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Research article

Simulation analysis, properties and applications on a new Burr XII model based on the Bell-X functionalities

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Abstract: In this article, we make mathematical and practical contributions to the Bell-X family of absolutely continuous distributions. As a main member of this family, a special distribution extending the modeling perspectives of the famous Burr XII (BXII) distribution is discussed in detail. It is called the Bell-Burr XII (BBXII) distribution. It stands apart from the other extended BXII distributions because of its flexibility in terms of functional shapes. On the theoretical side, a linear representation of the probability density function and the ordinary and incomplete moments are among the key properties studied in depth. Some commonly used entropy measures, namely Rényi, Havrda and Charvat, Arimoto, and Tsallis entropy, are derived. On the practical (inferential) side, the associated parameters are estimated using seven different frequentist estimation methods, namely the methods of maximum likelihood estimation, percentile estimation, least squares estimation, weighted least squares estimation, Cramér von-Mises estimation, Anderson-Darling estimation, and right-tail Anderson-Darling estimation. A simulation study utilizing all these methods is offered to highlight their effectiveness. Subsequently, the BBXII model is successfully used in comparisons with other comparable models to analyze data on patients with acute bone cancer and arthritis pain. A group acceptance sampling plan for truncated life tests is also proposed when an item's lifetime follows a BBXII distribution. Convincing results are obtained.

Keywords: Burr XII distribution; entropy measures; estimation; GASP; quality parameter; T-X family
Mathematics Subject Classification: 60E05, 62N05, 62F10, 62D05

1. Introduction

A variety of new generators, families of (absolutely continuous or discrete) distributions, and methods of inducing additional shape parameters into a baseline distribution have been added to the statistical literature, resulting in more flexible statistical models [1–3]. One of the most useful families of absolutely continuous distributions is without doubt the T-X family introduced by Alzaatreh et al. [4]. These developments provide practitioners with more flexible model choices for better fitting and eventually more accurate results in many applicable disciplines.

On the other hand, for many empirical and theoretical situations in life testing and reliability investigations, the Burr XII (BXII) distribution (initiated by Irving Burr [5]) offers a straightforward, elegant, and close-form solution. Numerous of its extensions have been elaborated in many references. See [6–13], to name a few. When the related probability density functions (PDFs) have several shapes, these lifetime models are useful for estimating the histogram shape of data. Several widely used distributions, including the logistic, gamma, and log-normal distributions, cover the BXII distribution [6]. The logistic and Weibull distributions are the asymptotic limit cases of the BXII distribution, and are highly well-liked for modeling lifetime data, such as those presenting monotonous hazard rates. Because of its skewed (left and right) PDF forms, the Weibull distribution might be a good starting point when modeling monotone hazard rates. It does not, however, offer an adequate parametric fit for modeling phenomena such as non-monotone hazard rates that are frequently observed in various applied areas. As opposed to equivalent models based on exponential tails, the BXII distribution includes algebraic tails, which are useful for modeling data that occurs less frequently.

On the mathematical side, the three-parameter BXII distribution [12, 13], or sometimes called the Singh-Maddala distribution [14], has the following cumulative distribution function (CDF):

$$G(x;\, a,\, b,\, k) = 1 - \left[1 + \left(\frac{x}{a}\right)^{b}\right]^{-k}, \tag{1.1}$$

for $a, b, k > 0$ and $x > 0$. As a matter of fact, Eq (1.1) turned to many commonly used distributions for specific values of the parameters. For example, if $b = 1$, it reduces to Pareto Type II distribution, or if $k = 1$, it reduces to the Fisk distribution, which is sometimes referred as the log-logistic distribution, a special case of Champernowne distribution. Furthermore, the paralogistic distribution can be obtained by replacing $b = k$ in Eq (1.1). By transforming Eq (1.1) as $G_*(x;\, a,\, b,\, k) = 1 - G(1/x;\, a,\, b,\, k)$, we get the CDF of a distribution called the Dagum distribution or Burr III distribution, or simply the inverse BXII distribution. The Dagum distribution is also called the generalized log-logistic distribution.

In this paper, motivated by the features of the T-X family and the willingness to propose a completely new extended BXII distribution, we first develop the Bell-X family of (absolutely continuous) distributions. It is based on a specific Bell-type distribution, which finds its first trace in the work of Castellares et al. [16]. To be more specific, it defines a discrete Bell distribution based on well-known Bell numbers [15] to achieve somewhat better fits than the Poisson model. Later, the Bell distribution was broadened by Fayomi et al. [17] to propose its generalized class analogy to the exponentiated Poisson generalized family of distributions, named the exponentiated Bell-generalized family, extending (in some sense) the Poisson-X family proposed by Tahir et al. [18]. In our study, we consider a natural sub-family to the one in [17], named the Bell-X family, while examining its significant mathematical features and taking into account its numerous properties and real-world applications.

After a general investigation of the Bell-X family, the new extended BXII distribution, named the Bell-BXII (BBXII) distribution, is presented, and several important properties are derived. Among them, the main functions of the distribution are quite manageable, with extremely flexible PDF and hazard rate function (HRF). In particular, it is demonstrated that the HRF can take on a variety of shapes, including unimodal, upside-down bathtub-shaped, increasing, and decreasing trends, which makes it suitable for real-world applications in various fields. On the statistical aspect, following the spirit of the references [19–24], various frequentist estimation techniques, including the methods of maximum likelihood estimation (MLE), percentile estimation (PE), least squares estimation (LSE), weighted least squares estimation (WLSE), Cramér von-Mises estimation (CME), Anderson-Darling estimation (ADE), and right-tail Anderson-Darling estimation (RTADE) are investigated. A simulation study utilizing all these techniques is offered to highlight their effectiveness. Application of the BBXII model to actual data in the fields of medicine and reliability show its powerfull practicality. It is compared with several top-ranked comparable models, including the exponentiated Poisson Burr-XII (EPBXII) [7], Kumaraswamy Burr-XII (KBXII) [8], Marshall-Olkin Burr-XII (MBXII) [9], Topp Leone Burr-XII (TLBXII) [10], beta Burr-XII (B-BXII) [11], and Burr-XII (BXII) [12] models. As a result of this study, we show that the BBXII model outperforms several well-known extended versions of the BXII model, such as the EPBXII, KBXII, MBXII, TLBXII, B-BXII, and BXII models. Last but not least, a group acceptance sampling plan (GASP) is designed when the product lifespan follows the BBXII model with a real data application.

The manuscript is structured into the eight following sections: Section 2 presents the Bell-X family, including some of its fundamental characteristics. Section 3 is focused on the BBXII distribution and some of its important properties. Section 4 shows the parameter estimation using seven different frequentist estimation methods. Simulation analysis is examined in Section 5. Real data applications are provided in Section 6. Section 7 focuses on the design of GASP when a product's life cycle follows the BBXII model. Finally, Section 8 is devoted to some concluding remarks. An appendix containing technical elements is given at the very end.

2. The Bell-X family with motivation

2.1. Construction

To begin, a restrospective on the family introduced in [17] is necessary. Let us start with a concrete modeling scenario. Suppose a company where each system consists of c parallel units that operate independently in K systems. If any one of these systems fails, the entire system will fail. The random variables $W_{i,1}, W_{i,2}, \ldots, W_{i,c}$, which represent the hazard rates of the parallel components of the ith system, are assumed to be independent and possessing the same standard uniform distribution. Furthermore, assume that K is a random variable that follows the truncated Bell distribution and is given by

$$\mathbb{P}(K = k) = \frac{\varsigma^k \exp(1 - e^{\varsigma}) B_k}{k!\,[1 - \exp(1 - e^{\varsigma})]}, \qquad k = 1, 2, \ldots \tag{2.1}$$

where B_k is the well-known Bell numbers as defined in [15] and $\varsigma > 0$. Let T be the first of the K operational systems' time to failure, so we can write $T = \min(V_1, \ldots, V_K)$, with

$V_i = \max(W_{i,1}, W_{i,2}, \dots, W_{i,c})$. Then the conditional CDF of T under $K = k$, say $R(t|K = k)$. Then we have

$$
\begin{aligned}
R(t|K = k) &= \mathbb{P}[T \le t|K = k] = 1 - \mathbb{P}[T > t|K = k] = 1 - [\mathbb{P}(V_1 > t)]^k \\
&= 1 - [1 - \mathbb{P}(W_{1,1} \le t, W_{1,2} \le t, \dots, W_{1,c} \le t)]^k \\
&= 1 - (1 - t^c)^k .
\end{aligned}
$$

The unconditional CDF of T is obtained as

$$
\begin{aligned}
R(t) = \mathbb{E}(R(t|K = k)|_{k=K}) &= \sum_{k=1}^{\infty} \frac{\varsigma^k \exp(1 - e^{\varsigma}) B_k}{k!\,[1 - \exp(1 - e^{\varsigma})]} \left[1 - (1 - t^c)^k\right] \\
&= \frac{1 - \exp\left(-e^{\varsigma}\left[1 - e^{-\varsigma t^c}\right]\right)}{1 - \exp(1 - e^{\varsigma})}. \qquad (2.2)
\end{aligned}
$$

The absolutely continuous Bell-distribution is thus defined. The PDF corresponding to Eq (2.2) is given by

$$
r(t) = \frac{\varsigma\, c\, t^{c-1} e^{\varsigma(1-t^c)} \exp\left(-e^{\varsigma}\left[1 - e^{-\varsigma t^c}\right]\right)}{1 - \exp(1 - e^{\varsigma})}, \qquad t \in [0, 1]. \qquad (2.3)
$$

It is understood that $r(t) = 0$ for $t \notin [0, 1]$.

Now, let us present in brief the basics of the T-X family. Let $r(t)$ be a PDF with support (a, b) for $a < b$ (possibly infinite). Let us consider a function $\Omega(t)$, $t \in [0, 1]$, with values into an interval $[a, b]$, such that $\Omega(t)$ is differential and non-decreasing by monotonically, and that $\Omega(0)$ and $\Omega(1)$ tends to a and b, respectively. Then the T-X family, as described by [4], is defined with the following CDF:

$$
F(x) = \int_a^{\Omega[G(x)]} r(t)\, dt, \quad x \in \mathbb{R}, \qquad (2.4)
$$

where $G(x)$ represents the baseline CDF of an absolutely continuous distribution. By simply taking $\Omega(t) = t$ and $r(t)$ as in Eq (2.3) with $\varsigma = 1$ to avoid the hyper parameterization problem (and $a = 0$), the CDF of the T-X family becomes

$$
F(x) = \frac{1 - \exp\left(-e\left[1 - e^{-G(x)^c}\right]\right)}{1 - \exp(1 - e)}, \qquad x \in \mathbb{R}. \qquad (2.5)
$$

The considered Bell-X family is thus defined by this CDF, which can be viewed as a sub-family of the one in [17] because of $\varsigma = 1$. It is of interest for the reasons listed below:

- It has an original functional structure.
- Thanks to the addition of only one shape parameter, i.e. c (combined with the originality of the functional structure), the Bell-X family is capable of making the baseline distribution more flexible by changing its functional form.
- All significant HRF shapes can be supported by the special sub-distributions of the Bell-X family, including the bathtub, constant, upside-down bathtub, increasing, decreasing-increasing-decreasing, and decreasing shapes. As a result, many applied fields can use its unique family to model various kinds of real-world data.

- Last but not least, compared to its direct model competitors, and the baseline model in particular, the unique sub-models derived from the Bell-X family may offer better fits.

To support these statements, we again refer to [17]. In complement to the CDF, the PDF corresponding to Eq (2.5) is given by

$$f(x) = \frac{c\, g(x)\, G(x)^{c-1}\, e^{1-G(x)^c} \exp\Big(-e\left[1 - e^{-G(x)^c}\right]\Big)}{1 - \exp(1-e)}, \tag{2.6}$$

where $g(x)$ represents the baseline PDF.

Among the other functions of interest, let us mention the survival and hazard rate functions.

2.2. Survival and hazard rate functions

We recall that the survival function (SF) provides the probability that a particular object of interest (patient or device) will survive past a certain time. It is sometimes called the reliability function (in engineering) or simply the survivor function (in human mortality). On his side, the HRF (or failure rate function) refers to the likelihood that a system will fail, assuming that failure has not already occurred at a certain time.

In the Bell-X family setting, the SF and HRF are given by

$$S(x) = \frac{\exp\Big(-e\left[1 - e^{-G(x)^c}\right]\Big) - \exp(1-e)}{1 - \exp(1-e)} \tag{2.7}$$

and

$$h(x) = \frac{c\, g(x)\, G(x)^{c-1} e^{1-G(x)^c} \exp\Big(-e\left[1 - e^{-G(x)^c}\right]\Big)}{\exp\Big(-e\left[1 - e^{-G(x)^c}\right]\Big) - \exp(1-e)}, \tag{2.8}$$

respectively. These functions will play a central role in various practical aspects, among others.

2.3. Quantile function

The QF of the Bell-X family, say $Q(p)$, is given by

$$Q(p) = G^{-1}\left[\left(1 - \log\Big\{\log\Big(1 - p\left[1 - \exp\big(1-e\big)\right]\Big) + e\Big\}\right)^{1/c}\right], \quad p \in [0,1], \tag{2.9}$$

where $G^{-1}(x)$ is the inverse function of $G(x)$, that is, the QF of the baseline distribution. The median of the Bell-X family can be obtained by setting $p = 0.5$ in Eq (2.9). In addition, the L-moments of the Bell-X family can be expressed by using it. For instance, the first four L-moments are as follows:

$$L_1 = \int_0^1 Q(p)\, dp,\ L_2 = \int_0^1 Q(p)\,(2p-1)\, dp,\ L_3 = \int_0^1 Q(p)\Big(6p^2 - 6p + 1\Big)\, dp$$

and

$$L_4 = \int_0^1 Q(p)\Big(20p^3 - 30p^2 + 12p - 1\Big)\, dp.$$

2.4. Useful expansion

Here we derive a linear functional representation of the Bell-X family, which is useful for acquiring several important properties. In some sense, we revisit a result established in [17].

Proposition 2.1. *A linear functional representation of the PDF and CDF are given by*

$$f(x) = \sum_{p=0}^{\infty} \nabla_p w_{c(p+1)}(x) \tag{2.10}$$

and

$$F(x) = \sum_{p=0}^{\infty} \nabla_p W_{c(p+1)}(x), \tag{2.11}$$

respectively, where

$$\nabla_p = \frac{e}{(p+1)!\,[1-\exp(1-e)]} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{(-1)^{v+u+p}}{v!}\binom{v}{u}(1+u)^p\, e^v, \tag{2.12}$$

$W_{c(p+1)}(x) = G(x)^{c(p+1)}$ *and* $w_{c(p+1)}(x) = c\,(p+1)\;g(x)\;G(x)^{c(p+1)-1}$ *are the CDF and PDF, respectively, of the exp-G family with power parameter* $c(p+1)$.

Proof. As the main general formulas, there are the binomial expansion and the power series for the exponential functions, given as

$$(1-\rho)^z = \sum_{u=0}^{\infty} (-1)^u \binom{z}{u} \rho^u, \tag{2.13}$$

where $\binom{z}{u}$ represents the generalized binomial coefficient (the formula remaining valid for ρ such that $|\rho| < 1$ only) and

$$\exp(-m\,x^n) = \sum_{v=0}^{\infty} (-1)^v\; m^v \frac{x^{vn}}{v!}, \tag{2.14}$$

for any real numbers m, n and x, respectively.

By using Eq (2.14) to the last term of Eq (2.6), we obtain

$$\exp\left(-e\left[1-e^{-G(x)^c}\right]\right) = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!}\left(e\left[1-e^{-G(x)^c}\right]\right)^v. \tag{2.15}$$

We can bound the final power term by using Eq (2.13), which gives

$$\left(e\left[1-e^{-G(x)^c}\right]\right)^v = e^v \sum_{u=0}^{\infty} (-1)^u \binom{v}{u} e^{-uG(x)^c}. \tag{2.16}$$

After simplification, Eq (2.6) reduces as follows:

$$f(x) = \frac{e^{v+1} c\, g(x)\, G(x)^{c-1}}{1-\exp(1-e)} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{(-1)^{v+u}}{v!}\binom{v}{u} e^{-(1+u)G(x)^c}. \tag{2.17}$$

Again using Eq (2.14) to the last term to Eq (2.17), we get

$$e^{-(1+u)G(x)^c} = \sum_{p=0}^{\infty} (-1)^p \frac{(1+u)^p}{p!} G(x)^{cp}, \tag{2.18}$$

and Eq (2.6) becomes

$$f(x) = \frac{e^{v+1}}{(p+1)\left[1-\exp(1-e)\right]} \sum_{p=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{(-1)^{v+u+p}}{v!p!}\binom{v}{u}(1+u)^p \times c\,(p+1)\; g(x)\,G(x)^{c(p+1)-1}.$$

The desired expansion is obtained for $f(x)$. The expansion for $F(x)$ is obtained upon integration. This ends the proof of Proposition 2.1. □

As a remark, the constant term in Eq (2.12) satisfies $\sum_{p=0}^{\infty} \nabla_p = 1$. Thanks to Proposition 2.1, it can be noted that all quantities of the following integral form: $\int_{-\infty}^{\infty} u(x)f(x)dx$ has a linear representation with simple coefficients.

Corollary 2.1. *Let $\delta > 0$. Then the following expansion holds:*

$$f(x)^\delta = \sum_{u=0}^{\infty} w_u g(x)^\delta\, G(x)^{cu+\delta(c-1)},$$

where

$$w_u = \frac{c^\delta e^\delta}{u!\left[1-\exp(1-e)\right]^\delta} \sum_{\varrho=0}^{\infty}\sum_{\omega=0}^{\infty} \frac{1}{\varrho!}(-1)^{\varrho+\omega+u}\binom{\varrho}{\omega}(\delta e)^\varrho\,(\omega+\delta)^u. \tag{2.19}$$

Corollary 2.1 will be useful to give mathematical expansion of complex measures, such as entropy measures, which are the subject of the next part.

2.5. *Entropy measures*

The goal of the entropy measure is to highlight a random variable's uncertainty variation. For a complete overview, we may refer to [26]. Here, we determine the expansions of four well-established entropy measures for the Bell-X family, namely the Rényi entropy, the Havrda and Charvat entropy, the Arimoto entropy, and the Tsallis entropy.

Rényi entropy: The Rényi entropy of an absolutely continuous distribution having a PDF $f(x)$ can be expressed as follows:

$$R_\delta(x) = \delta^* \log\left[\int_{-\infty}^{\infty} f(x)^\delta dx\right], \tag{2.20}$$

where $\delta^* = (1-\delta)^{-1}$, $\delta > 0$ and $\delta \neq 1$. Hereafter, such assumptions on δ will be supposed. In the context of the Bell-X family, by using Corollary 2.1, we can expand it as follows:

$$R_\delta(x) = \delta^* \log\left[\sum_{u=0}^{\infty} w_u \int_{-\infty}^{\infty} g(x)^\delta\, G(x)^{cu+\delta(c-1)} dx\right], \tag{2.21}$$

where w_u is defined in Eq (2.19). For a given absolutely continuous baseline distribution, the remaining integral term is calculable or can be found in many references dealing with the exp-G family.

Havrda and Charvat entropy: The Havrda and Charvat entropy of an absolutely continuous distribution having a PDF $f(x)$ can be expressed as follows:

$$HC_\delta(x) = \frac{1}{2^{1-\delta}-1}\left[\int_{-\infty}^{\infty} f(x)^\delta dx - 1\right]. \tag{2.22}$$

In the context of the Bell-X family, by using Corollary 2.1, in a similar way to the Rényi entropy, we can express it as

$$HC_\delta(x) = \frac{1}{2^{1-\delta}-1}\left(\sum_{u=0}^{\infty} w_u \int_{-\infty}^{\infty} g(x)^\delta\, G(x)^{cu+\delta(c-1)} dx - 1\right).$$

Arimoto entropy: The Arimoto entropy of an absolutely continuous distribution having a PDF $f(x)$ can be expressed as

$$A_\delta(x) = \frac{\delta}{1-\delta}\left[\left(\int_{-\infty}^{\infty} f(x)^\delta dx\right)^{1/\delta} - 1\right]. \tag{2.23}$$

In the context of the Bell-X family, by using Corollary 2.1, in a similar way to the Rényi entropy, we can express it as

$$A_\delta(x) = \frac{\delta}{1-\delta}\left[\left(\sum_{u=0}^{\infty} w_u \int_{-\infty}^{\infty} g(x)^\delta\, G(x)^{cu+\delta(c-1)} dx\right)^{1/\delta} - 1\right].$$

Tsallis entropy: The Tsallis entropy of an absolutely continuous distribution having a PDF $f(x)$ can be expressed as

$$T_\delta(x) = -\delta^*\left[1 - \int_{-\infty}^{\infty} f(x)^\delta dx\right]. \tag{2.24}$$

In the context of the Bell-X family, by using Corollary 2.1, in a similar way to the Rényi entropy, we can express it as

$$T_\delta(x) = -\delta^*\left[1 - \left(\sum_{u=0}^{\infty} w_u \int_{-\infty}^{\infty} g(x)^\delta\, G(x)^{cu+\delta(c-1)} dx\right)\right].$$

The expansions above provide the mathematical basis for more of the computation of the considered entropy measures. In particular, by taking the Rényi entropy as an example, the following approximation is conceptually valid:

$$\tilde{R}_{\Pi,\delta}(x) = \delta^* \log\left[\sum_{u=0}^{\Pi} w_u \int_{-\infty}^{\infty} g(x)^\delta\, G(x)^{cu+\delta(c-1)} dx\right] \approx R_\delta(x), \tag{2.25}$$

where Π denotes a large enough integer. Similar approximations can be used for the other entropy measures. An alternative evaluation apporach is the determination of bounds for such entropy measures. See, for example, [27].

The next section focuses on a special Bell-X distribution of interest, extending the scope of the BXII distribution, as sketched in the introductory section.

3. The BBXII distribution

3.1. Definition

Here, we define the BBXII distribution, using the Bell-X family and the three-parameter BXII distribution given by Zimmer et al. [12] as an absolutely continuous baseline distribution. First, we recall that the CDF and PDF of the (three-parameter) BXII distribution are indicated as

$$G(x;\, a,\, b,\, k) = 1 - \left[1 + \left(\frac{x}{a}\right)^{b}\right]^{-k}, \quad x > 0 \tag{3.1}$$

and

$$g(x;\, a,\, b,\, k) = k\, b\, a^{-b}\, x^{b-1} \left[1 + \left(\frac{x}{a}\right)^{b}\right]^{-k-1}, \quad x > 0, \tag{3.2}$$

respectively, with parameters a, b and $k > 0$. The CDF and PDF of the corresponding Bell-X distribution are obtained directly by setting Eqs (3.1) and (3.2) in Eqs (2.5) and (2.6), respectively. Hence, they are indicated as

$$F(x\,;a,\, b,\, k,\, c) = \frac{1 - \exp\left(-e\left[1 - e^{-\left(1-\left[1+(x/a)^{b}\right]^{-k}\right)^{c}}\right]\right)}{1 - \exp\left(1 - e\right)} \tag{3.3}$$

and

$$f(x\,;a,\, b,\, k,\, c) = c\, k\, b\, a^{-b}\, x^{b-1} \left[1 + \left(\frac{x}{a}\right)^{b}\right]^{-k-1} \left[1 - \left(1 + \left(\frac{x}{a}\right)^{b}\right)^{-k}\right]^{c-1} e^{1-\left[1-\left(1+(x/a)^{b}\right)^{-k}\right]^{c}}$$
$$\times \exp\left(-e\left\{1 - e^{-\left(1-\left[1+(x/a)^{b}\right]^{-k}\right)^{c}}\right\}\right) \left[1 - \exp\left(1 - e\right)\right]^{-1}, \tag{3.4}$$

respectively. The BBXII distribution is thus defined.

Graphical illustrations of the PDF are available in Figure 1. It is interesting to see all the possible shapes (almost symmetrical, spikes, bumps, decreasing, etc.), so much more flexible than those of the BXII distribution. This is a consequence of the original transform behind the Bell-X family and the new tuning parameter c.

The SF and HRF of the BBXII distribution are given by

$$S(x\,;a,\, b,\, k,\, c) = \frac{\exp\left(-e\left[1 - e^{-\left(1-\left[1+(x/a)^{b}\right]^{-k}\right)^{c}}\right]\right) - \exp\left(1 - e\right)}{1 - \exp\left(1 - e\right)}$$

and

$$h(x\,;a,\, b,\, k,\, c) = c\, k\, b\, a^{-b}\, x^{b-1} \left[1 + \left(\frac{x}{a}\right)^{b}\right]^{-k-1} \left[1 - \left(1 + \left(\frac{x}{a}\right)^{b}\right)^{-k}\right]^{c-1}$$
$$\times\ e^{1-\left[1-\left(1+(x/a)^{b}\right)^{-k}\right]^{c}} \exp\left\{-e\left[1 - e^{-\left[1-\left(1+(x/a)^{b}\right)^{-k}\right]^{c}}\right]\right\}$$

$$\times \left\{ \exp\left(-e\left[1-e^{-\left(1-\left[1+(x/a)^{b}\right]^{-k}\right)^{c}}\right]\right) - \exp(1-e) \right\}^{-1},$$

respectively.

Graphical illustrations of the HRF are given in Figure 2. We can distinguish unimodal, upside-down bathtub-shaped, increasing and decreasing trends for the shapes, making the HRF extremely flexible and useful for various statistical modeling based on lifetime data.

On the other hand, since the QF of the BXII distribution is given by

$$G^{-1}(p;\, a,\, b,\, k) = a\left[(1-p)^{-1/k} - 1\right]^{1/b},$$

using Eq (2.9), the QF of the BBXII distribution is derived as

$$Q(p\,;a,\, b,\, k,\, c) = a\left[\left(1-\left\{\left(1-\log\left\{\log\left[1-p\left(1-\exp\left(1-e\right)\right)\right]+e\right\}\right)^{1/c}\right\}\right)^{-1/k} - 1\right]^{1/b}, \quad p \in [0, 1]. \tag{3.5}$$

By setting $p = 0.5$, in Eq (3.5), the median of the BBXII distribution follows.

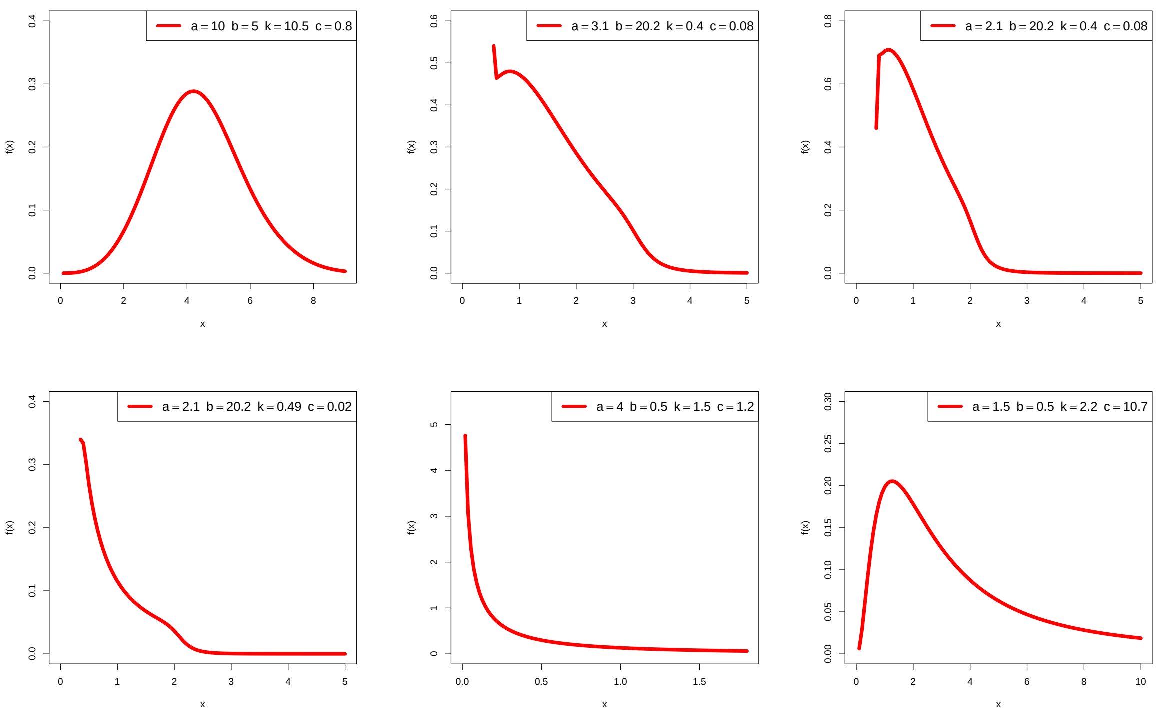


Figure 1. Graphical illustrations of the PDF of the BBXII distribution at some parametric values.

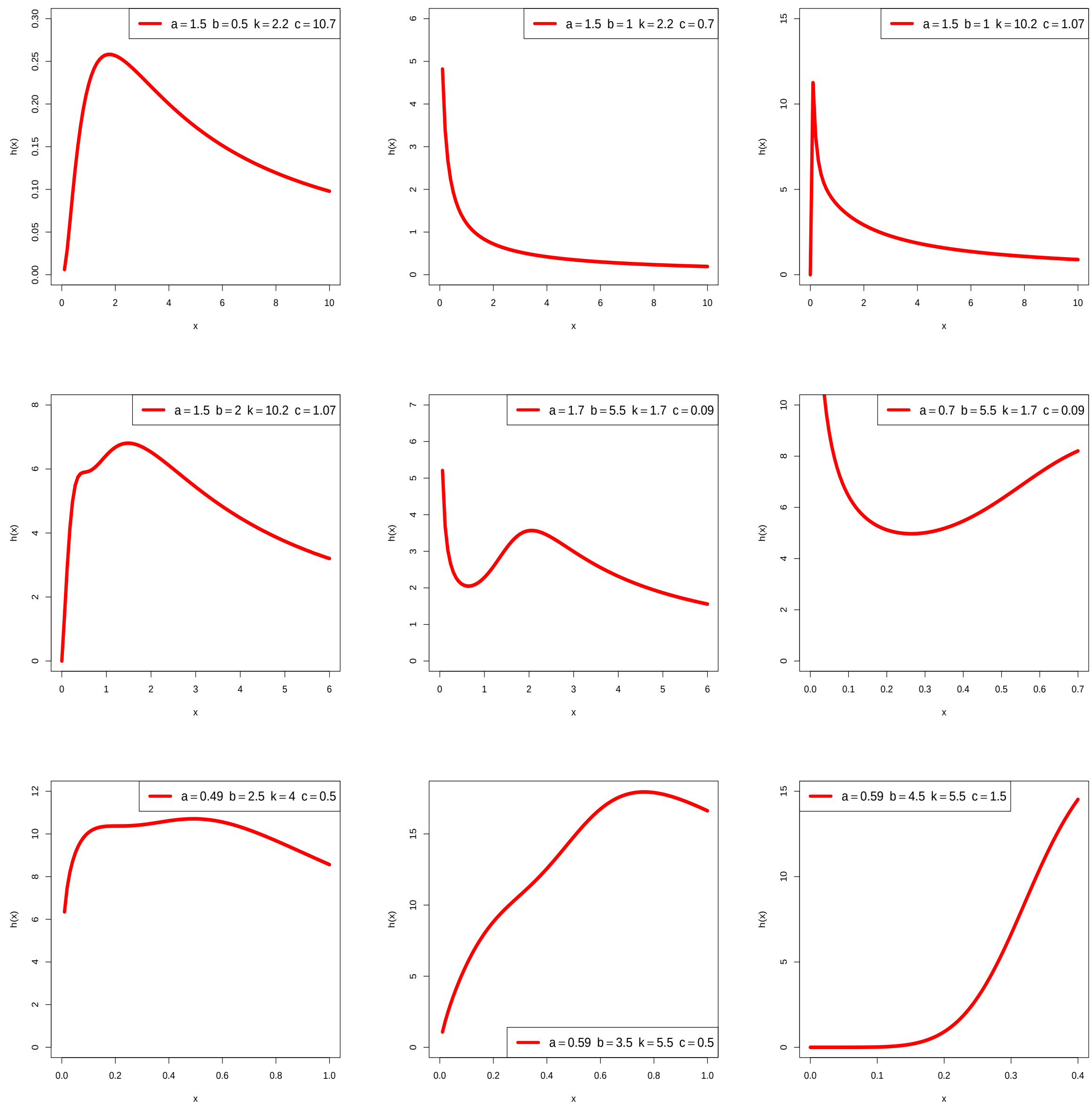


Figure 2. Graphical illustrations of HRF of the BBXII distribution at some parametric values.

3.2. Properties

Various mathematical properties of the BBXII distribution are now being investigated. We first derive a linear functional representation of the PDF, which will allow us to express various of its mathematical features, such as moments and entropy measures.

Proposition 3.1. *The PDF of the BBXII distribution can be expanded as*

$$f(x\,; a,\, b,\, k,\, c) = \sum_{\theta=0}^{\infty} \Delta_\theta\, g\,[x;\ a,\ b,\ k(\theta+1)]\,, \tag{3.6}$$

where

$$\varDelta_\theta = \sum_{p=0}^{\infty} \nabla_p \frac{c\,(p+1)}{\theta+1}\ (-1)^\theta \binom{c(p+1)-1}{\theta},$$

∇_p being defined by Eq (2.12)*, and $g\,[x;\ a,\ b,\ k(\theta+1)]$ is the PDF of the BXII distribution.*

Proof. By using Eq (2.10), the PDF can be expressed as follows:

$$f(x\,;a,\ b,\ k,\ c) = \sum_{p=0}^{\infty} \nabla_p c\,(p+1)\ k\,b\,a^{-b}\,x^{b-1}\left[1+\left(\frac{x}{a}\right)^b\right]^{-k-1}\left[1-\left[1+\left(\frac{x}{a}\right)^b\right]^{-k}\right]^{c(p+1)-1}. \tag{3.7}$$

Then, by applying Eq (2.13) to the last power term, we obtain

$$\left[1-\left[1+\left(\frac{x}{a}\right)^b\right]^{-k}\right]^{c(p+1)-1} = \sum_{\theta=0}^{\infty} (-1)^\theta \binom{c(p+1)-1}{\theta}\left[1+\left(\frac{x}{a}\right)^b\right]^{-\theta k}. \tag{3.8}$$

This yields the desired expansion, and ends the proof. □

From Proposition 3.1, we see that the PDF of the BBXII distribution can be expressed as an infinite general linear mixture of PDFs of the BXII distribution. Therefore, a number of significant properties of the BXII distribution can be transposed to derive those of the BBXII distribution. Some of them will be presented later.

3.3. Ordinary moments

As a consequence of Proposition 3.1, the rth ordinary moment of a random variable X that follows the BBXII distribution is given by

$$\mu'_r = \mathbb{E}(X^r) = a^r k \sum_{\theta=0}^{\infty} \varDelta_\theta(\theta+1)\beta\left[\left(k(\theta+1)-\frac{r}{b}\right),\left(\frac{r}{b}+1\right)\right], \tag{3.9}$$

for $r < bk$, where $\beta(a,b)$ represents the standard beta function. Setting $r = 1$ in Eq (3.9) yields the mean of X. As usual, the variance of X can be obtained as $\sigma^2 = \mu'_2 - (\mu'_1)^2$. By using Eq (3.9), the zth central moment and cumulant of X can be obtained as

$$\mu_z = E[(X-\mu'_1)^z] = \sum_{m=0}^{z} (-1)^m \binom{z}{m}(\mu'_1)^z \mu'_{z-m}$$

and

$$k_z = \mu'_z - \sum_{m=1}^{z-1} \binom{z-1}{m-1} k_m \mu'_{z-m}$$

with $k_1 = \mu'_1$, respectively (see [28]). Based on it, the skewness and kurtosis coefficients of X are defined by $CS = \mu_3/\sigma^3$ and $CK = \mu_4/\sigma^4$, respectively. The plot of the mean and variance of X, as well as its skewness and kurtosis coefficients are presented in Figure 3 by taking $a = 1$, $b = 1.5$ and $k = 2.5$.

As the values of parameters c and b increase, the skewness and kurtosis tend to decrease. Following the interaction between the parameters c and b, Figure 3 shows that when the values of c and b increase, the values of the mean and variance also tend to increase, but at a certain point the variance tends to decrease.

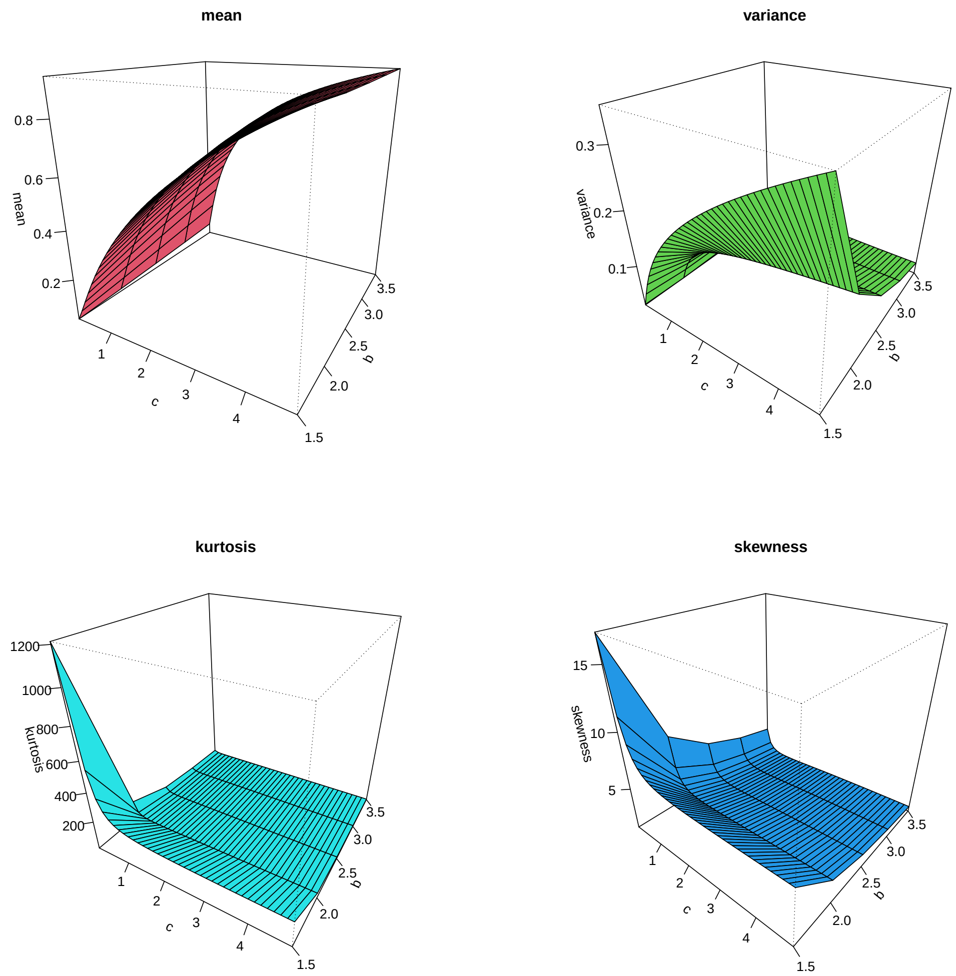


Figure 3. Graphical illustrations of the mean and variance, as well as the skewness and kurtosis coefficients of the BBXII distribution.

Table 1 presents the main variability measures of the BBXII distribution at some parametric values. The following six different and arbitrary combinations of parameters a, b, k, c are used, and the well-known relationship between the ordinary and non-central moments is used to find the central moments: $\omega_1 = [1.5, 1.2, 10.2, 1.5]$, $\omega_2 = [5.5, 1.2, 10.2, 4]$, $\omega_3 = [0.5, 1.2, 10.2, 0.5]$, $\omega_4 = [2.5, 2.2, 10.2, 0.5]$, $\omega_5 = [2.5, 2.2, 6, 10.5]$ and $\omega_6 = [0.5, 3.2, 1.6, 4.5]$. Based on Table 1, the values of CS and CK indicate that the BBXII distribution is right-skewed and leptokurtic at various combinations of parametric values.

Table 1. Numerical values of some moment measures of the BBXII distribution.

Measures	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
μ'_1	0.1541	1.1453	0.0184	0.3108	1.7036	0.5616
μ'_2	0.0448	1.7098	0.0015	0.1869	3.0505	0.3475
μ'_3	0.0220	3.4383	0.0003	0.1645	5.7871	0.2497
μ'_4	0.0166	9.5498	0.0001	0.1865	11.7558	0.2437
σ^2	0.0210	0.3980	0.0012	0.0903	0.1484	0.0321
σ	0.1450	0.6309	0.0344	0.3005	0.3852	0.1791
CV	0.9409	0.5508	1.8700	0.9672	0.2261	0.3189
μ_2	0.0210	0.3980	0.0012	0.0903	0.1484	0.0321
μ_3	0.0086	0.5682	0.0002	0.0503	0.0848	0.0185
μ_4	0.0077	2.0931	0.0001	0.0624	0.1719	0.0419
CS	2.8351	2.2625	4.4993	1.8528	1.4847	3.2246
CK	14.524	10.211	32.959	4.6441	4.8082	37.704

3.4. Incomplete moments

As a consequence of Proposition 3.1, the sth incomplete moment of a random variable X that follows the BBXII distribution is given by

$$\mu_s(x) = \mathbb{E}(X^s 1_{\{X\leq x\}}) = a^s k \sum_{\theta=0}^{\infty} \Delta_\theta(\theta+1)\beta_z\left[\left(k(\theta+1)-\frac{s}{b}\right),\left(\frac{s}{b}+1\right)\right], \tag{3.10}$$

where $z = a^b/(a^b + x^b)$ and $\beta_z(m, n) = \int_0^z p^{m-1}(1-p)^{n-1}dp$ is the incomplete beta function. The first incomplete moment can be obtained by taking $s = 1$ in Eq (3.10), among other incomplete moments of interest.

3.5. Entropy measures

We now aim to give the four expressions of the entropy measures, previously introduced in full generality for the Bell-X family, for the BBXII distribution.

Rényi entropy: Based on Eq (2.20) and Proposition 3.1, the Rényi entropy of the BBXII distribution can be expanded as follows:

$$R_\delta(x) = \delta^* \log\left\{\sum_{u=0}^{\infty}\sum_{z=0}^{\infty} w^*_{u,z}\beta\left[\frac{\delta(b-1)+1}{b},\frac{bk(\delta+z)+\delta-1}{b}\right]\right\}, \tag{3.11}$$

where

$$w^*_{u,z} = \left(k^\delta b^\delta a^{-b\delta}\right)\frac{a^{\delta(b-1)+1}}{b}w_u(-1)^z\binom{cu+\delta(c-1)}{z},$$

and w_u is given in Eq (2.19).

Havrda and Charvat entropy: Similarly, based on Eq (2.22) and Proposition 3.1, an expression of the Havrda and Charvat entropy is

$$HC_\delta(x) = \frac{1}{2^{1-\delta}-1}\left(\sum_{u=0}^{\infty}\sum_{z=0}^{\infty} w^*_{u,z}\beta\left[\frac{\delta(b-1)+1}{b},\frac{bk(\delta+z)+\delta-1}{b}\right]-1\right). \tag{3.12}$$

Arimoto: Similarly, based on Eq (2.22) and Proposition 3.1, an expression of the Arimoto entropy is

$$A_{\delta}(x) = \frac{\delta}{1-\delta}\left[\left(\sum_{u=0}^{\infty}\sum_{z=0}^{\infty} w_{u,z}^{*}\beta\left[\frac{\delta(b-1)+1}{b}, \frac{bk(\delta+z)+\delta-1}{b}\right]\right)^{1/\delta} - 1\right]. \tag{3.13}$$

Tsallis: Similarly, based on Eq (2.22) and Proposition 3.1, an expression of the Tsallis entropy is

$$T_{\delta}(x) = -\delta^{*}\left[1 - \sum_{u=0}^{\infty}\sum_{z=0}^{\infty} w_{u,z}^{*}\beta\left[\frac{\delta(b-1)+1}{b}, \frac{bk(\delta+z)+\delta-1}{b}\right]\right]. \tag{3.14}$$

The plot of the Rényi entropy (denoted as RE), Havrda and Charvat (denoted as HC), Arimoto and Tsallis are presented in Figure 4 by fixing the scale parameter as $a = 1$, and the following shape parameters: for the Rényi entropy $b = 1.5, k = 2.5$, for the Havrda and Charvat entropy: $b = 1.5, k = 2.5$, for the Arimoto entropy: $b = 1.5, k = 4.5$ and for the Tsallis entropy: $b = 1.5, k = 2.5$.

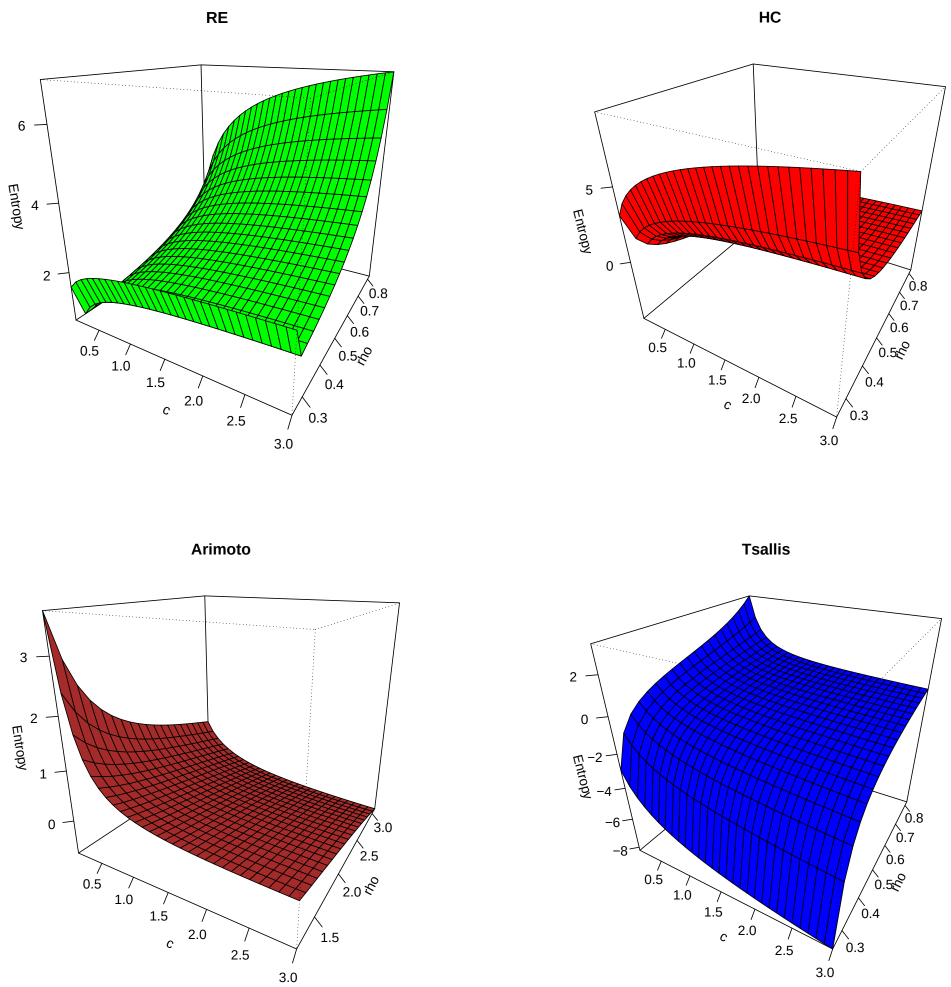


Figure 4. Graphical illustrations of entropy measures of the BBXII distribution.

As a remark, the interaction of the parameters c and δ resulted in Figure 4 shows that, as c and δ values increase, the value of RE tends to increase as well. In contrast, as c and δ values increase, the values of Arimoto entropy decrease. Various non-monotonic shapes are also observed in all the plots, illustrating the complexity of the uncertainty hidden behind the BBXII distribution.

4. Estimation

An essential component of modeling is parameter estimation. By maximizing or minimizing an objective function, system identification, inverse modeling, or parameter estimation are used to solve optimization problems and arrive at the optimal model parameters within the permissible range. Thus, by using data assimilation, parameter estimation refers to finding the optimal values for specific parameters in a numerical model. An accurate estimation of the system's parameters is necessary for a mathematical model to be able to anticipate a system's behavior in various scenarios. There are several approaches for estimating unknown parameters that have been described in the literature. Here, we apply seven well-known estimation techniques to estimate the a priori unknown parameters a, b, k and c of the BBXII model.

Hereafter, we consider $x_1,\ x_2,\ldots,x_n$ as the observations a random sample of size n from the BBXII distribution with parameters a, b, k and c. We will denote by $\mathbf{\Omega} = [a,\ b,\ k,\ c]^T$ the vector of parameters, and, for the sake of simplicity, the functions of the BBXII distribution will be sometimes written under the following forma: $F(x;\mathbf{\Omega})$ instead of $F(x;\ a,\ b,\ k,\ c)$.

4.1. Method of MLE

The log-likelihood function of $\mathbf{\Omega}$ is given by

$$
\begin{aligned}
l(\mathbf{\Omega}) = \sum_{i=1}^{n}\log[f(x_i;\mathbf{\Omega})] &= n\log(k\,c\,b) - nb\log(a) \\
&+ (b-1)\sum_{i=1}^{n}\log(x_i) - (k+1)\sum_{i=1}^{n}\log\left[1+\left(\frac{x_i}{a}\right)^b\right] \\
&+ (c-1)\sum_{i=1}^{n}\log\left(1-\left[1+\left(\frac{x_i}{a}\right)^b\right]^{-k}\right) + \sum_{i=1}^{n}\left(1-\left[1-\left[1+\left(\frac{x_i}{a}\right)^b\right]^{-k}\right]^c\right) \\
&- \sum_{i=1}^{n}e\left(1-e^{-\left[1-\left[1+\left(\frac{x_i}{a}\right)^b\right]^{-k}\right]^c}\right) - n\log\left[1-\exp(1-e)\right]. \qquad (4.1)
\end{aligned}
$$

The MLE estimates are given as the vector $\hat{\mathbf{\Omega}} = \left[\hat{a},\ \hat{b},\ \hat{k},\ \hat{c}\right]^T$ maximizing this log-likelihood function with respect to $\mathbf{\Omega}$. This maximization procedure can be done by finding the vector of parameters such that $\partial l(\mathbf{\Omega})/\partial a = 0$, $\partial l(\mathbf{\Omega})/\partial b = 0$, $\partial l(\mathbf{\Omega})/\partial k = 0$ and $\partial l(\mathbf{\Omega})/\partial c = 0$. Thus, we obtain a system of equations with no explicit solution. Consequently, it needs to use nonlinear numerical computational methods (quasi Newton-Raphson) to obtain (approximate values of) the MLE estimates.

4.2. Method of PE

The idea of the PE technique is to simultaneously solve the equations that follow from equating two values of the CDFs with their respective percentiles. Then we consider the CDF of the BBXII distribution given in Eq (3.3) and its QF is given in Eq (3.5). The PE method as described by [25] is based on the fact that $p_i = i/(n+1)$ must be an unbiased estimate of $F\left(x_{i:n};\boldsymbol{\Omega}\right)$, where $x_{1:n},\ x_{2:n},\ldots,x_{n:n}$ represent the ordered values of $x_1,\ldots,x_n$. That is, let us consider the following function:

$$P\left(\boldsymbol{\Omega}\right)=\sum_{i=1}^{n}\left[x_{i:n}-Q(p_i;\boldsymbol{\Omega})\right]^2. \tag{4.2}$$

The PE estimates are given as the vector of parameter values minimizing this function with respect to $\boldsymbol{\Omega}$. This minimization procedure can be done by finding the vector of parameters such that $\partial P\left(\boldsymbol{\Omega}\right)/\partial a = 0$, $\partial P\left(\boldsymbol{\Omega}\right)/\partial b = 0$, $\partial P\left(\boldsymbol{\Omega}\right)/\partial k = 0$ and $\partial P\left(\boldsymbol{\Omega}\right)/\partial c = 0$. These partial derivatives are complicated in the analytical sense, and they can be found in Appendix to illustrate this remark. Thus, as for the MLE estimates, we obtain a system of equations with no explicit solution, so it needs to use nonlinear numerical computational methods (quasi Newton-Raphson) to obtain the PE estimates.

4.3. Method of LSE and WLSE

Swain et al. [29] suggest the LSE and WLSE methods, initially utilized to estimate the parameters of the beta distribution. In the BBXII distribution setting, let us consider the two following functions:

$$S\left(\boldsymbol{\Omega}\right)=\sum_{i=1}^{n}\left[F(x_{i:n};\boldsymbol{\Omega})-\frac{i}{n+1}\right]^2 \tag{4.3}$$

and

$$Z(\boldsymbol{\Omega})=\sum_{i=1}^{n}w_i\left[F(x_{i:n};\boldsymbol{\Omega})-\frac{i}{n+1}\right]^2, \tag{4.4}$$

where $w_i = (n+1)^2(n+2)/[i(n-i+1)]$. The LSE estimates are given as the vector of parameter values minimizing the function $S\left(\boldsymbol{\Omega}\right)$ with respect to $\boldsymbol{\Omega}$, and the WLSE estimates are given as the vector of parameter values minimizing the function $Z(\boldsymbol{\Omega})$ with respect to $\boldsymbol{\Omega}$. Again, we can proceed with the partial derivatives of these functions, but in the end, the computer power is necessary to determine the estimates.

4.4. Method of CME

The CME method is a well-known parameter estimation technique that was first developed by [30]. In the setting of the BBXII distribution, let us consider the following function:

$$C(\boldsymbol{\Omega})=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n;\boldsymbol{\Omega}})-\frac{2i-1}{2n}\right]^2. \tag{4.5}$$

The CME estimates are given as the vector of parameter values minimizing the function $C(\boldsymbol{\Omega})$ with respect to $\boldsymbol{\Omega}$. Again, we can move forward with the partial derivatives of this function, but in the end, the estimations must be computed.

4.5. Method of ADE and RTADE

The ADE method was first presented by [31], along with a variant, the RTADE method. In the BBXII distribution setting, let us consider the two following functions:

$$A(\mathbf{\Omega}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log F(x_{i:n};\mathbf{\Omega}) + \log[S(x_{n+1-i:n};\mathbf{\Omega})]\right\} \tag{4.6}$$

and

$$R(\mathbf{\Omega}) = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{i:n};\mathbf{\Omega}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\ \log[S(x_{n+1-i:n};\mathbf{\Omega})]. \tag{4.7}$$

The ADE estimates are given as the vector of parameter values minimizing the function $A(\mathbf{\Omega})$ with respect to $\mathbf{\Omega}$, and the RTADE estimates are given as the vector of parameter values minimizing the function $R(\mathbf{\Omega})$ with respect to $\mathbf{\Omega}$. Again, we can proceed with the partial derivatives of these functions, but in the end, the computer power is necessary to determine the estimates.

5. Simulation analysis

This section includes a simulation analysis to illustrate the performance of the parameter estimates of the BBXII model over a defined replication $r = 1,000$. We make $N = 1,000$ samples of varying sizes (n = 20, 50, 100, 300) and combine the parameters a, b, k, c in various ways. For example, we take set I and set II as follows: [$a = 3.20$, $b = 1.02$, $k = 0.50$, $c = 2.15$] and [$a = 0.50$, $b = 1.30$, $k = 2.50$, $c = 1.15$], respectively. Using the MLE, PCE, LSE, WLSE, CME, ADE, and RTADE methods as described in the aforementioned section, the unknown parameters are estimated. The simulation study's findings demonstrated that, as sample size increases, mean square errors (MSEs) and biases decrease. Therefore, the confidence intervals for the BBXII model can be constructed using these estimates (asymptotically unbiased estimators). The results of the simulation analysis of average estimates (AEs) and MSEs in parentheses, as well as estimate biases, are shown in Tables 2 and 3. All estimation techniques are working effectively, but MLE seems to be stable. The R computation programming language is used to perform all of the calculations for the various estimation methods.

Table 2. Simulation result summary of the AEs and MSEs of various estimation methods.

	MLE	PCE	LSE	WLSE	CME	ADE	RTADE
				$n = 20$			
$\hat{a}$	3.158(0.076)	3.206(0.107)	2.796(0.496)	2.842(0.448)	2.890(0.358)	2.883(0.382)	2.852(0.449)
$\hat{b}$	1.027(0.042)	1.461(0.264)	1.029(0.052)	1.044(0.050)	1.099(0.061)	1.071(0.052)	1.089(0.061)
$\hat{k}$	0.518(0.002)	0.546(0.022)	0.459(0.007)	0.460(0.008)	0.458(0.007)	0.461(0.008)	0.456(0.007)
$\hat{c}$	1.979(0.030)	2.138(0.053)	2.089(0.020)	2.089(0.024)	2.110(0.013)	2.102(0.020)	2.107(0.015)
				$n = 50$			
$\hat{a}$	3.212(0.022)	3.225(0.099)	2.901(0.278)	2.920(0.263)	2.928(0.250)	2.938(0.243)	2.929(0.258)
$\hat{b}$	1.006(0.018)	1.501(0.278)	1.054(0.029)	1.061(0.028)	1.086(0.033)	1.068(0.027)	1.087(0.035)
$\hat{k}$	0.510(0.004)	0.571(0.021)	0.456(0.005)	0.459(0.006)	0.454(0.005)	0.459(0.006)	0.455(0.006)
$\hat{c}$	1.980(0.027)	2.146(0.054)	2.092(0.015)	2.101(0.014)	2.107(0.010)	2.103(0.015)	2.106(0.013)
				$n = 100$			
$\hat{a}$	3.233(0.007)	3.237(0.062)	2.931(0.205)	2.940(0.210)	2.944(0.192)	2.974(0.169)	2.960(0.177)
$\hat{b}$	1.003(0.009)	1.519(0.286)	1.063(0.019)	1.060(0.017)	1.080(0.021)	1.069(0.016)	1.093(0.024)
$\hat{k}$	0.508(0.002)	0.592(0.023)	0.455(0.005)	0.460(0.005)	0.454(0.004)	0.463(0.005)	0.456(0.005)
$\hat{c}$	1.981(0.029)	2.146(0.064)	2.090(0.014)	2.102(0.011)	2.096(0.012)	2.093(0.013)	2.093(0.013)
				$n = 300$			
$\hat{a}$	3.244(0.005)	3.259(0.025)	2.974(0.141)	2.987(0.145)	2.98(0.140)	3.000(0.128)	2.991(0.127)
$\hat{b}$	1.002(0.004)	1.540(0.290)	1.065(0.009)	1.057(0.007)	1.070(0.009)	1.061(0.008)	1.077(0.011)
$\hat{k}$	0.507(0.001)	0.612(0.025)	0.457(0.004)	0.466(0.003)	0.458(0.003)	0.463(0.003)	0.458(0.004)
$\hat{c}$	2.012(0.023)	2.167(0.030)	2.080(0.014)	2.099(0.011)	2.086(0.013)	2.095(0.013)	2.089(0.013)

Table 3. Simulation result summary of the AEs and MSEs of various estimation methods.

	MLE	PCE	LSE	WLSE	CME	ADE	RTADE
				$n = 20$			
$\hat{a}$	0.465(0.015)	0.437(0.061)	0.511(0.031)	0.495(0.022)	0.479(0.014)	0.483(0.016)	0.478(0.016)
$\hat{b}$	1.256(0.080)	1.352(0.122)	1.084(0.120)	1.107(0.110)	1.124(0.097)	1.121(0.101)	1.142(0.090)
$\hat{k}$	2.493(0.025)	2.544(0.036)	2.343(0.094)	2.375(0.082)	2.385(0.070)	2.383(0.076)	2.399(0.062)
$\hat{c}$	0.912(0.188)	0.539(0.556)	0.974(0.162)	0.971(0.171)	0.953(0.149)	0.960(0.157)	0.922(0.163)
				$n = 50$			
$\hat{a}$	0.450(0.008)	0.424(0.027)	0.472(0.008)	0.461(0.008)	0.462(0.006)	0.464(0.007)	0.463(0.006)
$\hat{b}$	1.223(0.071)	1.334(0.170)	1.111(0.095)	1.145(0.079)	1.115(0.091)	1.140(0.079)	1.173(0.068)
$\hat{k}$	2.488(0.022)	2.557(0.030)	2.374(0.065)	2.394(0.058)	2.392(0.060)	2.390(0.059)	2.407(0.052)
$\hat{c}$	0.908(0.145)	0.408(0.646)	0.948(0.127)	0.925(0.124)	0.953(0.121)	0.923(0.123)	0.875(0.135)
				$n = 100$			
$\hat{a}$	0.477(0.006)	0.415(0.016)	0.459(0.005)	0.452(0.005)	0.459(0.005)	0.453(0.005)	0.451(0.005)
$\hat{b}$	1.276(0.059)	1.231(0.270)	1.130(0.081)	1.160(0.066)	1.139(0.077)	1.131(0.080)	1.184(0.058)
$\hat{k}$	2.499(0.018)	2.572(0.015)	2.389(0.051)	2.436(0.029)	2.403(0.046)	2.409(0.045)	2.405(0.045)
$\hat{c}$	0.938(0.117)	0.309(0.760)	0.930(0.121)	0.910(0.114)	0.919(0.115)	0.939(0.113)	0.865(0.131)
				$n = 300$			
$\hat{a}$	0.490(0.005)	0.404(0.010)	0.450(0.004)	0.441(0.005)	0.450(0.004)	0.450(0.004)	0.449(0.004)
$\hat{b}$	1.284(0.046)	1.548(0.081)	1.132(0.077)	1.188(0.052)	1.131(0.075)	1.155(0.066)	1.201(0.048)
$\hat{k}$	2.488(0.014)	2.587(0.008)	2.409(0.041)	2.441(0.027)	2.414(0.038)	2.416(0.037)	2.415(0.035)
$\hat{c}$	0.987(0.090)	0.727(0.604)	0.937(0.111)	0.890(0.111)	0.936(0.108)	0.911(0.111)	0.847(0.128)

6. Applications

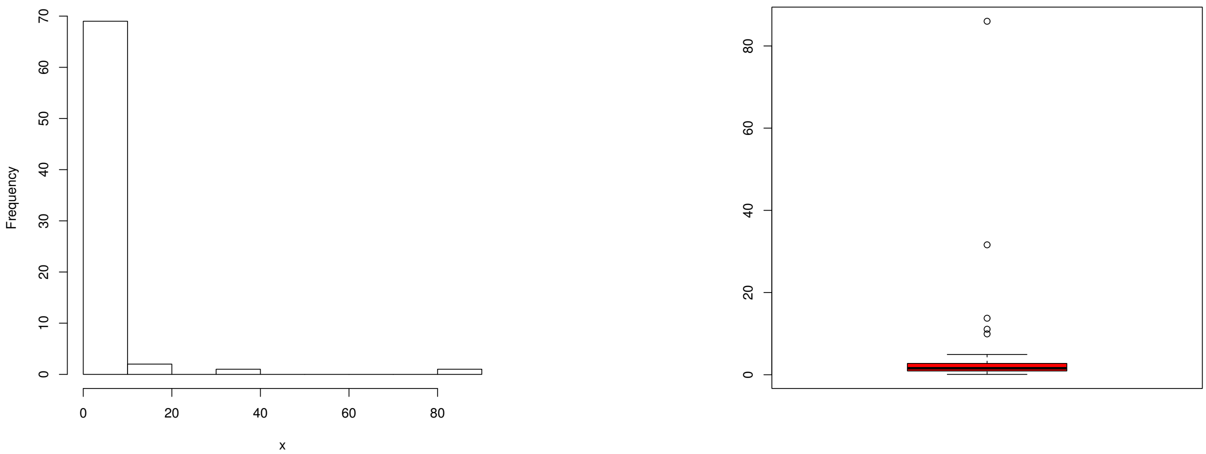
Here, we demonstrate the significance of the BBXII model by using two actual datasets. The BBXII model is contrasted with several well-known models, including the EPBXII [7], KBXII [8], MBXII [9], TLBXII [10], B-BXII [11], and BXII [12] models. Let us now present our first dataset under consideration. Based on 73 patients' acute bone cancer survival rates, expressed in days, the data are as follows: 0.090, 0.760, 1.810, 1.100, 3.720, 0.720, 2.490, 1.000, 0.530,0.660, 31.610, 0.600, 0.200, 1.610, 1.880, 0.700, 1.360, 0.430, 3.160, 1.570, 4.930, 11.070, 1.630, 1.390, 4.540, 3.120,86.010, 1.920, 0.920, 4.040, 1.160, 2.260, 0.200, 0.940, 1.820, 3.990, 1.460, 2.750, 1.380, 2.760, 1.860, 2.680, 1.760, 0.670, 1.290, 1.560, 2.830, 0.710, 1.480, 2.410, 0.660, 0.650, 2.360, 1.290, 13.750, 0.670, 3.700, 0.760, 3.630, 0.680,2.650, 0.950, 2.300, 2.570, 0.610, 3.930, 1.560, 1.290, 9.940, 1.670, 1.420, 4.180 and 1.370. This dataset is named Data-1. Patient survival times range from 0.090 days (minimum) to 86.01 days (maximum), respectively, while it takes 3.755 days on average to survive. The data are represented by an histogram and a boxplot in Figure 5.

It is clear from Figure 5 that the data are right-skewed with several extreme values, so far beyond a normal distribution representation. Recently, Mansour et al. [32] suggested the novel Burr XII distribution and fitted it to these data. He found the AD test of 0.66184 and the CM test of 0.1046 as accuracy measures for model selection. When the same dataset is used and our proposed model is fitted to it, the results reveal superior fitting because both measures are at their lowest values, with the AD test of 0.31107 and the CM test of 0.0422. The second dataset considered may be described as follows: A complete sample from a clinical study is provided in the second dataset, and 50 individuals with arthritis are chosen to report their relief time (in hours). Joint stiffness and pain are the main signs and symptoms of arthritis, and these symptoms usually get worse as people age. The two most prevalent kinds of arthritis are rheumatoid arthritis and osteoarthritis. Osteoarthritis results in the deterioration of cartilage, the tough, slippery tissue covering the ends of bones where they meet to create a joint. The following data are extracted from [33]: 0.700, 0.840, 0.580, 0.500, 0.550, 0.820, 0.590, 0.710, 0.720, 0.610, 0.570, 0.440, 0.440,0.620, 0.490, 0.540, 0.360, 0.360, 0.710, 0.350, 0.640, 0.840, 0.550, 0.730, 0.800, 0.870, 0.590, 0.290 ,0.750, 0.460, 0.460, 0.600, 0.600, 0.360, 0.520, 0.680, 0.750, 0.500, 0.610, 0.560, 0.490, 0.700, 0.340, 0.340, 0.840, 0.550, 0.800, 0.810, 0.290 and 0.710. This dataset is named Data-2. The mean relief time is 0.5906 hours. The data are represented by an histogram and a boxplot in Figure 6.

From Figure 6, we observe that the data are almost symmetrical and unimodal. The estimated parameters together with the AD, CM, and p-value (PV) for the Kolmogorov-Smirnov (KS) test are shown in Table 4. Seven different estimation techniques were used, and the analysis showed that MLE performed better, having the lowest results for the AD, CM, and KS tests and the highest PV.

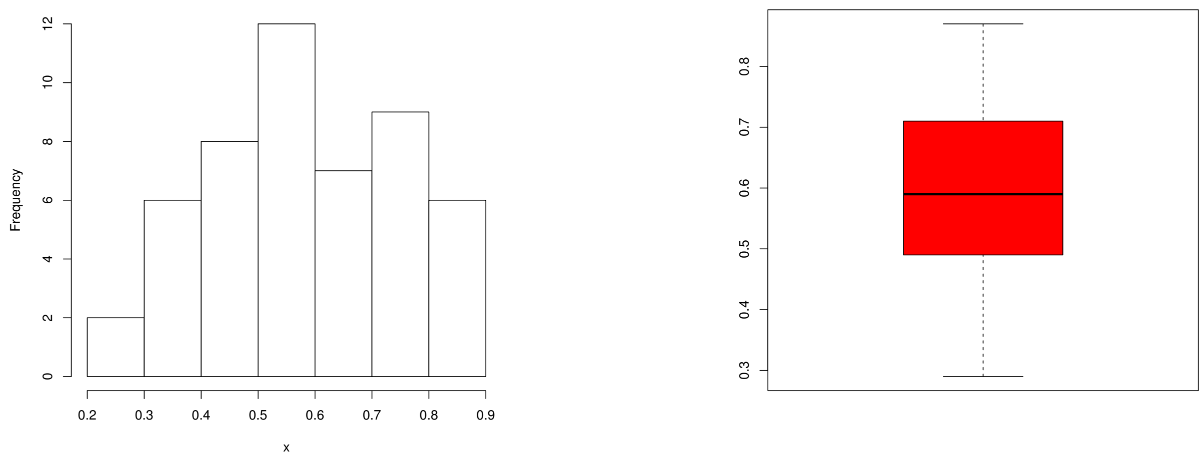
Tables 5 and 6 show the fitted models with estimated parameters via the MLE method, standard errors (SEs), and goodness of fit tests for both datasets. The general findings of the analysis revealed that the proposed BBXII model outperforms the competitive models for both datasets, due to least goodness of fit measures, AD [0.3107; 0.3432], CM [0.0422; 0.0445] and KS [0.0683; 0.0772] and higher PV [0.8849; 0.9267], respectively. As for the simulation part, the R programming language is used for all computational work.

Graphical illustrations of fitted Data-1 and Data-2 under the BBXII model can be found in Figures 7 and 8, respectively. We observe that all the data objects are well fitted by the estimated model. This visually confirms the numerical interpretation of the previous tables. We complete this graphical work by showing the probability-probability (P-P) plots of the considered estimation methods based on Data-1 and Data-2 in Figures 9 and 10, respectively.



(a) (b)

Figure 5. Plots of (a) the histogram and (b) the boxplot of Data-1.



(a) (b)

Figure 6. Plots of (a) the histogram and (b) the boxplot of Data-2.

Table 4. Result summary of the various estimation methods.

Method	$\hat{a}$	$\hat{b}$	$\hat{k}$	$\hat{c}$	$-2\hat{\ell}$	KS	PV
Data-1							
MLE	2.5035	3.6552	0.2245	0.4804	138.1650	0.0683	0.8849
PCE	2.0454	4.0976	0.1362	0.4534	138.9397	0.0854	0.6620
LSE	3.3182	0.7624	2.8585	4.7395	140.4668	0.0719	0.8450
WLSE	3.3856	0.9052	2.4535	3.3356	140.3859	0.0708	0.8577
CME	1.7302	0.9049	1.8272	4.2053	140.4170	0.0740	0.8184
ADE	1.8236	0.9569	1.6388	3.4315	139.8749	0.0778	0.7694
RTADE	2.0716	2.3838	0.4449	0.8747	138.4828	0.0747	0.8105
Data-2							
MLE	0.9616	23.1756	11.5502	0.1774	22.7833	0.0772	0.9267
PCE	0.9720	9.5006	3.2861	0.4314	21.0036	0.0773	0.9260
LSE	0.8233	7.5933	0.8197	0.5461	18.6692	0.0885	0.8282
WLSE	1.0367	4.5754	3.8267	1.0547	19.3969	0.0830	0.8816
CME	1.0159	5.2182	3.2196	0.8550	19.4207	0.0798	0.9075
ADE	1.0007	7.6908	3.6513	0.5395	20.5754	0.0812	0.8965
RTADE	0.9401	8.0645	2.6614	0.5544	20.6363	0.0853	0.8597

Table 5. The detailed statistical summary of the fitted models of Data-1.

Model	Estimates with SEs						AD	CM	KS	PV
BBXII	MLE	2.5035	3.6552	0.2245	0.4804	–	0.3107	0.0422	0.0683	0.8849
	SE	0.7714	2.3145	0.2354	0.3742	–				
EPBXII	MLE	1.9392	4.0796	0.2568	0.4118	–	0.3253	0.0436	0.0708	0.8578
	SE	0.5169	2.1910	0.2143	0.2829	–				
KBXII	MLE	1.6629	2.6563	0.0086	0.7348	30.4729	0.3607	0.0478	0.0713	0.8521
	SE	0.6503	0.6046	0.0076	0.2249	13.8601				
MBXII	MLE	3.7224	2.1189	0.3573	0.0559	–	0.3489	0.0473	0.0695	0.8737
	SE	2.7944	0.2649	0.3240	0.1116	–				
TLBXII	MLE	1.6525	4.2474	0.1380	0.3796	–	0.3404	0.0453	0.0727	0.8346
	SE	0.4565	2.3838	0.1146	0.2808	–				
B-BXII	MLE	1.6661	4.0125	0.0992	0.4047	2.6861	0.3354	0.0447	0.0722	0.8408
	SE	0.4492	2.1233	0.4321	0.2848	10.857				
BXII	MLE	1.2024	2.2157	0.6612	–	–	0.4566	0.0611	0.0718	0.8456
	SE	0.2877	0.3837	0.2174	–	–				

Table 6. The detailed statistical summary of the fitted models of Data-2.

Model		Estimates with SEs					AD	CM	KS	PV
BBXII	MLE	0.9616	23.1756	11.5502	0.1774	–	0.3432	0.0445	0.0772	0.9267
	SE	0.0280	0.0955	0.4674	0.0196	–				
EPBXII	MLE	0.9435	24.0679	15.5320	0.1320	–	0.5238	0.0793	0.1224	0.4426
	SE	0.0187	0.0682	0.3918	0.0171	–				
KBXII	MLE	2.1205	13.3822	5.3173	0.3157	85.075	0.3901	0.0477	0.0827	0.8834
	SE	0.1819	0.1424	0.5857	0.0332	45.650				
MBXII	MLE	1.5510	4.4574	43.574	0.8254	–	0.3922	0.0466	0.0881	0.8327
	SE	0.9578	0.9930	113.62	0.8336	–				
TLBXII	MLE	1.2645	6.0841	15.988	0.5810	–	0.4310	0.0584	0.0899	0.8143
	SE	0.7162	4.1118	44.349	0.5668	–				
B-BXII	MLE	1.6996	5.8230	11.789	0.6220	11.9329	0.4209	0.0561	0.0862	0.8511
	SE	2.2049	4.3171	76.911	0.6637	80.5337				
BXII	MLE	1.5490	4.2880	41.877	–	–	0.3944	0.0477	0.0851	0.8616
	SE	0.9584	0.4966	104.20	–	–				

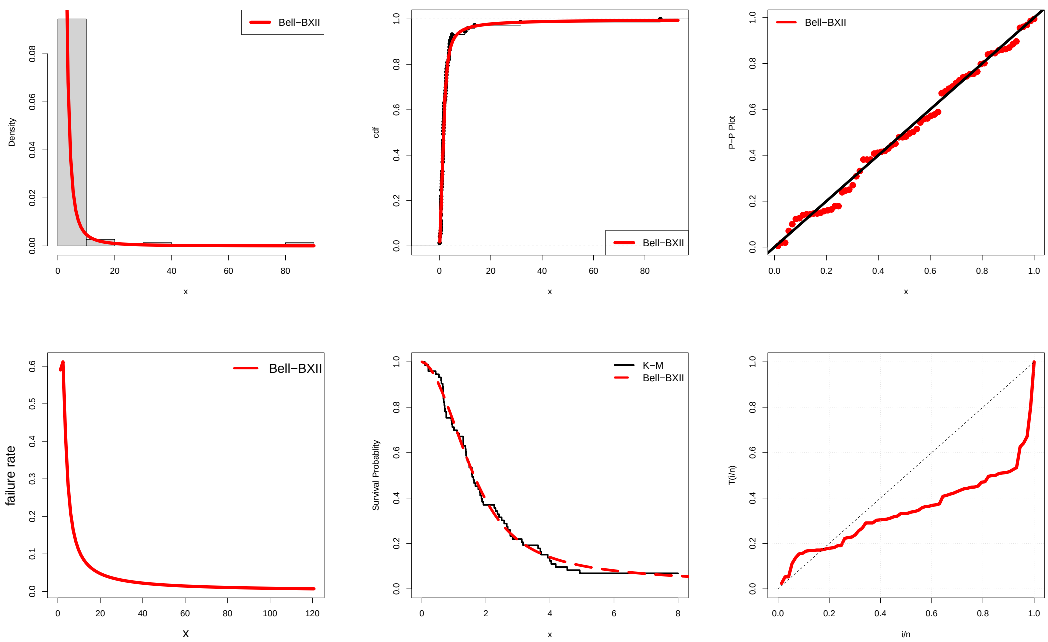


Figure 7. Graphical illustrations of fitted Data-1 under the BBXII model.

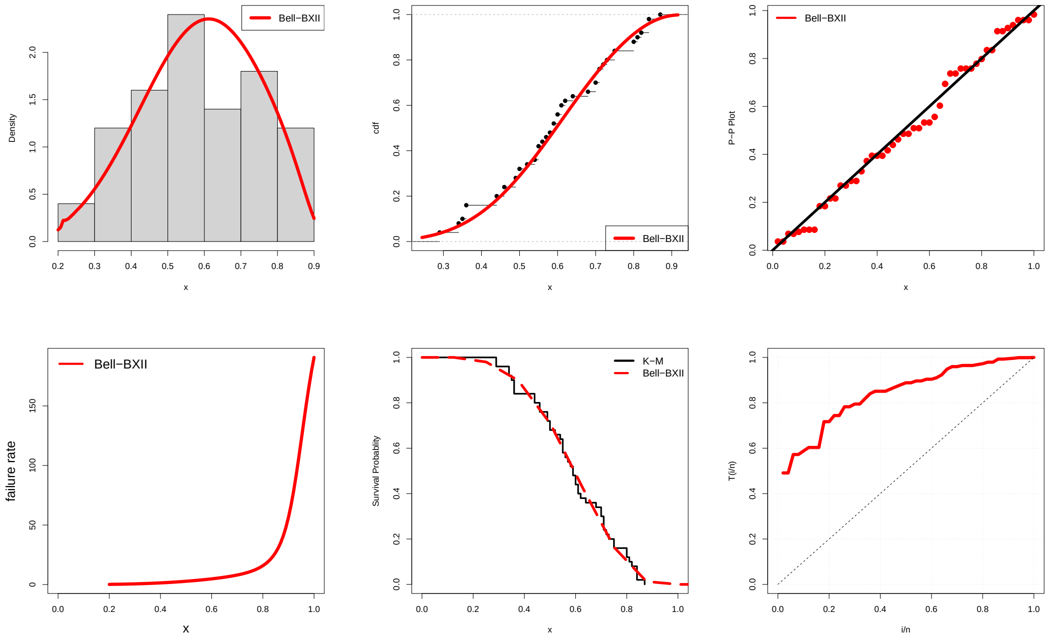


Figure 8. Graphical illustrations of fitted Data-2 under the BBXII model.

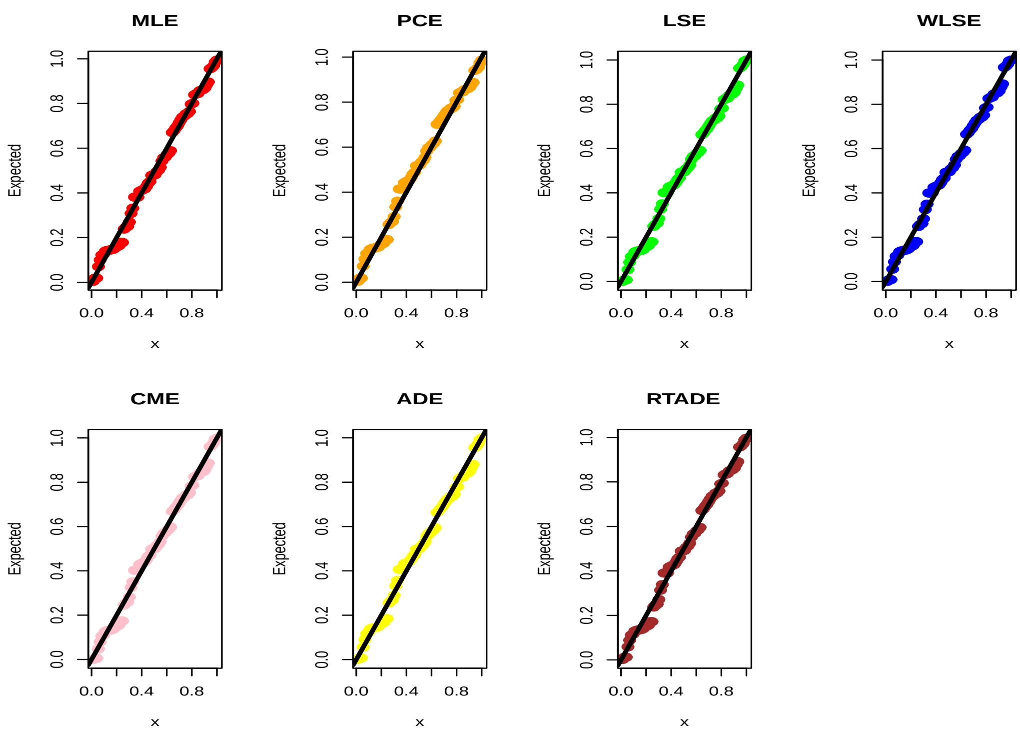


Figure 9. P-P plots of the BBXII model under various estimation methods for Data-1.

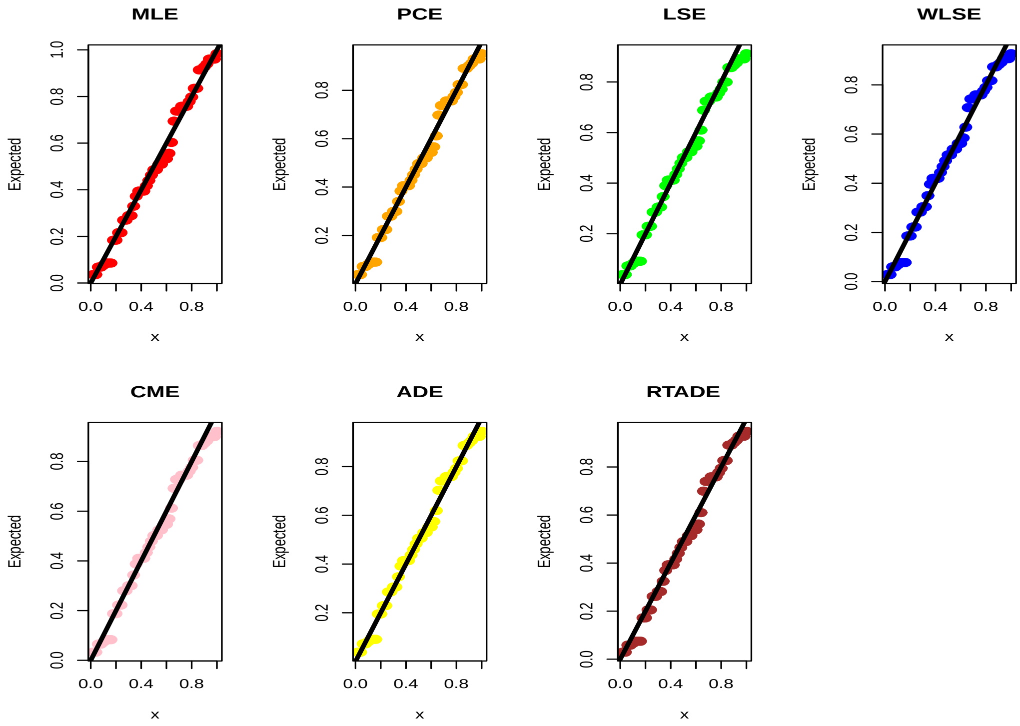


Figure 10. P-P plots of the BBXII model under various estimation methods for Data-2.

7. GASP based on the BBXII distribution

7.1. Motivation

In today's fiercely competitive global market, product quality is a key consideration for both manufacturers and customers. However, due to time and cost limitations, a complete inspection is often not considered. Consequently, a product's quality is ensured using the acceptance sampling technique and statistical process control. A statistical quality control or reliability test that evaluates whether it should accept or reject a given lot is called an acceptance sampling plan (ASP).

For a shortened life test, it is hard to choose an appropriate sample size from a large number of items/products being investigated. To do this, the smallest testing sample can be determined using ASP. These plans are crucial instruments to assess a product's reliability. Unlike a GASP, which inspects numerous items based on the number of testers accessible to the experimenter for testing, the ordinary ASP only inspects one item at a time. This demonstrates that GASP is superior to the traditional or ordinary sampling strategy since testing time and cost can be greatly reduced. Variable plans, characteristics plans, expedited plans, gradually progressive plans, and group plans are all examples of ASP [35–40]. However, the fundamental objective of these methods is to safeguard both the producer and the consumer while deciding lot sentencing (acceptance or rejection of a lot) using a small sample size. The creation of ASP is a topic that is commonly discussed regarding quality assurance and reliability for lot acceptance criteria.

7.2. Method

This part of the paper focuses on an optimization strategy for constrained situations. By utilizing numerous things on a single tester, the experimenter can cut down on both the expense and the length of the test time or experiment. Because samples are split up among various groups, GASP delivers a stricter product inspection than ordinary ASP. We adopt the median as a quality index since the QF function for the suggested BBXII model is close to being ideal. To ensure that the median life of the items in a lot, say m, is longer than the required life, let's say m_0, we are interested in constructing a sampling strategy. If there is strong evidence that $m > m_0$ at specific levels of producer and consumer risks, we will accept the lot. The lot's acceptance probability is given by

$$p_{a(p)} = \left(\sum_{i=0}^{c} \binom{r}{i} p^i (1-p)^{r-i} \right)^g , \tag{7.1}$$

where p is the probability that a group item will expire prior to t_0 and m is derived by substituting Eq (3.5) in Eq (3.3), that is,

$$m = a\left[\left(1 - \left\{ 1 - \log\left(\log\left[1 - \left(1 - e^{1-e} \right) p \right] + e \right) \right\}^{1/c} \right)^{-1/k} - 1 \right]^{1/b} . \tag{7.2}$$

Now, considering

$$\eta = \left[\left(1 - \left\{ 1 - \log\left(\log\left[1 - \left(1 - e^{1-e} \right) p \right] + e \right) \right\}^{1/c} \right)^{-1/k} - 1 \right]^{1/b} , \tag{7.3}$$

and setting $a = m/\eta$ and $t = a_1\, m_0$ in Eq (3.3). The likelihood of failure is given by

$$F(t) = \frac{1 - \exp\left\{ -e\left[1 - e^{-\left(1 - \left[1 + (\frac{a_1 \eta}{r_2})^b \right]^{-k} \right)^c} \right] \right\}}{1 - \exp\left(1 - e \right)}, \tag{7.4}$$

where a_1 and r_2 are predetermined, with $r_2 = m/m_0$. Hence, the likelihood of failure can be calculated for chosen b, k and c. Here, we evaluate the two failing probabilities associated with the consumer risk and the producer risk, denoted as p_1 and p_2, respectively, based on the BBXII model. We aim to evaluate the design parameters that simultaneously satisfy the following two equations, for a given specific value of the constraints b, k, c, a_1, r_2, β and γ:

$$p_{a\left(p_1 | \frac{m}{m_0} = r_1\right)} = \left(\sum_{i=0}^{c} \binom{r}{i} p_1^i (1-p_1)^{r-i} \right)^g \leq \beta \tag{7.5}$$

and

$$p_{a\left(p_2 | \frac{m}{m_0} = r_2\right)} = \left(\sum_{i=0}^{c} \binom{r}{i} p_2^i (1-p_2)^{r-i} \right)^g \geq 1 - \gamma, \tag{7.6}$$

where the average ratio of producer risk to consumer risk is represented by the numbers r_1 and r_2. Based on Eqs (7.5) and (7.6), the likelihood of failure under the BBXII model is given by

$$p_1 = \frac{1 - \exp\left\{ -e\left[1 - e^{-\left(1 - \left[1 + (a_1 \eta)^b \right]^{-k} \right)^c} \right] \right\}}{1 - \exp\left(1 - e \right)} \tag{7.7}$$

and

$$p_2 = \frac{1 - \exp\left\{-e\left[1 - e^{-\left(1-\left[1+(a_1\eta/r_2)^b\right]^{-k}\right)^c}\right]\right\}}{1 - \exp\left(1 - e\right)}. \quad (7.8)$$

Table 7 shows the design parameters at arbitrary values under the BBXII model.

Table 7. GASP of the BBXII model for the arbitrary values chosen as $a = 1.0$, $b = 1.5$, $c = 1$ and $k = 1.5$, minimizing design parameters.

		$r = 5$						$r = 10$					
		$a_1 = 0.5$			$a_1 = 1$			$a_1 = 0.5$			$a_1 = 1$		
β	r_2	g	c	$p(a)$	g	c	$p(a)$	g	c	$p(a)$	g	c	$p(a)$
0.25	2	94	3	0.957	44	4	0.961	19	4	0.9688	–	–	–
	4	14	2	0.993	3	2	0.9743	3	2	0.9845	1	3	0.987
	6	4	1	0.9837	1	1	0.9712	2	1	0.9658	1	2	0.9837
	8	4	1	0.9929	1	1	0.987	2	1	0.9848	1	2	0.9948
0.1	2	–	–	–	–	–	–	125	5	0.9814	–	–	–
	4	22	2	0.9891	4	2	0.9658	4	2	0.9794	2	3	0.9742
	6	6	1	0.9756	4	2	0.9933	2	1	0.9658	1	2	0.9837
	8	6	1	0.9894	2	1	0.9741	2	1	0.9848	1	2	0.9948
0.05	2	–	–	–	–	–	–	163	5	0.9758	–	–	–
	4	29	2	0.9857	5	2	0.9575	5	2	0.9743	2	3	0.9742
	6	7	1	0.9716	5	2	0.9916	5	2	0.9952	2	2	0.9677
	8	7	1	0.9876	2	1	0.9741	3	1	0.9773	2	2	0.9896
0.01	2	–	–	–	–	–	–	250	5	0.9631	–	–	–
	4	45	2	0.9783	23	2	0.9893	8	2	0.9592	3	3	0.9616
	6	11	1	0.9557	7	2	0.9883	8	2	0.9924	2	2	0.9677
	8	11	1	0.9807	3	1	0.9614	4	1	0.9698	2	2	0.9896

*Note: Cells for a very large sample size are marked with hyphens (–).

7.3. Real data application

Here, we consider carbon fiber breaking strength data for GASP illustration, recently used by [34], for designing GASP under the so-called MOKw-E model. The data are as follows: 1.1200, 0.1700, 0.6400, 4.3200, 1.2200, 0.3700, 1.1600, 1.4200, 0.0900, 1.6700, 0.1300, 0.2500, 0.0800, 0.0400, 2.3500, 0.2000, 0.7800, 0.3400, 1.0200, 0.1700, 1.7600, 2.3900, 0.5000, 1.3500, 3.3600, 0.4500, 0.9000, 2.9200, 6.5300, 1.6200, 7.4600, 3.1900, 2.4900, 1.4000, 7.4900, 0.5700, 0.1400, 0.6300, 5.2300, 0.7100, 0.6800, 0.1200, 0.0900, 3.4700, 5.9300, 1.8200, 4.2000, 7.2900, 3.1300 and 3.4100. The related dataset is named Data-3. The parameters are estimated using different methods in Table 8.

Table 8. Output summary of the various estimation methods.

Data-3							
Method	$\hat{a}$	$\hat{b}$	$\hat{k}$	$\hat{c}$	$-2\hat{\ell}$	KS	PV
MLE	8.4960	5.7675	0.8072	0.1399	87.688	0.0567	0.9971
PCE	18.775	1.6411	5.0243	0.5536	88.034	0.0785	0.9179
LSE	9.6398	0.5969	2.6646	2.0711	90.639	0.0966	0.7397
WLSE	20.689	1.2128	3.9184	0.7051	88.791	0.0616	0.9914
CME	6.0295	0.9582	1.6804	1.1553	89.785	0.0779	0.9214
ADE	2.6899	0.5068	2.1355	3.4264	91.406	0.0981	0.7212
RTADE	7.6934	0.9783	2.2681	1.2291	89.634	0.0996	0.7037

Since the MLE method yields better PV and KS tests, we use it for our data application of the considered GASP. The estimated parameters with SEs of the estimates under the BBXII model are given by $\hat{a} = 8.4960(0.03512)$, $\hat{b} = 5.7675(0.0251)$, $\hat{k} = 0.8072(0.0554)$ and $\hat{c} = 0.1399(0.0138)$. According to the KS test, there is a maximum difference of 0.0567 with a PV of 0.9971 between the original data and the data that is based on the BBXII distribution. Table 9 shows the design parameters based on the BBXII model. Here, $a_1 = [0.5, 1]$ and $r = [5, 10]$. The true median life r_2 ranges from 2, 4, 6, 8 with $\beta = [0.25, 0.1, 0.05, 0.01]$. When $\beta = 0.25$, $r_2 = 4$, $a_1 = 0.5$, the number of units needed to put on a life test should be 160 ($32 \times 5 = 160$), but when r is increased to 10 a considerable reduction can be observed, only 60 total units are needed to put on a life test.

Here, 10 groups would be preferred in this situation. The general finding from Table 9 clearly shows that when the r_2 grows, the number of groups for the considered GASP decreases, and the operating characteristic values increase. As it is obvious from Table 10, when $\beta = 0.1$, $a_1 = 1$ and $r = 10$, the design parameters decreased and operating curve values increased as r_2 increased.

The study ends with a graphical analysis of Data-3 analyzed with the BBXII model, to complete the related GASP. Figure 11 illustrates the nice fit of the BBXII model, whereas Figure 12 presents the P-P plots of the considered estimation methods.

Table 9. GASP of the BBXII model for $\hat{a}$ = 8.4960, $\hat{b}$ = 5.7675, $\hat{c}$ = 0.1399 and $\hat{k}$= 0.8072, minimizing design parameters with Data-3.

		$r = 5$						$r = 10$					
		$a_1 = 0.5$			$a_1 = 1$			$a_1 = 0.5$			$a_1 = 1$		
β	r_2	g	c	$p(a)$	g	c	$p(a)$	g	c	$p(a)$	g	c	$p(a)$
0.25	2	–	–	–	–	–	–	–	–	–	–	–	–
	4	32	3	0.9663	44	4	0.9838	6	4	0.9743	3	5	0.9778
	6	7	2	0.9547	7	3	0.983	3	3	0.9724	2	4	0.9782
	8	7	2	0.9757	3	2	0.9534	3	3	0.9874	1	3	0.9731
0.1	2	–	–	–	–	–	–	–	–	–	–	–	–
	4	588	4	0.9828	73	4	0.9732	10	4	0.9575	5	5	0.9633
	6	52	2	0.9834	12	3	0.971	5	3	0.9544	4	4	0.9675
	8	11	2	0.9621	12	3	0.9872	5	3	0.9791	3	4	0.9871
0.05	2	–	–	–	–	–	–	–	–	–	–	–	–
	4	764	4	0.9777	95	4	0.9653	40	5	0.9805	–	–	–
	6	68	3	0.9783	15	3	0.9639	13	4	0.9861	3	4	0.9569
	8	14	2	0.9520	15	3	0.984	6	3	0.9750	4	4	0.9828
0.01	2	–	–	–	–	–	–	–	–	–	–	–	–
	4	–	–	–	–	–	–	61	5	0.9704	–	–	–
	6	104	3	0.9671	146	4	0.9876	20	4	0.9787	10	5	0.9843
	8	104	3	0.9860	23	3	0.9756	9	3	0.9628	5	4	0.9785

*Note: Cells for a very large sample size are marked with hyphens (–).

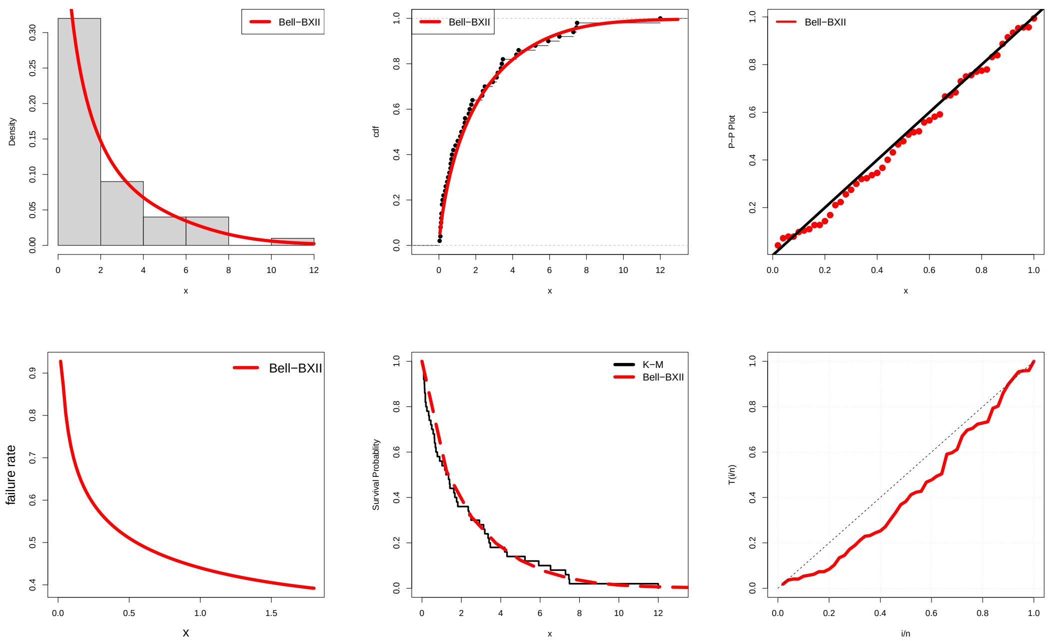


Figure 11. Graphical illustrations of fitted Data-3 under the BBXII model.

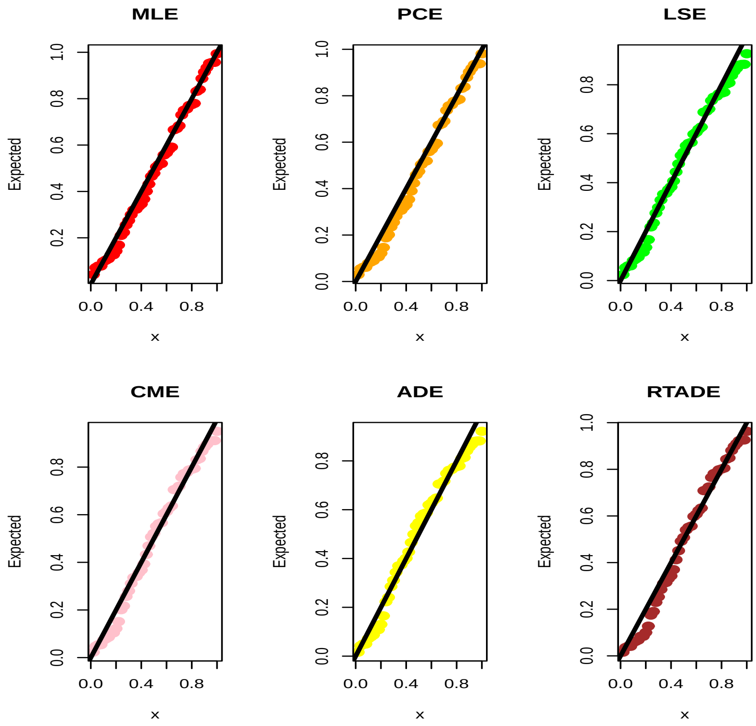


Figure 12. P-P plots of the BBXII model under various estimation methods for Data-3.

Table 10. The proposed GASP for the carbon fiber breaking strength.

r_2	2	4	6	8
n	–	50	40	30
g	–	5	4	3
c	–	5	4	4
$p(a)$	–	0.9633	0.9675	0.9871

8. Conclusions

We develop the Bell-X family, as a sub-family of the one introduced in [17]. A special distribution, called the BBXII distribution, extending the functionalities of the famous BXII distribution, is highlighted It is of particular interest because it can take on various shapes, such as unimodal, upside-down bathtub-shaped, reversed upside-down bathtub-shaped, increasing, and declining trends. Several important properties of the BBXII distribution are presented, such as useful functional expansions, quantiles, moments, skewness, kurtosis, and various entropy measures, such as Rényi, Havrda and Charvat, Arimoto, and Tsallis. Seven frequentist estimation methods with complete mathematical illustrations are presented for parameter estimation. Additionally, a group acceptance sampling plan is constructed based on the BBXII distribution, utilizing the median as a quality criterion. On the applied side, the BBXII model is successfully used for the analysis of acute bone cancer and arthritis relief time data. It outperforms some well-established and widely used extended

BXII models, such as the EPBXII [7], KBXII [8], MBXII [9], TLBXII [10], B-BXII [11], and BXII [12] models. When an item's lifetime follows a BBXII distribution, a group acceptance sampling strategy is also suggested for truncated life tests, which produces quite satisfying results. As a final comment, since the BXII distribution is commonly employed in various applied fields, including economics and finance, actuarial science, medicine, engineering, and quality control, the same fate is expected for the proposed BBXII model.

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Conflicts of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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Appendix

Partial derivatives of the log-likelihood

$$\frac{\partial l(\boldsymbol{\Omega})}{\partial a} = -\frac{nb}{a} + (k+1)\sum_{i=1}^{n} \frac{b x_i^b \left[1+\left(\frac{x_i}{a}\right)^b\right]^{-1}}{a^{b+1}} - (c-1)\sum_{i=1}^{n} \frac{k b x_i^b \left[a^b + (x_i)^b\right]^{-k-1}}{a^{-kb+1}\left[1-\left(1+\left(\frac{x_i}{a}\right)^b\right)^{-k}\right]}$$
$$+ \sum_{i=1}^{n} \frac{kbc\, x_i^b \left[x_i^b + a^b\right]^{-k-1}\left[-\left(\left(\frac{x_i}{a}\right)^b+1\right)^{-k}+1\right]^{c-1}}{a^{-kb+1}}$$
$$+ \sum_{i=1}^{n} \frac{k\,b\,c\, e^{-\left[-\left(\left(\frac{x_i}{a}\right)^b+1\right)^{-k}+1\right]^c+1} x_i^b \left[x_i^b + a^b\right]^{-k-1}\left[-\left(\left(\frac{x_i}{a}\right)^b+1\right)^{-k}+1\right]^{c-1}}{a^{-kb+1}},$$

$$\frac{\partial l(\boldsymbol{\Omega})}{\partial b} = \frac{n}{b} - n\log(a) + \sum_{i=1}^{n}\log(x_i) - (k+1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{a}\right)^b \log\left(\frac{x_i}{a}\right)}{1+\left(\frac{x_i}{a}\right)^b}$$

$$+ (c-1)\sum_{i=1}^{n} x_i^b \frac{k a^{kb}\log\left(\frac{x_i}{a}\right)\left[x_i^b + a^b\right]^{-k-1}}{1-\left[1+\left(\frac{x_i}{a}\right)^b\right]^{-k}}$$
$$- ck\sum_{i=1}^{n}\left(\frac{x_i}{a}\right)^b \log\left(\frac{x_i}{a}\right)\left[\left(\frac{x_i}{a}\right)^b+1\right]^{-k-1}\left[-\left(\left(\frac{x_i}{a}\right)^b+1\right)^{-k}+1\right]^{c-1}$$
$$- ck\sum_{i=1}^{n} e^{1-\left[-\left(\left(\frac{x_i}{a}\right)^b+1\right)^{-k}+1\right]^c}\left(\frac{x_i}{a}\right)^b \log\left(\frac{x_i}{a}\right)\left[\left(\frac{x_i}{a}\right)^b+1\right]^{-k-1}\left[-\left(\left(\frac{x_i}{a}\right)^b+1\right)^{-k}+1\right]^{c-1},$$

$$\frac{\partial l(\boldsymbol{\Omega})}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n}\log\left[1+\left(\frac{x_i}{a}\right)^b\right] + (c-1)\sum_{i=1}^{n}\frac{\log\left[\left(\frac{x_i}{a}\right)^b+1\right]\left[\left(\frac{x_i}{a}\right)^b+1\right]^{-k}}{1-\left[1+\left(\frac{x_i}{a}\right)^b\right]^{-k}}$$
$$- c\sum_{i=1}^{n}\log\left[\left(\frac{x_i}{a}\right)^b+1\right]\left[\left(\frac{x_i}{a}\right)^b+1\right]^{-k}\left[-\left(\left(\frac{x_i}{a}\right)^b+1\right)^{-k}+1\right]^{c-1}$$
$$- c\sum_{i=1}^{n} e^{1-\left[-\left(\left(\frac{x_i}{a}\right)^b+1\right)^{-k}+1\right]^c}\log\left[\left(\frac{x_i}{a}\right)^b+1\right]\left[\left(\frac{x_i}{a}\right)^b+1\right]^{-k}\left[-\left(\left(\frac{x_i}{a}\right)^b+1\right)^{-k}+1\right]^{c-1}$$

and

$$\frac{\partial l(\mathbf{\Omega})}{\partial c}=\frac{n}{c}+\sum_{i=1}^{n}\log\left[1-\left[1+\left(\frac{x_i}{a}\right)^{b}\right]^{-k}\right]-\sum_{i=1}^{n}\log\left[-\left(\left(\frac{x_i}{a}\right)^{b}+1\right)^{-k}+1\right]\left[-\left(\left(\frac{x_i}{a}\right)^{b}+1\right)^{-k}+1\right]^{c}$$
$$-\sum_{i=1}^{n}e^{1-\left[-\left(\left(\frac{x_i}{a}\right)^{b}+1\right)^{-k}+1\right]^{c}}\log\left[-\left(\left(\frac{x_i}{a}\right)^{b}+1\right)^{-k}+1\right]\left[-\left(\left(\frac{x_i}{a}\right)^{b}+1\right)^{-k}+1\right]^{c}.$$