Hindawi
Journal of Mathematics
Volume 2022, Article ID 4288286, 18 pages
https://doi.org/10.1155/2022/4288286

Research Article

A New Modified Kumaraswamy Distribution: Actuarial Measures and Applications

Ayed R. A. Alanzi[1], **M. Qaisar Rafique**[2], **M. H. Tahir**[2], **Waqas Sami**[3] and **Farrukh Jamal**[2]

[1]Department of Mathematics, College of Science and Arts in Gurayat, Jouf University, Gurayat 77454, Saudi Arabia
[2]Department of Statistics, Faculty of Computing, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan
[3]College of Nursing, QU Health, Qatar University, Doha, P.O. Box 2713, Qatar

Correspondence should be addressed to Farrukh Jamal; farrukh.jamal@iub.edu.pk

Received 1 February 2022; Revised 8 December 2022; Accepted 16 December 2022; Published 31 December 2022

Academic Editor: Jun Fan

In this paper, a new modified Kumaraswamy distribution is proposed, and some of its basic properties are presented, such as the mathematical expressions for the moments, probability weighted moments, order statistics, quantile function, reliability, and entropy measures. The parameter estimation is done via the maximum likelihood estimation method. In order to show the usefulness of the proposed model, some well-established actuarial measures such as value-at-risk, expected-shortfall, tail-value-at-risk, tail-variance, and tail-variance-premium are obtained. A simulation study is carried out to assess the performance of maximum likelihood estimates. The empirical analysis is carried out to show that our proposed model is better in performance as compared to other competitive models related to the extended Kumaraswamy model. Thus, insurance claim data and engineering related real-life data sets are considered to prove this claim.

1. Introduction

The discipline of actuaries, the actuarial statistics, has also received increased attention in statistical science with the existence of agricultural statistics, mathematical statistics, medical statistics, bio-statistics, computational statistics, reliability analysis, and survival analysis. The actuaries are always looking for ways to model insurance risk data using heavy-tailed and other models. According to some researchers, the insurance risk data may be unimodal [1], positively skewed [2], or having a longer tail [3]. It has also been claimed by many authors that the heavy-tailed distributions are better for estimating risk from insurance risk data and sometimes perform better as compared to other existing models. In order to improve risk assessment, there is always a need for a flexible model that can provide better estimates of well-established actuarial measures, and also provide a better goodness-of-fit to actuarial data sets. Such adaptable models may entice more researchers and practitioners, who are always on the lookout for ways to reduce their losses in terms of insurance risk or risk returns.

Modern distribution theory also emphasizes on the development or proposal of new models, which can be extended, generalized, or modified. Some new models which are applied to claimed data sets have been reported in recent literature, for example, Ahmad et al. [4] defined the exponentiated power Weibull distribution, which is based on heavy-tailed models and has applications in medical care insurance and vehicle insurance. Then, Afify et al. [5] proposed a new heavy-tailed exponential distribution with application to unemployment claim data. Furthermore, some new unit models have been developed to model different phenomenons in [6–20].

P. Kumaraswamy introduced the well-known Kumaraswamy (Kw) distribution in 1980 with the application to hydrology. The probability density function (pdf) and cumulative distribution function (cdf) for unit support (0, 1) are given by the following equations:

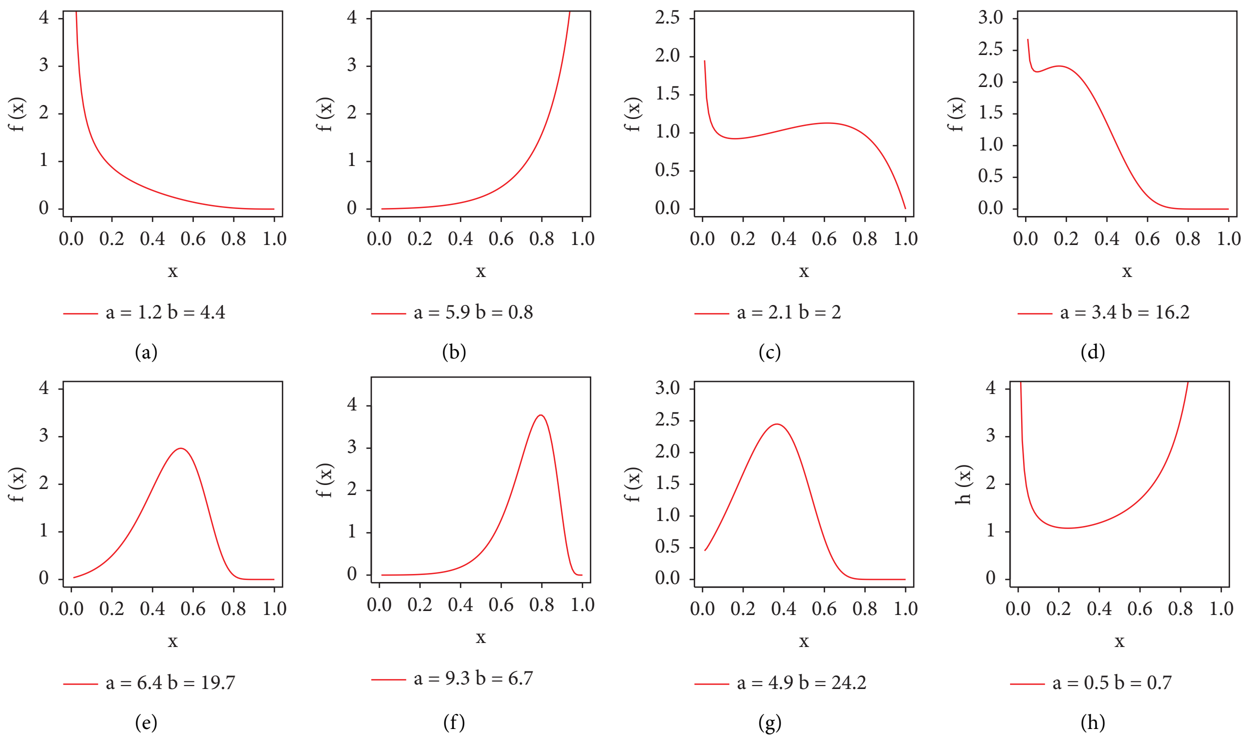


Figure 1: Plots of MKw pdf for some different parametric values.

$$f(t) = ab^{a-1}\left(1-t^{a}\right)^{b-1}, t \in (0,1), \tag{1}$$

and

$$F(t) = 1-\left(1-t^{a}\right)^{b}, \tag{2}$$

respectively, where both parameters $a > 0$ and $b > 0$ are shapes parameters, and the corresponding rv (rv) having pdf (1) is denoted by $t \sim \mathrm{Kw}(a, b)$. The Kw model exhibits flexible shapes such as unimodal (symmetrical, left-skewed, and right-skewed), bathtub, J, and reversed-J (uni-antimodal). The hazard rate shapes of the Kumaraswamy model are increasing and bathtub-shaped which are useful for investigations of the lifetime and reliability phenomenon. Only a few authors [21, 22] have explored some more properties of the Kw model that are not addressed in the original novel paper. If we consider the Kw model to have unit support, then a few models for Kw's power function distribution have been reported in the literature [23]. The Kumaraswamy exponentiated Weibull was studied by [24]. In addition to that, Tahir et al. and Ramzan et al. [25, 26] also presented new Kumaraswamy models and extended generalized inverse Kumaraswamy models, respectively. There is also a modified Kw distribution introduced by Alshkaki [27]. However, none of the authors has investigated actuarial data using the Kw distribution or some of its modified versions.

As a result, in this article, we attempted to bridge the gap by employing the proposed Kw model to assess actuarial data, and report computation results of actuarial measures. Thus, we propose the first "transformation of the Kw distribution for a new unit distribution" based on novel variable transformation that can be written as "$t = (1-\log y)^{-1}$" (further details will be provided later). More specifically, we modified the functionalities of the former Kw distribution in a totally new way, giving new possibilities for the pdf (a quick look to Figure 1 shows a lot) flexibility on the basis of our model. We investigate different phenomenon, including those in actuarial science and engineering to complete the objective of our paper.

The organization of our paper is as follows: In Section 2, the proposal for a new modified Kumaraswamy distribution (MKw) is presented, while in Section 3 some basic mathematical properties of the proposed MKw model are discussed, including the linear representation of the pdf, the quantile function, the expression of moments, probability weighted moments, the pdf of order statistics, stress-strength-reliability, and entropies. In Sections 4 and 5, the parameter estimation of MKw is dealt, and then a simulation study is conducted to assess the parameters performance of the proposed model. In Section 6, some well-established actuarial measures such as value-at-risk (VaR), expected shortfall (ES), tail-variance (TV), tail-value-at-risk (TVaR), and tail-variance premium (TVP) are obtained. The empirical investigation is carried out in Section 7, where the usefulness of the proposed MKw model is shown by analyzing five real-life data sets. Section 8 concludes our paper final remarks.

2. New Modified Kumaraswamy Distribution

The new modified Kw (MKw) distribution is derived from the Kw distribution by using the following original variable: $t = (1-\log y)^{-1}$ in the cdf. Hence, based on equation (2),

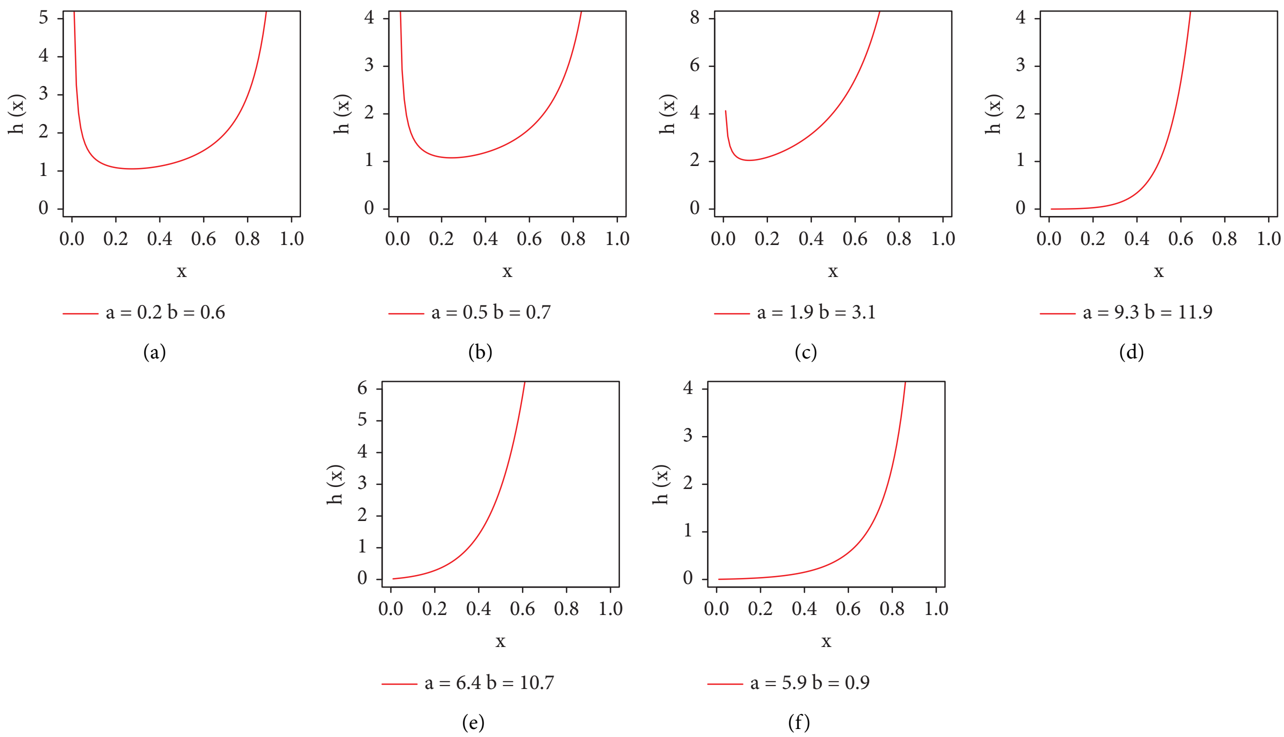


Figure 2: Plots of MKw hrf for some different parametric values.

the cdf and pdf are, respectively, given by the following equations:

$$F_{\mathrm{MKw}}(y) = F\left((1-\log y)^{-1}\right) = 1-\left[1-(1-\log y)^{-a}\right]^{b}, \tag{3}$$

$$f_{\mathrm{MKw}}(y) = aby^{-1}(1-\log y)^{-(a+1)}\left[1-(1-\log y)^{-a}\right]^{b-1}, y \in (0,1). \tag{4}$$

To the best of our knowledge, it is the first "transformation for a new unit distribution" based on the transformation $t = (1-\log y)^{-1}$, opening some new horizon of modeling. The rv associated with pdf (4) is denoted by $Y \sim \mathrm{MKw}(a,b)$, having same shape parameters $a>0$ and $b>0$. The survival hazard rate and cumulative hazard rate functions of MKw model are, respectively, given by the following equations:

$$S_{\mathrm{MKw}}(y) = \left[1-(1-\log y)^{-a}\right]^{b},$$

$$h_{\mathrm{MKw}}(y) = \frac{ab(1-\log y)^{-(a+1)}}{y\left[1-(1-\log y)^{-a}\right]}, \tag{5}$$

$$H_{\mathrm{MKw}}(y) = -b\ln\left[1-(1-\log y)^{-a}\right].$$

The possible shapes of the pdf and hrf of the newly proposed model are displayed in Figures 1 and 2. The pdf of MKw distribution exhibits flexible shapes such as unimodal (right-skewed, symmetrical, and left-skewed), reversed-*J*, *J*, and bathtub (uniantimodal). The hrf of MKw distribution exhibits flexible shapes such as increasing function and bathtub shape.

3. Properties of the MKw Distribution

Several mathematical properties of the newly investigated MKw distribution are reported in the following section.

3.1. Quantile Function. The quantile function (qf) is important for obtaining information about the median and other positional measures. Furthermore, qf is also an important tool for generating random variates. The qf of the proposed family after inverting equation (3) becomes as follows:

$$Q_Y(u;a,b) = \exp\left[1-\left\{1-(1-u)^{1/b}\right\}^{-1/a}\right], u \in (0,1). \tag{6}$$

3.2. Analytical Shapes of the pdf and hrf. Ignoring the dependence of parameters, the shapes of the pdf as well as hrf can be viewed analytically. The solutions of the following equations give the critical point of the pdf as follows:

$$-\frac{1}{y}+\frac{a+1}{y(1-\log y)}-\frac{a(b-1)[1-\log y]^{-(a+1)}}{1-[1-\log y]^{-a}}=0, \tag{7}$$

and the solutions of the following equations give the critical point of the hrf as follows:

$$-\frac{1}{y}+\frac{a+1}{y(1-\log y)}-\frac{a[1-\log y]^{-(a+1)}}{y\{1-[1-\log y]^{-a}\}}=0. \tag{8}$$

It can be noted that the parameter b has no influence on the solutions of the above equations.

3.3. Expansion of the MKw pdf. We derive the linear expansion of the MKw pdf by using the generalized binomial expansion $(1-z)^p=\sum_{i=0}^{\infty}(-1)^i\binom{p}{i}z^i$ twice in equation (3), which becomes as follows:

$$F_{\mathrm{MKw}}(y)=1-\sum_{i=0}^{\infty}(-1)^i\binom{b}{i}\sum_{j=0}^{\infty}\binom{-ia}{j}[-\log y]^j. \tag{9}$$

By separating the null values for the indices, we obtain the following equation:

$$F_{\mathrm{MKw}}(y)=-\sum_{i=1}^{\infty}(-1)^i\binom{b}{i}-\sum_{i,j=1}^{\infty}(-1)^i\binom{b}{i}\binom{-ia}{j}[-\log y]^j. \tag{10}$$

Furthermore, we use a result given by Castellares and Lemonte (2014, Proposition 2), which states that

$$[-\log(1-z)]^{\delta}=\sum_{m=0}^{\infty}\rho_m(\delta)z^{m+\delta}, \tag{11}$$

where $\delta\in\mathbb{R}$, $|z|<1$, $\rho_0(\delta)=1$, $\rho_m(\delta)=\delta\psi_{m-1}(m+\delta-1)$ for $m\geq 1$, and $\psi_m(\cdot)$ are Stirling polynomials. The first four polynomials are $\psi_0(w)=1/2$, $\psi_1(w)=(2+3w)/24$, $\psi_2(w)=(w+w^2)/48$, and $\psi_3(w)=(-8-10w+15w^2+15w^3)/5760$.

By using equation (11) with y in place of $(1-z)$, we rewrite equation (10) as follows:

$$F_{\mathrm{MKw}}(y)=\sum_{i=1}^{\infty}(-1)^{i+1}\binom{b}{i}-\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}v_{m,j}[1-y]^{m+j}, \tag{12}$$

where $v_{m,j}=v_{m,j}(a,b)=\sum_{i=1}^{\infty}(-1)^i\binom{b}{i}\binom{-ia}{j}\rho_m(j)$ (for $m\geq 0$ and $j\geq 1$).

Changing indices $s=m+j$, we can rewrite $F(x)$ as follows:

$$F_{\mathrm{MKw}}(y)=\sum_{i=1}^{\infty}(-1)^{i+1}\binom{b}{i}-\sum_{m=0}^{\infty}\sum_{s=m+1}^{\infty}v_{m,s-m}[1-y]^s, \tag{13}$$

and by interchanging the sums, we obtain the following equation:

$$F_{\mathrm{MKw}}(y)=\sum_{i=1}^{\infty}(-1)^{i+1}\binom{b}{i}-\sum_{s=1}^{\infty}\sum_{m=0}^{s-1}v_{m,s-m}[1-y]^s. \tag{14}$$

By expanding through binomial and interchanging the sums, we can write as follows:

$$F_{\mathrm{MKw}}(y)=\sum_{i=1}^{\infty}(-1)^{i+1}\binom{b}{i}-\sum_{s=1}^{\infty}\sum_{l=0}^{s}(-1)^l\binom{s}{l}t_s y^l, \tag{15}$$

where $t_s=\sum_{m=0}^{s-1}v_{m,s-m}$ (for $s\geq 1$).

After interchanging sums, we get the following equation:

$$F_{\mathrm{MKw}}(y)=\sum_{i=1}^{\infty}(-1)^{i+1}\binom{b}{i}+\sum_{l=0}^{\infty}\varpi_l y^l, \tag{16}$$

where (for $l\geq 0$)

$$\begin{aligned}\varpi_l&=\sum_{s=\delta_l}^{\infty}(-1)^{l+1}\binom{s}{l}t_s,\\ \delta_0&=1,\text{and}\\ \delta_l&=l,\ (for\,(l\geq 1)).\end{aligned} \tag{17}$$

By differentiating $F_{\mathrm{MKw}}(y)$, we have the following equation:

$$f_{\mathrm{MKw}}(y)=\sum_{l=0}^{\infty}\varpi_l l y^{l-1}. \tag{18}$$

3.4. Moments. The r th raw or ordinary moment of Y, say $\mathbb{E}(Y^r)$, can be yielded by using the following definition:

$$\mu_r'=\mathbb{E}(Y^r)=\int_0^{\infty}y^r f_{\mathrm{MKw}}(y)\,\mathrm{d}y. \tag{19}$$

By using equation (18), the r th moment expression for the MKw distribution will be as follows:

$$\mu_r'=\sum_{l=0}^{\infty}\varpi_l l\int_0^1 y^{r+l-1}\mathrm{d}y, \tag{20}$$

and

$$\mu_r'=\sum_{l=0}^{\infty}\varpi_l l B(r+l,1), \tag{21}$$

where $B(.)$ denotes the beta function of the first kind. Furthermore, the actual or mean moments and cumulants of Y yielded from equation (21) are as follows:

$$\mu_r = \sum_{s=0}^{n} (-1)^s \binom{r}{s} \mu_1'^s \mu_{r-s}',$$
$$\kappa_r = \mu_n' \sum_{s=1}^{r-1} \binom{r-1}{s-1} \kappa_s \mu_{r-s}'. \tag{22}$$

Here, $\kappa_1 = \mu_1'$. However, by using the relationship between mean moments and ordinary moments, the measure of skewness as well as measure of kurtosis can be obtained. The r th descending factorial moment of Y (for $r = 1, 2, \ldots$) is as follows:

$$\mu_r' = \mathbb{E}\left[Y^{(r)}\right] = \mathbb{E}\left[Y(Y-1)\times\cdots\times(Y-r+1)\right],$$
$$= \sum_{k=0}^{r} s(r,k)\mu_k', \tag{23}$$

where $s(r,k) = (k!)^{-1}|\mathrm{d}^k k^{(r)}/\mathrm{d}y^k|_{y=0}$ is the first kind Stirling number.

Table 1 provides the results of the first four raw moments, variance, skewness, and kurtosis under different parametric values (a,b). The graphical illustration of skewness and kurtosis is shown in Figures 3 and 4 depending on the parameters a and b.

The r th incomplete moment of MKw distribution can be expressed as follows:

$$I_r(y) = \sum_{l=0}^{\infty} \varpi_l \frac{l y^{r+l}}{r+l}. \tag{24}$$

The incomplete moments are used to compute the well-known curves, namely, the Bonferroni and Lorenz curves. Incomplete moments can also be used to calculate mean waiting time and mean residual life.

3.5. Probability Weighted Moments. The authors [28] introduced the idea of computing probability waiting moments (PWMs). PWMs are the expected function of any rv with existing means. For $r \geq 1, q \geq 0$, the (r,q) th PWMs of Y is defined by the following equation:

$$\rho_{r,q} = \mathbb{E}\left[Y^r F(y)^q\right] = \int_0^{\infty} y^r F_{\mathrm{MKw}}(y)^q f_{\mathrm{MKw}}(y)\mathrm{d}y. \tag{25}$$

Inserting equations (3) and (4) in equation (25), we have the following equation:

$$\rho_{r,q} = ab\int_0^1 y^r y^{-1}\left[1-\log y\right]^{-(a+1)}\left\{1-\left[1-\log y\right]^{-a}\right\}^{b-1} \times\left[1-\left\{1-\left[1-\log y\right]^{-a}\right\}^b\right]^q \mathrm{d}y. \tag{26}$$

After using binomial series expansions by using similar fashion in the expansion of the pdf of the MKw distribution, then the expression for $\rho_{r,q}$ can be expressed as follows:

$$\rho_{r,q} = \sum_{l=0}^{\infty} \eta_l \int_0^1 y^{l+r-1}\mathrm{d}y,$$
$$= \sum_{l=0}^{\infty} \eta_l \frac{1}{l+r}, \tag{27}$$

where

$$\eta_l = \sum_{i,j,k=0}^{\infty}\sum_{s=1}^{\infty} (-1)^{i+j+k+l} ab \binom{q}{i}\binom{b(i+1)-1}{j}\binom{-ai-a-1}{k}\binom{s}{l} t_s. \tag{28}$$

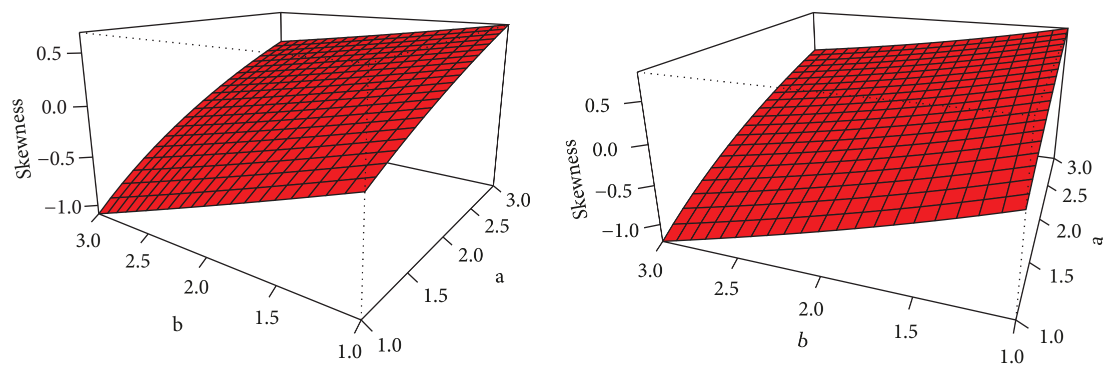
3.6. Order Statistics. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the MKw (a,b) distribution. Then, the pdf of the r th order statistics is as follows:

$$f_{r;n}(y) = \frac{n!}{(r-1)!(n-r)!}\sum_{i=0}^{\infty}(-1)^i\binom{n-r}{i} F_{\mathrm{MKw}}(y)^{i+r-1} f_{\mathrm{MKw}}(y). \tag{29}$$

Inserting equations (3) and (4) in equation (29), we get the following equation:

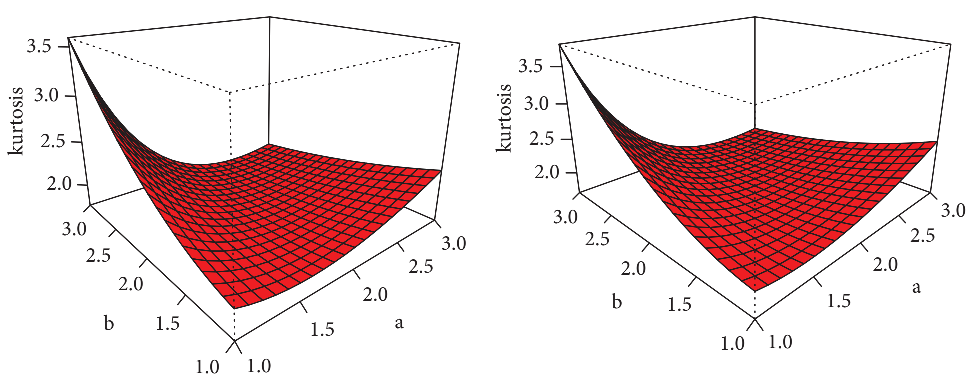
Table 1: The rth moments, variance, skewness, and kurtosis of MKw (a, b) distribution for different parameter values.

Parameters	μ_1'	μ_2'	μ_3'	μ_4'	Variance	Skewness	Kurtosis
(3.1, 2.5)	0.5305	0.3416	0.2399	0.1778	0.0602	0.1222	2.1939
(1.8, 2.4)	0.3449	0.1907	0.1222	0.0851	0.0717	0.1304	1.9760
(4.1, 2.7)	0.6108	0.4195	0.3066	0.2336	0.0464	0.4019	2.7795
(3.5, 1.5)	0.6678	0.5010	0.3977	0.3273	0.0550	0.6465	2.9301
(2.9, 5.3)	0.3516	0.1703	0.0945	0.0570	0.0467	0.0318	2.0953
(3.5, 4.2)	0.4729	0.2726	0.1729	0.1166	0.0490	0.0460	2.2029



(a) (b)

Figure 3: Plots of MKw skewness for some parametric values.



(a) (b)

Figure 4: Plots of MKw kurtosis for some parametric values.

After simplification, we get n th order statistic density as follows:

$$f_{r;n}(y) = \frac{n!}{(r-1)!(n-r)!} \sum_{i=0}^{\infty} (-1)^i a b y^{-1} [1 - \log y]^{-(a+1)} \{1 - [1 - \log y]^{-a}\}^{b-1} \times \left[1 - \{1 - [1 - \log y]^{-a}\}^b\right]^{i+r-1}. \tag{30}$$

$$f_{r;n}(y) = \sum_{p=0}^{\infty} \eta_p y^p, \tag{31}$$

where

$$\eta_k = \sum_{i,j,k,l=0}^{\infty}\sum_{s=1}^{\infty}(-1)^{i+j+k+l+p}\binom{n-r}{i}\binom{i+r-1}{j}\binom{b(j+1)-1}{l}\times\binom{-a(k+1)-1}{l}ab\binom{s}{p}t_s\frac{n!}{(r-1)!(n-r)!}. \tag{32}$$

The order statistics of the MKw distribution are expressed in terms of linear expansion. To study the distributional behavior of the set of observations, we can use a minimum and maximum (min-max) plot of the order statistics. Figure 5 represents the min-max plot that depends on extreme order statistics, and it is introduced to capture all information not only about the tails of the distribution, but also about the whole distribution of the data.

3.7. Reliability. Reliability is an important measure, and several applications are documented in the fields of economics, physical science, and engineering. Reliability enables us to determine the failure probability at a certain point in time. Let say Y_1 and Y_2 be the two *rv* following the MKw distribution. The component fails if the applied stress exceeds its strength, if $Y_1 > Y_2$ the component will perform satisfactorily. The reliability is defined by the following expression:

$$\begin{aligned}P(Y_1>Y_2) &= \int_0^1 f_2(y)[1-F_1(y)]\mathrm{d}y,\\ P(Y_1>Y_2) &= \int_0^1 a_2b_2y^{-1}[1-\log y]^{-(a_2+1)}\{1-[1-\log y]^{-a_2}\}^{b_2-1}\{1-[1-\log y]^{-a_1}\}^{b_1}\mathrm{d}y.\end{aligned} \tag{33}$$

Let $a_1 = a_2 = a$, then the above equation will be as follows:

$$P(Y_1>Y_2) = \int_0^1 ab_2y^{-1}[1-\log y]^{-(a+1)}\{1-[1-\log y]^{-a}\}^{b_2-1}\{1-[1-\log y]^{-a}\}^{b_1}\mathrm{d}y. \tag{34}$$

After solving, it gives the result as follows:

$$P(y_1>y_2) = \sum_{l=0}^{\infty}\frac{1}{l}W_l, \tag{35}$$

where

$$W_l = \sum_{i,j=0}^{\infty}\sum_{s=1}^{\infty}ab_2(-1)^{i+j+l}\binom{b_1+b_2-1}{i}\binom{-a(1+i)-1}{j}\binom{s}{l}t_s. \tag{36}$$

3.8. Entropy. Entropy measures are important for highlighting the uncertainty variation of the *rv*. Assume Y is a *rv* with pdf $f(y)$. The two important entropy measures, namely, Rényi and Shannon entropies, can be yielded by the following expressions.

3.8.1. Rényi Entropy. The Rényi entropy is defined by the following equation:

$$I(\delta) = \frac{1}{1-\delta}\log[I(\delta)], \tag{37}$$

where $I(\delta) = \int_{-\infty}^{\infty} f_{\mathrm{MKw}}(y)^{\delta}(y)\mathrm{d}y, \delta>0$, and $\delta\neq 1$.

Inserting equation (4) in $f_{\mathrm{MKw}}(y)^{\delta}(y)$ as follows:

$$f_{\mathrm{MKw}}(y)^{\delta}(y) = \left[aby^{-1}[1-\log y]^{-(a+1)}\{1-[1-\log y]^{-a}]^{b-1}\right]^{\delta}. \tag{38}$$

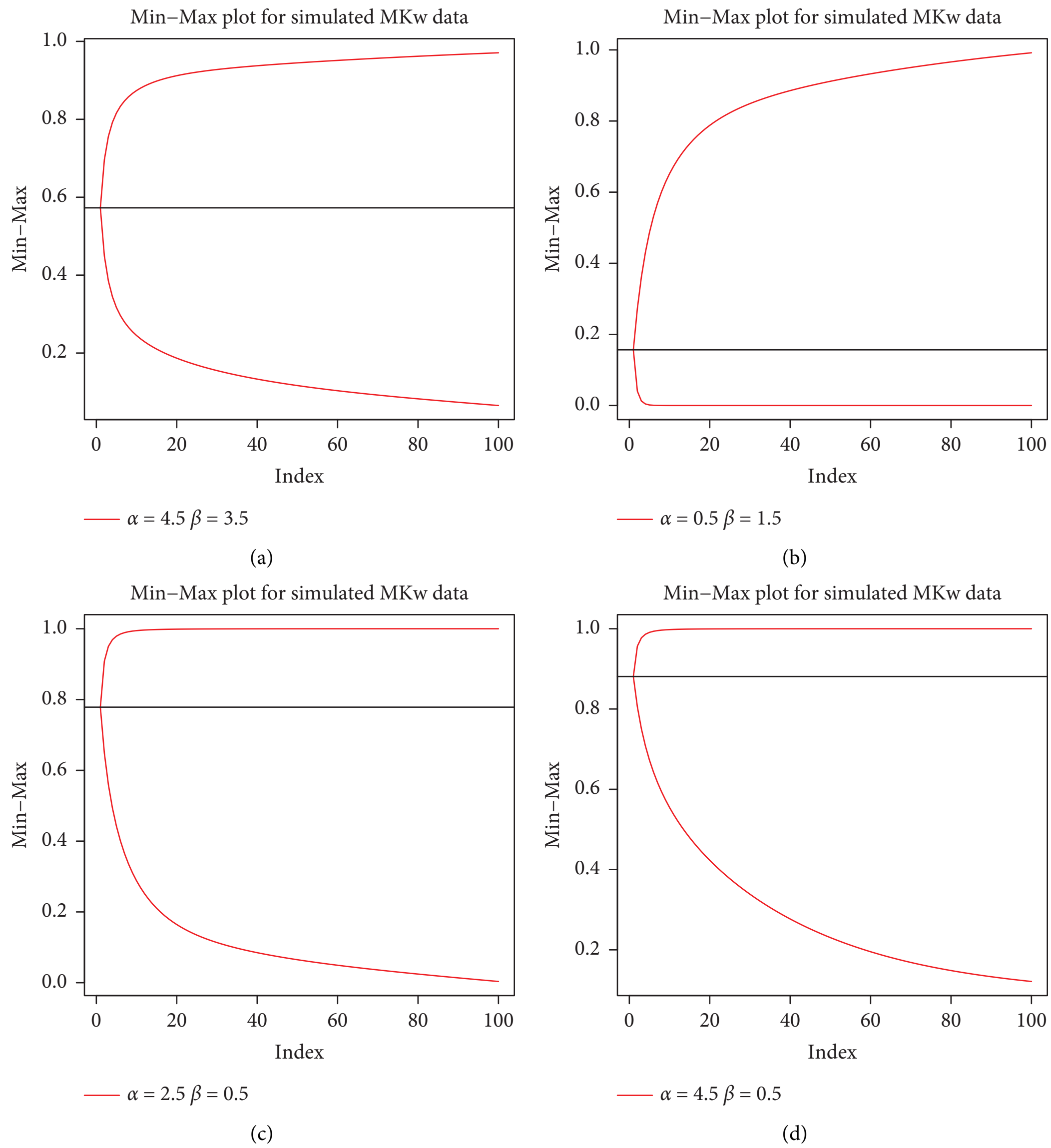


Figure 5: Min-max plot of order statistics of the MKw model for some parametric values.

By using the similar binomial series expansion as in Section 3.3, we have the following equation:

$$f_{\text{MKw}}(y)^{\delta}(y) = \sum_{l=0}^{\infty} \frac{1}{l-\delta+1}\omega_l. \tag{39}$$

After incorporating the result in equation (37), the expression for Rényi entropy is reduced as follows:

$$I_{\delta}(f) = \frac{1}{1-\delta}\log\left[\sum_{l=0}^{\infty}\omega_l \frac{1}{l-\delta+1}\right], \tag{40}$$

where

$$\omega_l = \sum_{i,j=0}^{\infty}\sum_{s=1}^{\infty}(-1)^{i+j+l}\binom{\delta(b-1)}{i}\binom{-ai-a\delta-\delta}{j}\binom{s}{l}t_s. \tag{41}$$

3.8.2. Shannon Entropy. The Shannon entropy is obtained as follows:

$$
\begin{aligned}
\rho_y &= \mathbb{E}\{-\log[f(y)]\} = \mathbb{E}\left[-\log\left\{aby^{-1}[1-\log y]^{-(a+1)}\{1-[1-\log y]^{-a}\}^{b-1}\right\}\right],\\
\mathbb{E}[-\log(ab)] &= -\log(ab),\\
\mathbb{E}[\log y] &= \sum_{l=0}^{\infty}\varpi_l\frac{1}{l},
\end{aligned}
\tag{42}
$$

where

$$
\begin{aligned}
\varpi_l &= ab\sum_{i,j=0}^{\infty}\sum_{s=1}^{\infty}(-1)^{i+j+l}\binom{b-1}{i}\binom{-a(i+1)-1}{j}\binom{s}{l}t_s,\\
(a+1)\mathbb{E}[\log\{1-\log y\}] &= \sum_{l=0}^{\infty}\varpi_l\frac{1}{l},
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
\varpi_l &= (a+1)ab\sum_{i,j=0}^{\infty}\sum_{k,s=1}^{\infty}\frac{(-1)^{i+j+l+2k+1}}{k}\binom{b-1}{i}\binom{-a(i+1)-1}{j}\binom{s}{l}t_s,\\
(b-1)\mathbb{E}\left[\log\left(1-\{1-\log y\}^{-a}\right)\right] &= \sum_{l=0}^{\infty}\varpi_l\frac{1}{l},
\end{aligned}
\tag{44}
$$

in which

$$
\varpi_l = (b-1)ab\sum_{i,j=0}^{\infty}\sum_{k,s=1}^{\infty}\frac{(-1)^{i+j+l+2k+1}}{k}\binom{b-1}{i}\binom{-a(k+i+1)-1}{j}\binom{s}{l}t_s. \tag{45}
$$

4. Estimation

In this section, we estimate the unknown parameters of the MKw model using the widely used estimation method known as maximum likelihood estimation (MLE). There are several advantages of MLE over other estimation methods, for instance, maximum likelihood estimates fulfill the required properties that can be used in constructing confidence intervals, as well as delivering a simple approximation that is very handy while working with the finite sample. The well-known R package called "adequacymodel" is implemented to estimate the unknown parameters in the application section. The log-likelihood function $\ell(\Omega)$ for the vector of parameters $\Omega = (a,b)^{\top}$ can be expressed as follows:

$$L(\boldsymbol{\Omega}) = n\log(ab) - \sum_{i=1}^{n}\log(y_i) + (b-1)\sum_{i=1}^{n}\log\{1-[1-\log y_i]\}^{-a}, \\ -(a+1)\sum_{i=1}^{n}\log[1-\log y_i]. \tag{46}$$

The score components are as follows:

$$\frac{\partial L}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\log[1-\log y_i] + (b-1)\sum_{i=1}^{n}\frac{[1-\log y_i]^{-a}\log[1-\log y_i]}{1-[1-\log y_i]^{-a}}, \\ \frac{\partial L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\{1-[1-\log y_i]\}^{-a}. \tag{47}$$

By setting these equations to zero and solving them simultaneously, the MLE of the model parameters is obtained.

5. Simulation Study

This section yielded a simulation study in order to test the performance of MLEs in the newly proposed MKw distribution. N is replicated 1000 times with various sample sizes, $n = 50$, 100, 200, 300, 400, and 500 of the MKw model by taking $a = 1.90, b = 1.40$; $a = 2.10, b = 2.40$; $a = 3.10$, $b = 2.90$; $a = 3.50, b = 3.90$; and $a = 3.0, b = 3.0$.

The calculation of estimates is based on the bias, mean square error (MSE), and average estimate (AE) of the MLEs of the model parameters, namely,

$$\text{Bias}(\widehat{\alpha}) = \sum_{i=1}^{N}\frac{\widehat{\alpha}_i}{N} - \alpha, \\ \text{MSE}(\widehat{\alpha}) = \sum_{i=1}^{N}\frac{(\widehat{\alpha}_i - \alpha)^2}{N}. \tag{48}$$

The R programming language is used for the empirical study, and the results of Tables 2–6 show that as sample sizes increase, both the mean square error and the bias reduce. Thus, MLEs perform well in evaluating the parameters of the MKw distribution.

6. Actuarial Measures

6.1. Value-at-Risk. The Value-at-Risk (VaR), also known as quantile risk or simply "VaR," is extensively used as a standard final market risk measure. It plays an important role in many business decisions; the uncertainty regarding foreign markets, commodity prices, and government policies can significantly affect firm earnings. The loss portfolio value is defined by a level of confidence, such as $q = (90\%, 95\%, \text{or} 99\%)$. For the MKw model, VaR is defined by the following expression:

$$\%\ Q_Y(q; a, b) = \exp\left[1 - \left\{1-(1-q)^{1/b}\right\}^{-(1/a)}\right], q \in (0,1). \tag{49}$$

6.2. Expected-Shortfall. One of the other measures is called expected shortfall (ES), which is considered as a better measure than VaR introduced by [29]. The ES can be yielded by the following equation:

$$\text{ES}_q(y) = \frac{1}{q}\int_0^q \text{VaR}_y\, \mathrm{d}y, \tag{50}$$

for $0 < q < 1$. Then, we have the following equation:

$$\text{ES}_q(y) = \frac{1}{q}\int_0^q \exp\left[1 - \left\{1-(1-y)^{1/b}\right\}^{-(1/a)}\right]\mathrm{d}y. \tag{51}$$

6.3. Tail-Value-at-Risk. The problem of risk measurement is one of the most important problems in risk management. Tail-value-at-risk (TVaR) or conditional tail expectation is an important measure in finance and insurance that is defined as the expected value of the loss if the loss is greater than the VaR. Its mathematical expression is as follows:

$$\text{TVaR}_q(y) = \frac{1}{1-q}\int_{\text{VaR}_q}^{1} y f_{\text{MKw}}(y)\mathrm{d}y. \tag{52}$$

By inserting equation (18) in equation (52), we get the TVaR as follows:

$$\text{TVaR}_q(y) = \frac{1}{1-q}\sum_{l=0}^{\infty}\varpi_l l(l+1)^{-1}\left[1 - \text{VaR}_q^{(l+1)}\right]. \tag{53}$$

6.4. Tail-Variance. The tail-variance (TV) is defined by the following expression:

$$\text{TV}_q(y) = \mathbb{E}\left[Y^2 | Y > y_q\right] - \left[\text{TVaR}_q\right]^2. \tag{54}$$

Consider $I = \mathbb{E}[Y^2 | Y > y_q]$. Then, we have the following equation:

Table 2: Biases, MSEs, and AE for scenario-I.

	$n = 50$		$n = 100$		$n = 200$		$n = 300$		$n = 400$		$n = 500$	
	a	b	a	b	a	b	a	b	a	b	a	b
Bias	0.088	0.088	0.041	0.051	0.023	0.023	0.012	0.015	0.007	0.011	0.009	0.010
MSE	0.122	0.108	0.055	0.045	0.027	0.021	0.017	0.014	1.907	1.410	0.010	0.008
AE	1.988	1.488	1.941	1.451	1.923	1.423	1.912	1.415	0.014	0.010	1.909	1.410

Table 3: Biases, MSEs, and AE for scenario-II.

	$n = 50$		$n = 100$		$n = 200$		$n = 300$		$n = 400$		$n = 500$	
	a	b	a	b	a	b	a	b	a	b	a	b
Bias	0.079	0.189	0.045	0.081	0.023	0.028	0.013	0.027	0.010	0.017	0.009	0.015
MSE	0.115	0.422	0.056	0.165	0.027	0.071	0.017	0.050	0.013	0.037	0.010	0.030
AE	2.179	2.589	2.145	2.481	2.113	2.428	2.113	2.432	2.110	2.427	2.109	2.415

Table 4: Biases, MSEs, and AE for scenario-III.

	$n = 50$		$n = 100$		$n = 200$		$n = 300$		$n = 400$		$n = 500$	
	a	b	a	b	a	b	a	b	a	b	a	b
Bias	0.096	0.235	0.059	0.126	0.037	0.048	0.020	0.042	0.010	0.027	0.007	0.022
MSE	0.251	0.691	0.105	0.270	0.052	0.115	0.034	0.076	0.025	0.058	0.020	0.045
AE	3.196	3.135	3.159	3.026	3.117	2.948	3.120	2.942	3.110	2.927	3.091	2.922

Table 5: Biases, MSEs, and AE for scenario-IV.

	$n = 50$		$n = 100$		$n = 200$		$n = 300$		$n = 400$		$n = 500$	
	a	b	a	b	a	b	a	b	a	b	a	b
Bias	0.124	0.353	0.054	0.161	0.020	0.069	0.017	0.056	0.015	0.045	0.006	0.028
MSE	0.292	1.554	0.133	0.584	0.064	0.267	0.044	0.176	0.031	0.115	0.013	0.075
AE	3.624	4.253	3.554	4.061	3.517	3.969	3.520	3.956	3.515	3.945	3.501	3.903

Table 6: Biases, MSEs, and AE for scenario-V.

	$n = 50$		$n = 100$		$n = 200$		$n = 300$		$n = 400$		$n = 500$	
	a	b	a	b	a	b	a	b	a	b	a	b
Bias	0.102	0.235	0.066	0.138	0.025	0.052	0.020	0.043	0.010	0.021	0.008	0.013
MSE	0.215	0.732	0.112	0.331	0.048	0.129	0.032	0.084	0.025	0.065	0.021	0.050
AE	3.102	3.235	3.066	3.185	3.025	3.052	3.020	3.043	3.010	3.021	3.008	3.013

$$
\begin{aligned}
I &= \mathrm{TVaR}_q(y) = \frac{1}{1-q}\int_{\mathrm{VaR}_q}^{1} y^2 f_{\mathrm{MKw}}(y)\mathrm{d}y,\\
I &= \frac{1}{1-q}\sum_{l=0}^{\infty}\varpi_l l(l+1)^{-2}\left[1-\mathrm{VaR}_q^{(l+2)}\right].
\end{aligned}
\tag{55}
$$

Substituting equation (53) and equation (55) in equation (54), we obtain the expression for TV for the MKw model.

6.5. Tail-Variance Premium. The Tail-variance premium (TVP) is a mixture of both central tendency and dispersion statistics. It is defined by the following expression:

$$
\mathrm{TVP}_q(Y) = \mathrm{TVaR}_q + \delta\mathrm{TV}_q, \tag{56}
$$

where $0 < \delta < 1$. Using expression equation (53) and equation (54) in equation (56), we obtain the TVP for MKw model.

A sample of 100 is randomly drawn and the effect of shape and scale parameters of the proposed models are underlined for both risk measures. Various combinations of the scale and shape parameters are executed $\mathrm{I} = [a = 2.1, b = 4.2]$, $\mathrm{II} = [a = 1.8, b = 5.1]$, $\mathrm{III} = [a = 1.1, b = 3.5]$, and $\mathrm{IV} = [a = 3.8, b = 6.1]$, and change in the curve of VaR and ES are illustrated in Figure 6.

7. Applications

In this section the proposed MKw model is compared to its counterpart models Gamma Kumaraswamy(GaKw) [30], Size-Biased Kumaraswamy (SBKw) [31], Kumaraswamy

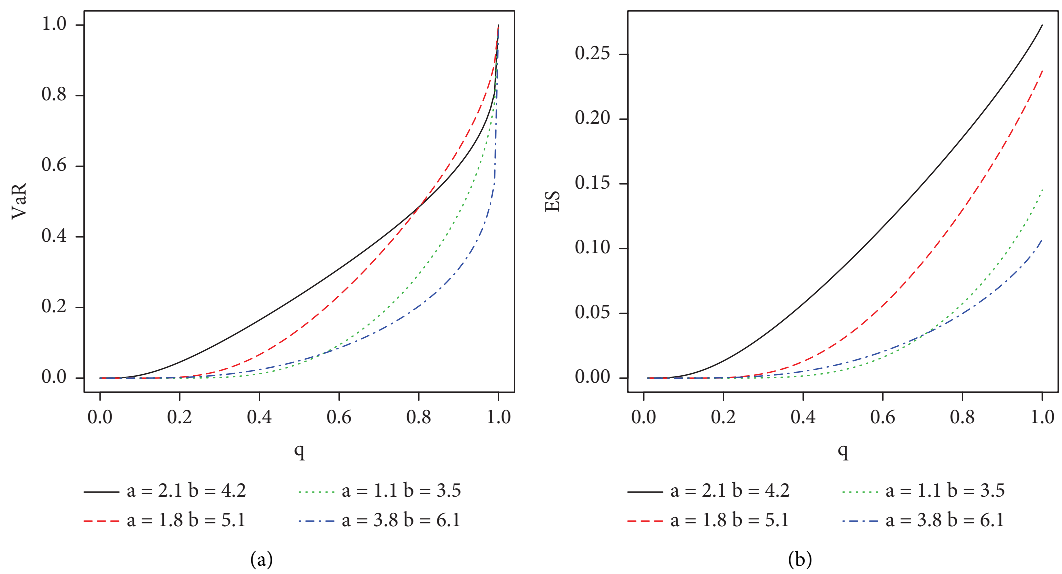


Figure 6: Plot of (a) VaR and (b) ES of the MKw model for some parametric values.

Table 7: MLEs and their standard errors (in parentheses) for data set 1.

Distribution	a	b	α	β
MKw	6.0882 (0.6881)	380.3097 (244.1106)	—	—
Kw	2.0774 (0.2548)	33.1374 (13.9216)	—	—
GaKw	0.0050 (0.0006)	0.4761 (0.2325)	0.2183 (0.4126)	38.2676 (26.4870)
SBKw	1.4472 (0.2530)	19.9669 (7.5763)	—	—
Beta	2.6826 (0.5072)	13.8658 (2.8280)	—	—

Table 8: MLEs and their standard errors (in parentheses) for data set 2.

Distribution	a	b	α	β
MKw	5.9715 (0.6781)	397.1149 (258.2666)	—	—
Kw	1.9586 (0.2441)	31.2634 (13.1620)	—	—
GaKw	0.0118 (0.0032)	0.3421 (0.0526)	0.0456 (0.0166)	58.1448 (11.3249)
SBKw	1.3155 (0.2417)	18.1894 (6.8386)	—	—
Beta	2.4004 (0.4511)	13.5216 (2.7704)	—	—

(Kw), and Beta by using the five data sets. The detailed description of the data sets are given below.

Data Set 1: Drilling Data 1: The first data set based on 50 observation of holes having diameter 12 mm and thickness of sheet 3.15 mm. The data set is also used by [32]. The following are the data observation: 0.040, 0.020, 0.060, 0.120, 0.140, 0.080, 0.220, 0.120, 0.080, 0.260, 0.240, 0.040, 0.140, 0.160, 0.080, 0.260, 0.320, 0.280, 0.140, 0.160, 0.240, 0.220, 0.120, 0.180, 0.240, 0.320, 0.160, 0.140, 0.080, 0.160, 0.240, 0.160, 0.320, 0.180, 0.240, 0.220, 0.160, 0.120, 0.240, 0.060, 0.020, 0.180, 0.220, 0.140, 0.060, 0.040, 0.140, 0.260, 0.180, and 0.160. Data Set 2: Drilling Data 2: The second data set is based on 50 observations of holes having diameter 9 mm and thickness of sheet 2 mm. The data set is also used by [32]. The following are the data observation: 0.060, 0.120, 0.140, 0.040, 0.140, 0.160, 0.080, 0.260, 0.320, 0.220, 0.160, 0.120, 0.240, 0.060, 0.020, 0.180, 0.220, 0.140, 0.220, 0.160, 0.120, 0.240, 0.060, 0.020, 0.180, 0.220, 0.140, 0.020, 0.180, 0.220, 0.140, 0.060, 0.040, 0.140, 0.220, 0.140, 0.060, 0.040, 0.160, 0.240, 0.160, 0.320, 0.180, 0.240, 0.220, 0.040, 0.140, 0.260, 0.180, and 0.160.

Data Set 3: Milk Production Data: The third data revealed the overall yield production of 107 cows at first birth of SINDI race. The data set is also used by [33]. The following are the data observation: 0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, and 0.2747.

Table 9: MLEs and their standard errors (in parentheses) for data set 3.

Distribution	a	b	α	β
MKw	4.3358 (0.3836)	6.9958 (1.3323)	—	—
Kw	2.1949 (0.2224)	3.4363 (0.5820)	—	—
GaKw	5.7675 (3.1728)	0.1429 (0.3857)	0.0246 (0.0784)	0.3087 (0.2001)
SBKw	1.3874 (0.2340)	3.0666 (0.4894)	—	—
Beta	2.4125 (0.3145)	2.8297 (0.3744)	—	—

Table 10: MLEs and their standard errors (in parentheses) for data set 4.

Distribution	a	b	α	β
MKw	13.5478 (1.2846)	2300.6454 (1625.7869)	—	—
Kw	7.4038 (0.7572)	311.4870 (175.9728)	—	—
GaKw	0.02961 (0.01302)	0.4896 (0.0805)	0.06987 (0.0336)	74.1031 (18.4789)
SBKw	6.8374 (0.7688)	236.1314 (133.0104)	—	—
Beta	16.8271 (3.0993)	22.2029 (4.1042)	—	—

Table 11: MLEs and their standard errors (in parentheses) for data set 5.

Distribution	a	b	α	β
MKw	10.6993 (1.3748)	5.2102 (1.2979)	—	—
Kw	8.3089 (1.1153)	3.9795 (0.9392)	—	—
GaKw	0.0156 (0.0094)	0.0673 (0.0105)	0.0085 (0.0007)	55.0201 (11.8362)
SBKw	7.4980 (1.1251)	3.8346 (0.8871)	—	—
Beta	11.4662 (2.1510)	3.1426 (0.5562)	—	—

Table 12: The statistics AIC, CAIC, BIC, HQIC, CvM, AD, and *K-S* for data set 1.

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	CvM	AD	*K-S*	KS *P* value
MKw	−57.0040	−110.0079	−109.7526	−106.1839	−108.5517	0.0738	0.4483	0.0911	0.8005
Kw	−56.0687	−108.1374	−107.8820	−104.3133	−106.6811	0.1023	0.6243	0.1103	0.5777
GaKw	−55.8026	−103.6052	−102.7164	−95.9572	−100.6928	0.1122	0.6821	0.1213	0.4537
SBKw	−55.2067	−106.4134	−106.1581	−102.5893	−104.9572	0.1269	0.7708	0.1237	0.4290
Beta	−54.6067	−105.2133	−104.9580	−101.3893	−103.7571	0.1479	0.8926	0.1415	0.2697

Data Set 4: Unemployment Claim Data 1: The usefulness of the proposed MKw model is determined by taking into the account a heavy tailed real data sets from insurance field. The given data was used by [5] and consisted of 58 values related to the monthly metrics on the unemployment insurance: 0.188, 0.202, 0.195, 0.385, 0.489, 0.545, 0.541, 0.535, 0.521, 0.508, 0.512, 0.507, 0.519, 0.493, 0.487, 0.460, 0.490, 0.460, 0.490, 0.500, 0.400, 0.350, 0.370, 0.410, 0.400, 0.400, 0.410, 0.400, 0.420, 0.450, 0.450, 0.420, 0.390, 0.340, 0.360, 0.400, 0.440, 0.390, 0.410, 0.450, 0.460, 0.470, 0.490, 0.460, 0.410, 0.390, 0.400, 0.440, 0.420, 0.420, 0.450, 0.470, 0.530, 0.420, 0.490, 0.440, 0.420, and 0.400.

Data Set 5: Unemployment Claim Data 2: 0.823, 0.864, 0.816, 0.841, 0.831, 0.833, 0.894, 0.869, 0.866, 0.860, 0.837, 0.826, 0.804, 0.809, 0.758, 0.770, 0.778, 0.707, 0.814, 0.825, 0.906, 0.924, 0.927, 0.920, 0.770, 0.544, 0.550, 0.608, 0.630, 0.650, 0.820, 0.873, 0.900, 0.916, 0.899, 0.862, 0.695, 0.650, 0.751, 0.862, 0.702, 0.530, 0.764, 0.898, 0.897, 0.908, 0.902, 0.879, 0.645, 0.739, 0.765, 0.803, 0.708, 0.669, 0.561, 0.579, 0.701, and 0.839.

We used the MLE method in order to find the unknown values of the MKw parameters. Several goodness-of-fit (GoF), namely, Akaike information criterion (AIC), corrected AIC (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Cramér-von Mises (CvM), Anderson–Darling (AD) and Kolmogrov–Smirnov (KS) measures were used to elect an adequate model.

Tables7–11 list the MLEs and standard error for the MKw model and other competitive distributions such as, Gamma Kumaraswamy(GaKw) [30], Size-Biased Kumaraswamy (SBKw) [31], Kumaraswamy (Kw), and Beta. While the AIC, CAIC, BIC, HQIC, and other GoFs for the MKw model and other competitive models (GaKw, SBKw, Kw, and Beta) for data sets 1, 2, 3, 4, and 5, respectively. The values of the GoFs in Tables 12–16 indicate that the MKw model shows small values of the GoFs, and thus provides the best fit as compared to the other models. The plots in Figures 7–11 also support our claim.

7.1. Numerical Illustration of VaR and ES. Here we demonstrate the numerical as well as graphical presentation of the two important risk measures, VaR and ES. The comparative

Table 13: The statistics AIC, CAIC, BIC, HQIC, CvM, AD, and K-S for data set 2.

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	CvM	AD	K-S	KS P value
MKw	−58.9481	−113.8963	−113.6409	−110.0722	−112.4400	0.1246	0.7590	0.1323	0.3458
Kw	−57.5214	−111.0428	−110.7875	−107.2188	−109.5866	0.2068	1.1717	0.1693	0.1139
GaKw	−56.1787	−104.3573	−103.4684	−96.7092	−101.4449	0.2391	1.3370	0.1824	0.0717
SBKw	−56.4011	−108.8023	−108.5469	−104.9782	−107.3460	0.2531	1.4122	0.1843	0.0670
Beta	−55.9312	−107.8624	−107.6071	−104.0384	−106.4062	0.2768	1.5347	0.1981	0.0396

Table 14: The statistics AIC, CAIC, BIC, HQIC, CvM, AD, and K-S for data set 3.

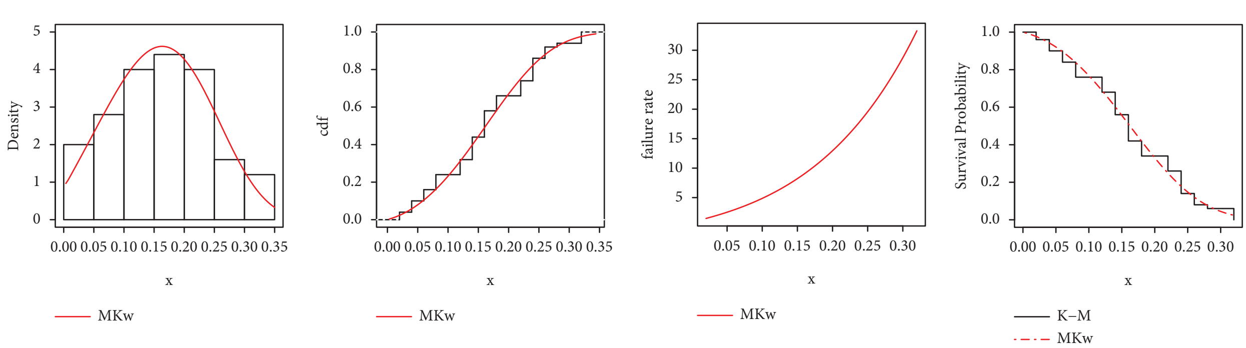
Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	CvM	AD	K-S	KS P value
MKw	−28.5855	−53.1709	−53.0556	−47.8253	−51.0039	0.0470	0.3097	0.0549	0.9033
Kw	−25.3947	−46.7894	−46.6740	−41.4437	−44.6223	0.1561	1.0090	0.0763	0.5626
GaKw	−27.6338	−47.2676	−46.8754	−36.5763	−42.9335	0.0916	0.5807	0.0747	0.5895
SBKw	−24.2495	−44.4989	−44.3836	−39.1533	−42.3319	0.1918	1.2271	0.0813	0.4788
Beta	−23.7772	−43.5545	−43.4391	−38.2088	−41.3874	0.2083	1.3263	0.0910	0.3384

Table 15: The statistics AIC, CAIC, BIC, HQIC, CvM, AD, and K-S for data set 4.

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	CvM	AD	K-S	KS P value
MKw	−74.2589	−144.5178	−144.2996	−140.3969	−142.9126	0.1066	0.7794	0.1116	0.4658
Kw	−72.7587	−141.5174	−141.2992	−137.3965	−139.9122	0.1165	0.9517	0.1156	0.4205
GaKw	−69.0701	−130.1402	−129.3854	−121.8984	−126.9298	0.1791	1.4926	0.1404	0.2033
SBKw	−72.3566	−140.7132	−140.4950	−136.5923	−139.1080	0.1199	0.9939	0.1137	0.4411
Beta	−65.5272	−127.0544	−126.8362	−122.9335	−125.4492	0.2678	2.1072	0.1686	0.0738

Table 16: The statistics AIC, CAIC, BIC, HQIC, CvM, AD, and K-S for data set 5.

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	CvM	AD	K-S	KS P value
MKw	−53.0937	−102.1874	−101.9692	−98.0665	−100.5823	0.0644	0.4753	0.0762	0.8893
Kw	−52.6921	−101.3841	−101.1659	−97.2632	−99.77896	0.0837	0.5819	0.0895	0.7421
GaKw	−52.9883	−97.97656	−97.22184	−89.7348	−94.76622	0.0659	0.4849	0.0781	0.8712
SBKw	−52.5935	−101.187	−100.9688	−97.0661	−99.58185	0.0870	0.6009	0.0920	0.7097
Beta	−52.0797	−100.1593	−99.94115	−96.0385	−98.55416	0.1064	0.7122	0.1047	0.5485



(a) (b) (c) (d)

Figure 7: Plots of estimated pdf, estimated cdf, estimated hrf, and failure rate for data set 1.

study of VaR and ES of the proposed MKw model with its counterparts (Kw, SBKw, and Beta models) is performed by taking MLEs estimates of the parameters for the models in both data sets. It is worth-emphasizing that a model with higher values of the risk measures is said to have a heavier tail. Tables 17 and 18 provide the numerical illustration of the VaR and ES for four models of data 4 and 5 and yield that the MKw model has higher values of both the risk measures as

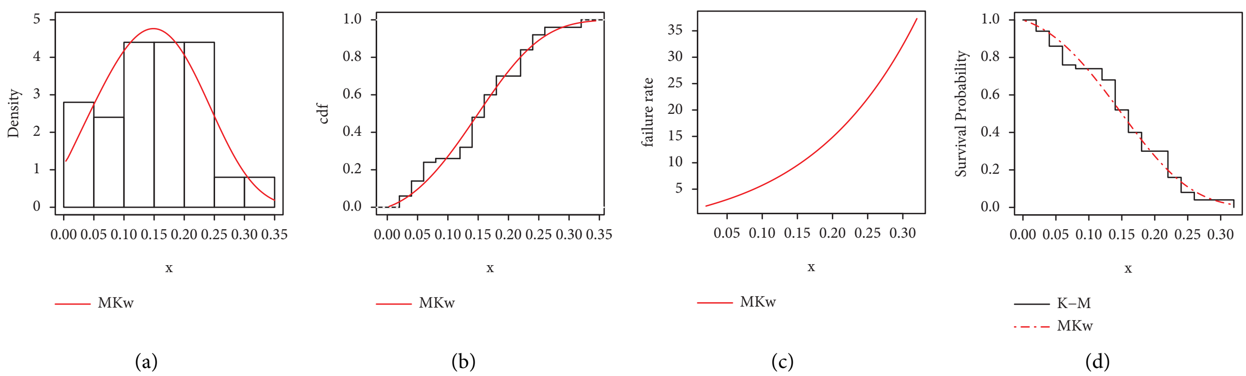


Figure 8: Plots of estimated pdf, estimated cdf, estimated hrf, and failure rate for data set 2.

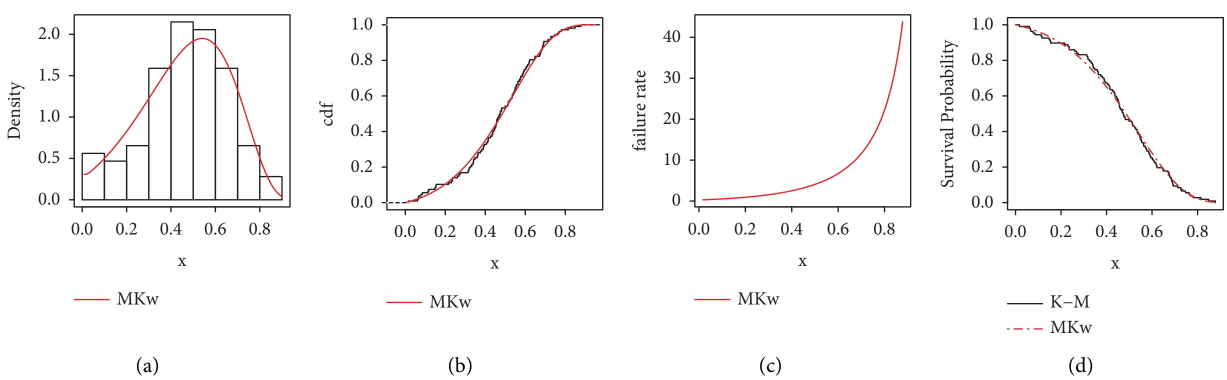


Figure 9: Plots of estimated pdf, estimated cdf, estimated hrf, and failure rate for data set 3.

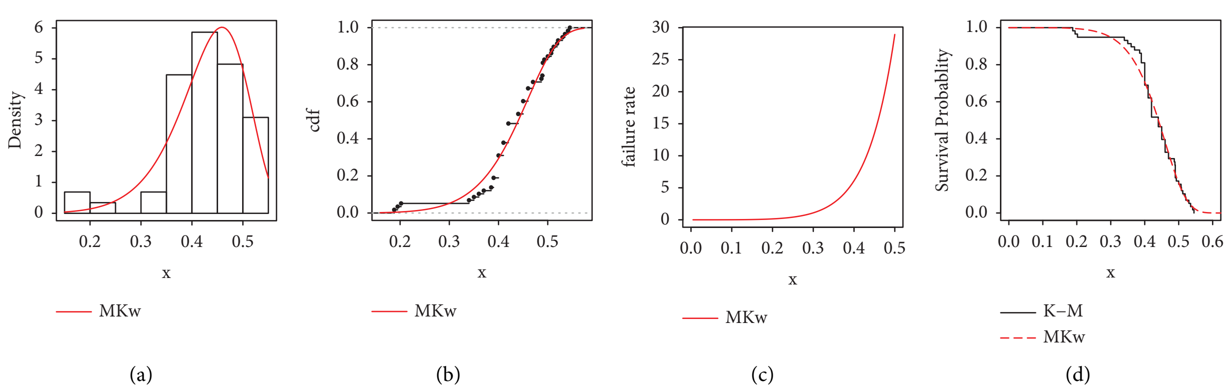


Figure 10: Plots of estimated pdf, estimated cdf, estimated hrf, and failure rate for data set 4.

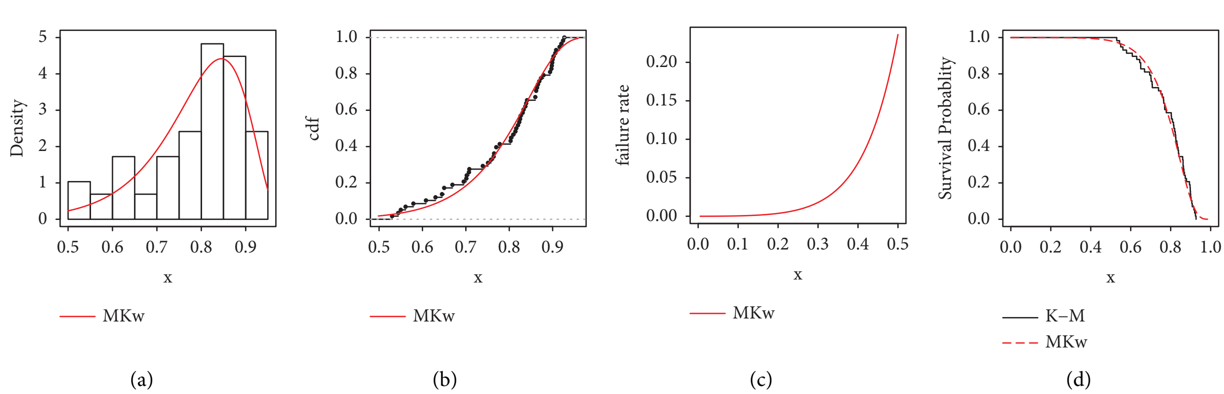


Figure 11: Plots of estimated pdf, estimated cdf, estimated hrf, and failure rate for data set 5.

Table 17: Numerical illustration of VaR and ES data set 4.

	VaR				ES			
q	MKw	Kw	SBKw	Beta	MKw	Kw	SBKw	Beta
0.55	0.46462	0.13487	0.44590	0.38257	0.39855	0.13475	0.38377	0.32340
0.60	0.47272	0.13489	0.45424	0.39193	0.40439	0.13476	0.38930	0.32872
0.65	0.48078	0.13491	0.46266	0.40166	0.40996	0.13477	0.39461	0.33395
0.70	0.48896	0.13494	0.47132	0.41197	0.41531	0.13478	0.39978	0.33915
0.75	0.49743	0.13496	0.48043	0.42315	0.42050	0.13480	0.40485	0.34437
0.80	0.50647	0.13498	0.49030	0.43566	0.42558	0.13481	0.40988	0.34968
0.85	0.51650	0.13500	0.50144	0.45029	0.43063	0.13482	0.41493	0.35515
0.90	0.52842	0.13502	0.51494	0.46877	0.43572	0.13483	0.42009	0.36093
0.95	0.54477	0.13504	0.53393	0.49622	0.44100	0.13484	0.42555	0.36726
0.99	0.58559	0.13506	0.56667	0.54748	0.44563	0.13485	0.43046	0.37329

Table 18: Numerical illustration of VaR and ES data set 5.

	VaR				ES			
q	MKw	Kw	SBKw	Beta	MKw	Kw	SBKw	Beta
0.55	0.84093	0.10671	0.81009	0.81124	0.73779	0.09838	0.71068	0.71145
0.60	0.85232	0.10822	0.82236	0.82411	0.74686	0.09914	0.71948	0.72030
0.65	0.86342	0.10974	0.83448	0.83690	0.75540	0.09990	0.72786	0.72878
0.70	0.87441	0.11125	0.84667	0.84980	0.76351	0.10065	0.73591	0.73696
0.75	0.88549	0.11277	0.85913	0.86303	0.77127	0.10141	0.74371	0.74492
0.80	0.89692	0.11428	0.87220	0.87692	0.77877	0.10217	0.75132	0.75273
0.85	0.90912	0.11580	0.88636	0.89198	0.78607	0.10292	0.75884	0.76047
0.90	0.92285	0.11731	0.90259	0.90918	0.79328	0.10368	0.76637	0.76824
0.95	0.94019	0.11882	0.92356	0.93116	0.80053	0.10444	0.77406	0.77621
0.99	0.96187	0.12004	0.95380	0.96187	0.80659	0.10504	0.78061	0.78300

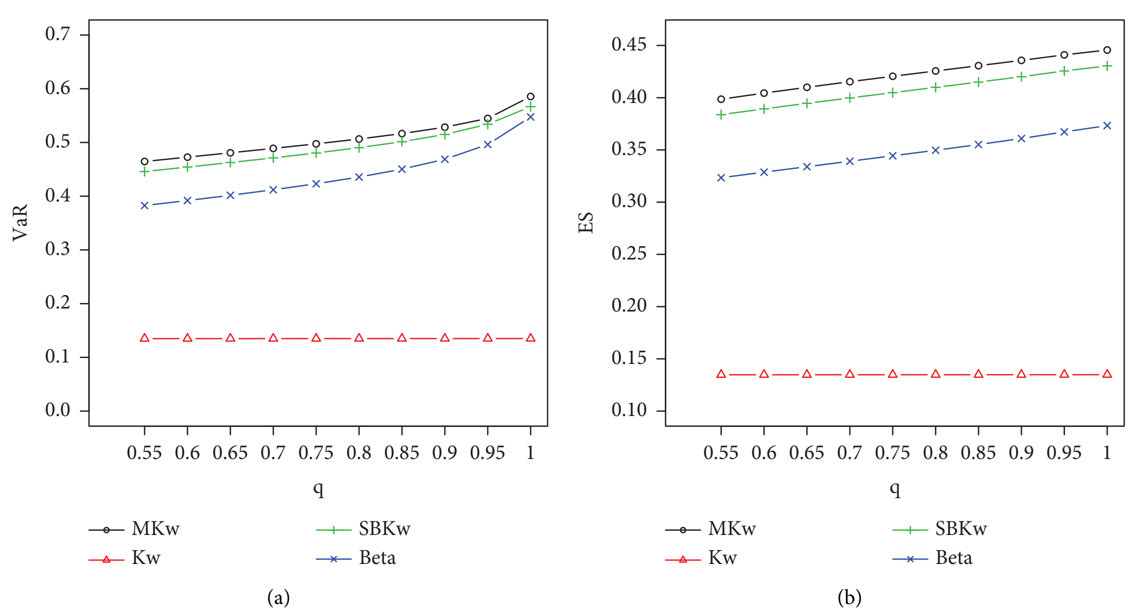


Figure 12: Plot of (a) VaR and (b) ES of MKw and Kw model data 4.

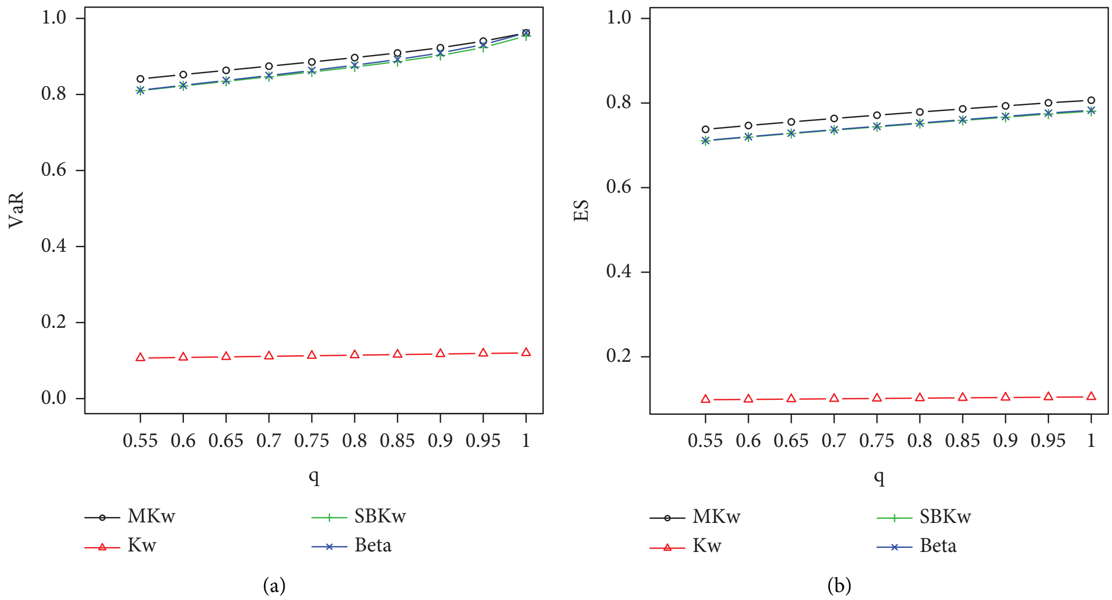


FIGURE 13: Plot of (a) VaR and (b) ES of MKw and Kw model data 5.

compared to their counterparts (Kw, SBKw, and Beta models). The graphical demonstration of the models from Figures 12 and 13, also revealed that the proposed model has heavier tail than Kw, SBKw, and Beta model. The readers are referred to [34] for detailed discussion of VaR and ES and their computation by using an *R* package.

It is clear that, the MKw model provides a better fit than the other tested models, because it has the smallest value among AIC, CAIC, BIC, HQIC, CvM, AD, and *K-S*.

8. Concluding Remarks

We proposed a modified Kumaraswamy distribution by modification $[1 - \log y]^{-1}$ for (0, 1). We reported some mathematical properties of the modified Kumaraswamy distribution. We solved the quantile function, which helped in the simulation study. We simulated some parameter values, which showed that the model's behavior was good. We also analyze this distribution with well-known models such as Gamma Kumaraswamy, Size-Biased Kumaraswamy, Kumaraswamy, and Beta using well-established GoF test-statistics for five real-life data sets including insurance claim data. We observed that our model performed better than the other comparative models on the basis of numerical results, GoFs, and graphical measures. We hope that the proposed modified distribution will get great attention from researchers.

Data Availability

The data used in the article are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] K. Cooray and M. M. A. Ananda, "Modeling actuarial data with a composite lognormal–Pareto model," *Scandinavian Actuarial Journal*, vol. 2005, no. 5, Article ID 03461230510009763, 334 pages, 2005.

[2] S. A. Klugman, H. H. Panjer, and G. E. Willmot, *Loss Models: From Data to Decisions*, Wiley, New York, NY, USA, 2012.

[3] R. Ibragimov and A. Prokhorov, *Heavy Tails and Copulas: Topics in Dependence Modelling in Economics and Finance*, World Scientific, Singapore, 2017.

[4] Z. A. Ahmad, E. M. Mahmoudi, and G. G. Hamedani, "A family of loss distributions with an application to the vehicle insurance loss data," *Pakistan Journal of Statistics and Operation Research*, vol. 16, pp. 731–744, 2019.

[5] A. Z. Afify, A. M. Gemeay, and N. A. Ibrahim, "The heavy-tailed exponential distribution: risk measures, estimation, and application to actuarial data," *Mathematics*, vol. 8, p. 1276, 2020.

[6] M. C. Korkmaz, C. Chesneau, and Z. S. Korkmaz, "A new alternative quantile regression model for the bounded response with educational measurements applications of OECD countries," *Journal of Applied Statistics*, vol. 50, no. 1, pp. 131–154, 2023.

[7] K. Karakaya, M. C. Korkmaz, C. Chesneau, and G. G. Hamedani, "A new alternative unit-Lindley distribution with increasing failure rate," *Scientia Iranica*, vol. 10, 2022.

[8] M. C. Korkmaz, "A new heavy-tailed distribution defined on the bounded interval: the logit slash distribution and its application," *Journal of Applied Statistics*, vol. 47, no. 12, pp. 2097–2119, 2020.

[9] M. C. Korkmaz, C. Chesneau, and Z. S. Korkmaz, "On the arcsecant hyperbolic normal distribution. Properties, quantile regression modeling and applications," *Symmetry*, vol. 13, no. 1, p. 117, 2021.

[10] M. C. Korkmaz, E. Altun, M. Alizadeh, and M. El-Morshedy, "The log exponential-power distribution: properties,

estimations and quantile regression model," *Mathematics*, vol. 9, no. 21, p. 2634, 2021.

[11] M. C. Korkmaz, "The unit generalized half normal distribution: a new bounded distribution with inference and application," *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, vol. 82, no. 2, pp. 133–140, 2020.

[12] M. C. Korkmaz and Z. S. Korkmaz, "The unit log–log distribution: a new unit distribution with alternative quantile regression modeling and educational measurements applications," *Journal of Applied Statistics*, vol. 2021, Article ID 2001442, 20 pages, 2021.

[13] M. C. Korkmaz, C. Chesneau, and Z. S. Korkmaz, "Transmuted unit Rayleigh quantile regression model: alternative to beta and Kumaraswamy quantile regression models," *University Politehnica of Bucharest Scientific Bulletin–Series A–Applied Mathematics and Physics*, vol. 83, pp. 149–158, 2021.

[14] M. C. Korkmaz and C. Chesneau, "On the unit Burr-XII distribution with the quantile regression modeling and applications," *Computational and Applied Mathematics*, vol. 40, no. 1, pp. 29–26, 2021.

[15] M. C. Korkmaz, C. Chesneau, and Z. S. Korkmaz, "The unit folded normal distribution: a new unit probability distribution with the estimation procedures, quantile regression modeling and educational attainment applications," *Journal of Reliability and Statistical Studies*, vol. 15, pp. 261–298, 2022.

[16] J. Mazucheli, M. C. Korkmaz, A. F. B. Menezes, and V. Leiva, "The unit generalized half-normal quantile regression model: formulation, estimation, diagnostics, and numerical applications," *Soft Computing*, vol. 2022, Article ID 07278, 17 pages, 2022.

[17] F. A. Bhatti, National College of Business Administration and Economics Lahore Pakistan, A. Ali, G. G. Hamedani, M. Ç. Korkmaz, and M. Ahmad, "The unit generalized log Burr XII distribution: properties and application," *AIMS Mathematics*, vol. 6, no. 9, pp. 10222–10252, 2021.

[18] J. Mazucheli, B. Alves, M. C. Korkmaz, and V. Leiva, "Vasicek quantile and mean regression models for bounded data: new formulation, mathematical derivations, and numerical applications," *Mathematics*, vol. 10, no. 9, p. 1389, 2022.

[19] M. C. Korkmaz, E. Altun, C. Chesneau, and H. M. Yousof, "On the unit-Chen distribution with associated quantile regression and applications," *Mathematica Slovaca*, vol. 72, no. 3, pp. 765–786, 2022.

[20] S. Gunduz and M. C. Korkmaz, "A new unit distribution based on the unbounded Johnson distribution rule: the unit Johnson SU distribution," *Pakistan Journal of Statistics and Operation Research*, vol. 16, pp. 471–490, 2020.

[21] M. Garg, "On distribution of order statistics from Kumaraswamy distribution," *Kyungpook Mathematical Journal*, vol. 48, no. 3, pp. 411–417, 2008.

[22] S. Nadarajah, "On the distribution of Kumaraswamy," *Journal of Hydrology*, vol. 348, no. 3-4, pp. 568-569, 2008.

[23] N. Bursa and G. Ozel, "The exponentiated Kumaraswamy-power function distribution," *Hacettepe Journal of Mathematics and Statistics*, vol. 46, no. 2, pp. 1–19, 2017.

[24] G. M. Cordeiro, A. Saboor, M. N. Khan, G. Ozel, and M. A. Pascoa, "The Kumaraswamy exponential-Weibull distribution: theory and applications," *Hacettepe journal of mathematics and statistics*, vol. 45, no. 76, pp. 1–1229, 2015.

[25] M. H. Tahir, M. A. Hussain, G. M. Cordeiro, M. El-Morshedy, and M. S. Eliwa, "A new Kumaraswamy generalized family of distributions with properties, applications, and bivariate extension," *Mathematics*, vol. 8, no. 11, p. 1989, 2020.

[26] Q. Ramzan, S. Qamar, M. Amin, H. M. Alshanbari, A. Nazeer, and A. Elhassanein, "On the extended generalized inverted Kumaraswamy distribution," *Computational Intelligence and Neuroscience*, vol. 2022, Article ID 1612959, 18 pages, 2022.

[27] R. Alshkaki, "A generalized modification of the Kumaraswamy distribution for modeling and analyzing real-life data," *Statistics, Optimization & Information Computing*, vol. 8, no. 2, pp. 521–548, 2020.

[28] J. A. Greenwood, J. M. Landwehr, N. C. Matalas, and J. R. Wallis, "Probability weighted moments: definition and relation to parameters of several distributions expressable in inverse form," *Water Resources Research*, vol. 15, no. 5, pp. 1049–1054, 1979.

[29] P. Artzner, F. Delbaen, J. M. Eber, and D. Heath, "Coherent measures of risk," *Mathematical Finance*, vol. 9, no. 3, pp. 203–228, 1999.

[30] I. Ghosh and G. G. Hamedani, "The Gamma–Kumaraswamy distribution: an alternative to Gamma distribution," *Communications in Statistics - Theory and Methods*, vol. 47, no. 9, pp. 2056–2072, 2018.

[31] D. Sharma and T. K. Chakrabarty, "On size biased Kumaraswamy distribution," *Statistics, Optimization & Information Computing*, vol. 4, no. 3, pp. 252–264, 2016.

[32] R. Dasgupta, "On the distribution of Burr with applications," *Sankhya B*, vol. 73, pp. 1–19, 2011.

[33] G. Moutinho Cordeiro and R. dos Santos Brito, "The beta power distribution," *Brazilian Journal of Probability and Statistics*, vol. 26, no. 1, pp. 88–112, 2012.

[34] S. Chan, S. Nadarajah, and E. Afuecheta, "An r package for value at risk and expected shortfall," *Communications in Statistics - Simulation and Computation*, vol. 45, no. 9, pp. 3416–3434, 2016.