to X$ by*

$$T(x) = \begin{cases} 2x, & x < 0, \\ x, & x \geq 0. \end{cases}$$

For $x = -3$ and $y = 4$, we get $d(Tx, Ty) > d(x, y)$. Let $F \in \mathcal{F}$. By (F_1), we have

$$\tau + F(d(Tx, Ty)) > F(d(x, y)), \quad \text{for } \tau > 0.$$

Thus, T is not an F-contraction. Now, we show that T is an F_t-contraction. Clearly, $F(\alpha) = \ln(\alpha) + \alpha \in \mathcal{F}$ for $\alpha \in (0, \infty)$. Set $\tau = \frac{1}{2}$. We show that Equation (1) is satisfied. Let $x, y \in X$ such that $x \neq Tx, y \neq Ty$ and $x < y$. Then, $y - x \geq 1$ and $x < y < 0$. Further, $d(x, Tx) = -x$ and $d(y, Ty) = -y$. In addition,

$$\begin{aligned} F(d(x, Tx)) - F(d(y, Ty)) &= [\ln(-x) - x] - [\ln(-y) - y] \\ &= \ln(\frac{x}{y}) + (y - x) \geq 1 > \frac{1}{2} = \tau. \end{aligned}$$

Thus, $\tau + F(d(y, Ty)) \leq F(d(x, Tx))$. This shows that T is an F_t-contraction.

Example 6. *Let $A = \{0, 1, 2, 3, 4, 5\}$ and $B = (5, 10)$. Endow $X = A \cup B$ with the usual metric of $\mathbb{R}$ and the natural ordering $\leq$. Define $T : X \to X$ by*

$$T(x) = \begin{cases} -\frac{2x}{5} + 7, & x \in A, \\ x, & x \in B. \end{cases}$$

Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \ln(\alpha) + \alpha$. Clearly, $F \in \mathcal{F}$. It can be easily proved that T is an F_t-contraction.

Our first fixed point result is:

Theorem 1. *Let $(X, d, \preceq)$ be an ordered metric space having t-property. Let $T : X \to X$ be an F_t-contraction. Suppose that T is non-decreasing and there exists $x_0 \in X$ such that $x_0 \preceq T(x_0)$. Then, T has a fixed point in X.*

Proof. By assumption, we have $x_0 \in X$ such that $x_0 \preceq T(x_0)$. If $x_0 = T(x_0)$, the proof is completed. Otherwise, choose $x_1 = T(x_0)$ such that $x_0 \prec x_1$. By monotonicity of T, we have $T(x_0) \preceq T(x_1)$, that is, $x_1 \preceq T(x_1)$. If $x_1 = T(x_1)$, the proof is completed. Otherwise, choose $x_2 = T(x_1)$ such that $x_1 \prec x_2$. Again, by monotonicity of T, we have $T(x_1) \preceq T(x_2)$. Continuing this process, we get a strictly increasing sequence $\{x_n\}$ in X such that $x_{n+1} = T(x_n)$. As $x_0 \prec x_1$, by Equation (1), we have

$$\tau + F(d(x_1, T(x_1))) \leq F(d(x_0, T(x_0))). \tag{2}$$

Again, since $x_1 \prec x_2$, by Equation (1), we have

$$\tau + F(d(x_2, T(x_2))) \leq F(d(x_1, T(x_1))). \tag{3}$$

From Equations (2) and (3), we get

$$F(d(x_2, T(x_2))) \leq F(d(x_1, T(x_1))) - \tau \leq F(d(x_0, T(x_0))) - 2\tau.$$

Continuing in this process, we get

$$F(d(x_n, T(x_n))) \leq F(d(x_{n-1}, T(x_{n-1}))) - \tau \leq \cdots \leq F(d(x_0, T(x_0))) - n\tau. \tag{4}$$

Denote $\lambda_n = d(x_n, T(x_n))$ for $n \in \mathbb{N}$. From Equation (4), we obtain

$$F(\lambda_n) \leq F(\lambda_{n-1}) - \tau \leq ... \leq F(\lambda_0) - n\tau. \tag{5}$$

We get $\lim\limits_{n\to\infty} F(\lambda_n) = -\infty$. Using property (F_2),

$$\lim_{n\to\infty} \lambda_n = 0.$$

By (F_3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \lambda_n^k F(\lambda_n) = 0.$$

By Equation (5), we have for all n

$$\lambda_n^k F(\lambda_n) - \lambda_n^k F(\lambda_0) \leq -\lambda_n^k n\tau \leq 0.$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} n\lambda_n^k = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that $n\lambda_n^k \leq 1$ for all $n \geq n_1$, that is,

$$\lambda_n \leq \frac{1}{n^{1/k}}, \tag{6}$$

for all $n \geq n_1$. Now, we show that $\{x_n\}$ is a Cauchy sequence. Let $n, m \in \mathbb{N}$ with $n_1 \leq n < m$. Using Equation (6), one writes

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &= d(x_n, T(x_n)) + d(x_{n+1}, T(x_{n+1})) + ... + d(x_{m-1}, T(x_{m-1})) \\ &= \lambda_n + \lambda_{n+1} + ... + \lambda_{m-1} \\ &= \sum_{i=n}^{m-1} \lambda_i \leq \sum_{i=n}^{\infty} \lambda_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Taking $n \to \infty$, we get $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$. Thus, $\{x_n\}$ is a strictly increasing Cauchy sequence in X, which has t-property. Therefore, there exists $u \in X$ such that $x_n \prec u$. If $T(u) = u$, the proof is completed. Otherwise, by Equation (1), we have $\tau + F(d(u, T(u))) \leq F(d(x_n, T(x_n)))$. Using Equation (4), we get

$$F(d(u, T(u))) \leq F(d(x_0, T(x_0))) - (n+1)\tau.$$

At the limit, $F(d(u, T(u))) = -\infty$. By (F_2), we have $T(u) = u$. Thus, u is a fixed point of T in X. □

Now, we report some examples to illustrate our obtained result. The first example clarifies Theorem 1 where the mapping T is not an F-contraction.

Example 7. *Let $A = \{a_n : a_{n+1} = 4a_n + 1$ for $n \geq 0$ and $a_0 = -1\}$ and $B = (-1, 0] \cap \mathbb{Q}$. Take $X = A \cup B$, so $X = \{..., -43, -11, -3, -1\} \cup B$. Endow X with the usual metric on $\mathbb{R}$ and the natural ordering $\leq$. Clearly, $(X, d, \preceq)$ is not complete but has the t-property. Define $T : X \to X$ by*

$$T(x) = \begin{cases} 4x + 1, & if x \in A, \\ x, & if x \in B. \end{cases}$$

Obviously, T is non-decreasing. Now, it remains to prove that T satisfies Equation (1). Letting $x, y \in X$ with $x < y$, $x \neq T(x)$ and $y \neq T(y)$, we have $x < y \leq -1$. Then, $d(x, T(x)) = -(3x + 1)$, $d(y, T(y)) = -(3y + 1)$ and $y - x \geq 2$. If we take $F(\alpha) = \ln(\alpha) + \alpha \in \mathcal{F}$ and $\tau = 1 > 0$. Then, $\tau + F(d(y, T(y))) \leq F(d(x, T(x)))$, i.e., T is an F_t-contraction. Hence, all the conditions of Theorem 1 are satisfied. B is the set of fixed points of T.

Example 8. *Let $X = \mathbb{R}^2$ be endowed with the Euclidean metric. Consider: $(x, y) \preceq (u, v)$ iff $x \leq u$ and $y \leq v$. Then, $(X, d, \preceq)$ is an ordered metric space having t-property. Take $A = \{..., -8, -6, -4, -2\}$. Let $E \subset X$ be defined by $E = \{(a, b) : a \in \mathbb{R}$ and $b \in A\}$. Clearly, for all $(a, b), (c, d) \in E$ such that $(a, b) \prec (c, d)$, we have $d - b \geq 2$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T(x, y) = \begin{cases} (x, 1), & if (x, y) \in E \\ (x, y), & if (x, y) \in E^c. \end{cases}$$

Clearly, T is non-decreasing. We show that T satisfies Equation (1). Let $x = (x_1, y_1), y = (x_2, y_2) \in X$ such that $x \prec y$, $y \neq T(y)$ and $x \neq T(x)$. Then, $x, y \in E$ and $y_2 - y_1 \geq 2$. In addition,

$$d(y, T(y)) = d((x_2, y_2), (x_1, 1)) = 1 - y_2,$$

and

$$d(x, T(x)) = d((x_1, y_1), (x_1, 1)) = 1 - y_1.$$

Since $y_1 < y_2 < 0$, we have $(1 - y_1) > (1 - y_2)$. Take $F(\alpha) = \ln(\alpha) + \alpha \in \mathcal{F}$ and $\tau = 1$. We have

$$\begin{aligned} F(d(x, T(x))) - F(d(y, T(y))) &= [\ln(1 - y_1) + (1 - y_1)] - [\ln(1 - y_2) + (1 - y_2)] \\ &= \ln(\frac{1 - y_1}{1 - y_2}) + (y_2 - y_1) \geq 2 > 1 = \tau. \end{aligned}$$

Hence, for all $x, y \in X$ with $x \prec y$, $x \neq T(x)$ and $y \neq T(y)$, we have

$$\tau + F(d(y, T(y))) \leq F(d(x, T(x))).$$

Thus, all the conditions of Theorem 1 are satisfied. Any element of E^c is a fixed point of T.

The following example clarifies Theorem 1, where the space is not complete.

Example 9. *Let $X = C[0, 1]$ be equipped with the metric d defined as $d(f, g) = \int_a^b | f - g | dx$. For $f, g \in X$, $f \preceq g$ iff $f(t) \leq g(t)$ for each $t \in [0, 1]$. Note that $(X, d, \preceq)$ is an ordered metric space having t-property, but it is not complete. Let $B = \{f_n(t) : f_{n+1}(t) = 3f_n(t) + 1$ for $n \geq 0$ and $f_0(t) = -1$, for each $t \in [0, 1]\}$ be a subset of X. Define $T : X \to X$ by*

$$T(f(t)) = \begin{cases} 3f(t) + 1, & f(t) \in B \\ f(t), & f(t) \in B^c, \end{cases}$$

for $t \in [0,1]$*. Clearly,* T *is non-decreasing. We prove that* T *is an* F_t*-contraction. Let* $f = f(t), g = g(t) \in X$ *with* $f \prec g$*,* $f \neq T(f)$ *and* $g \neq T(g)$*. Then,* $f, g \in B$ *and* $g(t) - f(t) \geq 1$ *for each* $t \in [0,1]$*. We have*

$$d(f, T(f)) = -(2f(t) + 1),$$

and

$$d(g, T(g)) = -(2g(t) + 1).$$

Consider $F(\alpha) = \ln(\alpha) + \alpha$ *and* $\tau = 1$*. We have*

$$\begin{aligned} F(d(f,T(f))) - F(d(g,T(g))) &= [\ln\{-(2f(t)+1)\} - (2f(t)+1)] - [\ln\{-(2g(t)+1)\} - (2g(t)+1)] \\ &= \ln\left(\frac{-(2f(t)+1)}{-(2g(t)+1)}\right) + 2(g(t) - f(t)) \geq 2 > 1 = \tau. \end{aligned}$$

Thus, for all $f, g \in X$ *with* $f \prec g$*,* $f \neq T(f)$ *and* $g \neq T(g)$*, we have*

$$\tau + F(d(g, T(g))) \leq F(d(f, T(f))).$$

Hence, all the conditions of Theorem 1 are satisfied. B^c *is the set of fixed points of* T*.*

Definition 5. *A function* $\psi : [0,\infty) \to [0,\infty)$ *is said to be a sublinear altering distance function, if it satisfies the following:*
1. ψ *is monotonic increasing and continuous.*
2. $\psi(t) = 0$ *iff* $t = 0$*.*
3. $\psi(a+b) \leq \psi(a) + \psi(b)$*, for any* $a, b \in [0,\infty)$*.*

Example 10. *The map* $\psi : [0,\infty) \to [0,\infty)$ *defined by* $f(x) = ax$ *(*$a > 0$*) is a sublinear altering function.*

Example 11. *Let us define* $\psi : [0,\infty) \to [0,\infty)$ *defined by* $f(x) = \sqrt{x}$*. Then,* ψ *is a sublinear altering function.*

Definition 6. *Let* $(X, d, \preceq)$ *be an ordered metric space and* $T : X \to X$ *be a self-mapping.* T *is said an* (ψ, ϕ, F_t)*-contraction, if for* $F \in \mathcal{F}$*, there exists* $\tau > 0$ *such that for all* $x, y \in X$ *with* $x \neq Tx$*,* $y \neq Ty$ *and* $x \prec y$*, we have*

$$\tau + F[\psi(d(y, T(y))] \leq F[\psi(d(x, T(x))) - \phi(d(x, T(x)))], \tag{7}$$

where ψ *is a sublinear altering function,* $\phi : [0,\infty) \to [0,\infty)$ *is such that* $\phi(t) = 0$ *iff* $t = 0$ *and* $\psi(t) > \phi(t)$ *for each* $t > 0$*.*

Our second fixed point result is:

Theorem 2. *Let* $(X, d, \preceq)$ *be an ordered metric space having t-property. Let* $T : X \to X$ *be an* (ψ, ϕ, F_t)*-contraction. Suppose that* T *is non-decreasing and there exists* $x_0 \in X$ *such that* $x_0 \preceq T(x_0)$*. Then,* T *has a fixed point in* X*.*

Proof. Let $x_0 \in X$ be such that $x_0 \preceq T(x_0)$. If $x_0 = T(x_0)$, the proof is completed. Otherwise, choose $x_1 = T(x_0)$ such that $x_0 \prec x_1$. Proceeding similarly as Theorem 1, we get a strictly increasing sequence $\{x_n\}$ in X such that $x_{n+1} = T(x_n)$. As $x_0 \prec x_1$, by Equation (7),

$$\tau + F[\psi(d(x_1, T(x_1))] \leq F[\psi(d(x_0, T(x_0))) - \phi(d(x_0, T(x_0)))].$$

By a property of ϕ, we get

$$F[\psi(d(x_1, T(x_1))] \leq F[\psi(d(x_0, T(x_0)))] - \tau.$$

Since $x_1 \prec x_2$, by Equation (7), we have

$$\tau + F[\psi(d(x_2, T(x_2))] \le F[\psi(d(x_1, T(x_1))) - \phi(d(x_1, T(x_1)))].$$

Again,

$$F[\psi(d(x_2, T(x_2))] \le F[\psi(d(x_0, T(x_0)))] - 2\tau.$$

Continuing in the same way, we get

$$F[\psi(d(x_n, T(x_n))] \le F[\psi(d(x_0, T(x_0)))] - n\tau. \tag{8}$$

Denote $\gamma_n = \psi(d(x_n, T(x_n)))$ for $n \in \mathbb{N}$. By (8), we obtain

$$F(\gamma_n) \le F(\gamma_{n-1}) - \tau \le ... \le F(\gamma_0) - n\tau. \tag{9}$$

We get $\lim_{n\to\infty} F(\gamma_n) = -\infty$. By (F_2), we have

$$\lim_{n\to\infty} \gamma_n = 0. \tag{10}$$

Using (F_3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0. \tag{11}$$

By Equation (9), we have for all n

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \le -\gamma_n^k n\tau \le 0. \tag{12}$$

Letting $n \to \infty$ in Equation (12) and using Equations (10) and (11), we get

$$\lim_{n\to\infty} n\gamma_n^k = 0.$$

Hence, there exists $n_1 \in \mathbb{N}$ such that $n\gamma_n^k \le 1$ for all $n \ge n_1$, i.e.,

$$\gamma_n \le \frac{1}{n^{1/k}}. \tag{13}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Using the triangular inequality, properties of ψ and Equation (13), we have

$$\begin{aligned}
\psi(d(x_n, x_m)) &\le \psi[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m)] \\
&\le \psi[d(x_n, T(x_n))] + \psi[d(x_{n+1}, T(x_{n+1}))] + ... + \psi[d(x_{m-1}, T(x_{m-1}))] \\
&= \gamma_n + \gamma_{n+1} + ... + \gamma_{m-1} \\
&= \sum_{i=n}^{m-1} \gamma_i \le \sum_{i=n}^{\infty} \gamma_i \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.
\end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} \psi[d(x_n, x_m)] = 0$. By properties of ψ, we get $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Thus, $\{x_n\}$ is a strictly increasing Cauchy sequence in X, which has the t-property, so there exists $u \in X$ such that $x_n \prec u$. If $T(u) = u$, the proof is completed. Otherwise, by Equation (7), we have

$$\tau + F[\psi(d(u, T(u))] \le F[\psi(d(x_n, T(x_n))) - \phi(d(x_n, T(x_n)))],$$

From Equation (8),

$$F[\psi(d(u,T(u))] \leq F[\psi(d(x_0,T(x_0))) - (n+1)\tau.$$

Hence, $\lim_{n\to\infty} F[\psi(d(u,T(u))] = -\infty$. By (F_2), we have $\lim_{n\to\infty} \psi(d(u,T(u)) = 0$. This implies that $d(u,T(u)) = 0$, *i.e.*, $T(u) = u$. Hence, u is a fixed point of T in X. $\square$

Example 12. *Let $A = \{a_n : \ a_{n+1} = 4a_n + 1$ for $n \geq 0$ and $a_0 = -1\}$ and $B = (-1,0] \cap \mathbb{Q}$. Take $X = A \cup B$, then $X = \{..., -43, -11, -3, -1\} \cup B$. Endow X with the usual metric on $\mathbb{R}$ and the natural ordering $\leq$. Clearly, $(X,d,\leq)$ has the t-property, but it is not complete. Define $T : X \to X$ by*

$$T(x) = \begin{cases} 4x+1, & if x \in A, \\ x, & if x \in B. \end{cases}$$

Then, T is non-decreasing. Further, define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = 4t$ and $\phi(t) = 2t$. We prove that Equation (7) is satisfied. Take $F(\alpha) = \ln(\alpha) + \alpha \in \mathcal{F}$ and $\tau = 4 > 0$. Let $x, y \in X$ such that $x < y$, $x \neq T(x)$ and $y \neq T(y)$. Then, $x < y \leq -1$, $d(x,T(x)) = -(3x+1)$, $d(y,T(y)) = -(3y+1)$ and $6y - 3x \geq 3$. In addition,

$$\begin{aligned} F[\psi(d(x,T(x))) - \phi(d(x,T(x)))] - F[\psi(d(y,T(y))] &= F[-2(3x+1)] - F[-4(3y+1)], \\ &= \{\ln[-2(3x+1)] - 2(3x+1)\} \\ &\quad -\{\ln[-4(3y+1)] - 4(3y+1)\}, \\ &= \ln[\frac{-2(3x+1)}{-4(3y+1)}] - 6x + 12y + 2, \\ &\geq (6y-3x) + 1, \\ &\geq 4 = \tau. \end{aligned}$$

Thus, all the conditions of Theorem 2 hold, so there exists a fixed point of T in X.

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