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Reliability analysis of the Stress-Strength model from truncated Pareto distribution based on progressive Type-II censored samples.

BY

Hadeel Mohammed Ali

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COMMITTEE PAGE

The members of the Committee approve the Project of
Hadeel Mohammed Ali defended on [Defense Date].

Dr. Reza Pakyari
Thesis/Dissertation Supervisor

Dr. Faiz Elfaki
Committee Member

Dr. Saddam Akber Abbasi
Committee Member

Prof. Abdelouahed Hamdi
Committee Member

Approved:

Ahmed Elzatahry, Dean, College of Arts and Sciences

ABSTRACT

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Title: Reliability analysis of Stress-Strength model from truncated Pareto distribution based on progressively Type-II censored samples

Supervisor of Project: Dr. Reza Pakyari.

In this project, we studied the stress strength reliability (SSR) models. The stress–strength model has many applications in engineering problems, for example the strength of a building being subjected to earthquake, the strength of a rocket motor being greater than its working pressure, and the strength of a bridge.

We estimated the reliability parameter using maximum likelihood estimation method in three cases (arbitrary case, common truncated case, and common resilience parameter case). We computed the maximum likelihood estimator (MLE) of the reliability parameter R and studied the properties of the estimator of the parameter R using a great amount of simulation studies and illustrate our method through some real data examples. Moreover, we compute the generalized confidence intervals passed on pivotal quantities. We computed the bootstrap confidence intervals. We found that, the confidence interval is wider in the arbitrary parameter case, and that there is no large difference between the estimators of reliability parameter using different methods.

DEDICATION

I present this work to my husband, my family, and especially my children. Also, for my supervisor.

ACKNOWLEDGMENTS

"I would like to acknowledge the support of Qatar University for providing all the needs to achieve the requirements of this study"

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CHAPTER 1: INTRODUCTION

1.1 Stress Strength Reliability (SSR)

The stress strength variable's reliability is defined as the probability that the random strength variable X is greater than the stress random variable Y, i.e., $R = P(X > Y)$.

Individuals fail when the stress exceeds their resistance. Thus, an individual's dependability, written as $R = P(X > Y)$, is the probability that the strength variable resists the stress variable.

The stress–strength model has various applications in engineering problems, including the strength of a building being subjected to the earthquake design, the strength of a rocket motor being greater than its working pressure, and the strength of a bridge.

In recent years, numerous studies on the stress–strength model have been conducted. Numerous applications of the stress-strength model involve engineering or military challenges, where it is also known as the load-strength model.

There are, however, many applications in medicine or psychology involving the comparison of two random variables, representing, for instance, the effect of a certain drug or treatment delivered to two groups (control and test), thus, reliability has a broader meaning.

Many examples can mention about SSR. Suppose that the bridge that crosses the cars represents the strength variable, and the cars that cross the bridge represent the stress variable. Here, the bridge's bearing strength must be greater than the pressure of cars on it, that is, R = P (X > Y), if the opposite is true, then the bridge will collapse.

Consider another example in the field of engineering. If we assume that the mobile needs a certain amount of voltage when charging it, if we charge it more often, it will lead to its failure, so the device will remain working if its strength is greater than its stress.

The reliability R can be calculated as probability where strength variable is greater than stress variable, so we can obtain this probability based on the joint probability distribution of two independent random variables as follows:

$$R = P\ (X > Y).$$

$$= \int_0^\infty \int_0^x f_y(y) f_x(x)\, dy dx,$$

$$= \int_0^\infty \left[\int_0^x f_y(y) dy\right] f_x(x) dx.$$

where X and Y are two continuous independent random variables and f(x) is the density function of random variable X, and f(y) is the density function of random variable Y.

1.2 Truncated Distributions

Truncated distributions are useful for determining when the minimum and maximum values of a random variable are bounded, and this can be done for a variety of reasons. This is a common occurrence in fields related to reliability. It's possible, for instance, that the warranty doesn't cover failures that take place during the warranty period. Items may be replaced after a set amount of time under the replacement policy so that defects are not ignored. When an autonomous recording device is used to collect data on failure or reliability life tests, the lives less than L and greater than T, where (L< T) cannot be assessed in any way, depending on the instrument's resolution or other environment.

Another example is that an upper truncated model can be used to represent empirical wind speed data, for instance, when modeling the distribution of wind speed in a meteorological study. This is because the observed wind speed data has been confined to a maximum value.

The estimation of truncated distribution parameters and related applications have been explored by different authors. A truncated Weibull distribution's parametric analysis techniques were considered by Zhang and Xie (2011). Aban et al. (2006) formulated the idea of maximum likelihood estimators (MLEs). Nadarajah (2009) offered outline for truncated distribution models.

For a continuous random variable $\mu \leq T \leq \nu$, the following is the expression of the double truncated using CDF and PDF functions, which can be calculated as follows:

$$F_{TD}(t) = \frac{F(t)-F(\mu)}{F(\nu)-F(\mu)} \;,\; f_{TD}(t) = \frac{f(t)}{F(\nu)-F(\mu)} \text{ where } \mu \leq t \leq \nu \,. \quad (1)$$

where μ is the lower truncated parameter and ν is the upper truncated parameter of time t.

1.2.1 Lower truncated distribution

When $\nu \to \infty$, the double truncated distribution is reduced to a lower truncated model where the PDF and CDF are given as follows:

$$F_{TD}(t) = \frac{F(t)-F(\mu)}{1-F(\mu)} \text{ and } f_{TD}(t) = \frac{f(t)}{1-F(\mu)},\ \mu \le t \le \infty\,. \quad (2)$$

1.2.2 Upper truncated distribution

When $\mu \to -\infty$, the double truncated distribution is reduced to an upper truncated model where the PDF and CDF are given as follows:

$$F_{TD}(t) = \frac{F(t)}{F(\nu)} \text{ and } f_{TD}(t) = \frac{f(t)}{F(\nu)}, -\infty \le t \le \nu\ . \quad (3)$$

1.3 The Pareto Distribution

Vilferdo Pareto (1897) presented the Pareto distribution for the first time in 1897. Since then, several researchers in the fields of industry and economics have used the Pareto distribution or variants of it. Some Pareto distribution applications and estimations are discussed by Harris (1968), Malik (1970), Kern (1983), Zaninetti and Ferraro (2008), and Arnold (2014).

The truncated Pareto distribution with the following probability density function will be considered as follows:

$$f(x) = \frac{\alpha\,\mu^{\alpha}}{x^{\alpha+1}}, \qquad 0 < \mu \le x \ and\ \alpha > 0 \quad .$$

Where μ is the truncated parameter and α is the shape parameter.

The cumulative distribution CFD, F(x) is given by:

$$F(x) = 1 - \left(\frac{\mu}{\mathrm{x}}\right)^{\alpha}, \qquad 0 < \mu \le x \ and\ \alpha > 0\ .$$

The figure shows the PDF and CDF for the truncated Pareto distribution with $1 \leq X$.

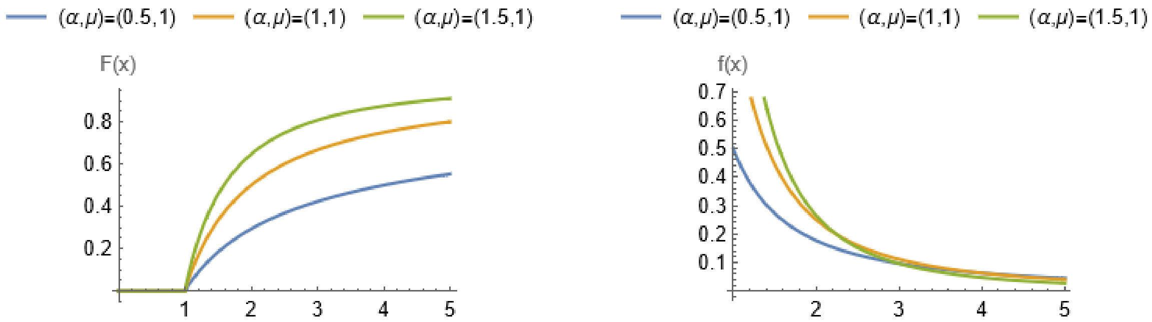

Figure 1. CDF and PDF for the lower truncated pareto distribution.

The mean value $E(X)$ exists if $\alpha > 1$ and is given by

$$E(X) = \frac{\alpha}{\alpha - 1}\mu \,.$$

The variance $V(X)$ exists if $\alpha > 2$ is given by

$$V(X) = \frac{\alpha}{(\alpha - 1)^2(\alpha - 2)}\mu^2.$$

1.4 Censored Data

The challenge of evaluating time-to-event data emerges, despite statistical equipment, in many practical disciplines, such as health, science, global health, epidemiology, education, economics, and demographics.

There are two kinds of samples: complete and incomplete or censored data, if the researcher doesn't know the exact failure time, the sample is incomplete or censored, while complete samples are known in life testing and reliability, a sample is referred to as a "complete sample" when the researcher records the failure rates of all units during the lifespan test.

Due to cost and time constraints, most experimenters do not witness all failure times and instead view a subset of sample units before terminating the experiment. This form of data is known as "censored data".

We do not know the precise moment at which this incident occurred, also known as the time at which it was witnessed or survived. Because even though we know the total amount of time someone was followed up on, we do not know how long they lived.

Censoring occurs in most of the prospective observational research. During the research period, a participant may become separated from follow-up records and the event of interest may not occur before the conclusion of the follow-up term.

The participant may need to withdraw from the study owing to a negative drug reaction, another pressing personal obligation, or any number of other potential scenarios.

Censoring is a feature unique to time-to-event data that occurs when specific timespan events are known to have occurred. Accurate information regarding the remaining life expectancy is available.

Censored data is any data for which the actual occurrence time is unknown. Right censored, left censored, and interval censored are the three types of censored data.

Left censoring: If a study subject's expected lifespan is shorter than the censoring period, the subject's data will be considered left censoring because the event of interest has already occurred for this subject.

By "interval censoring," we indicate that the range across which a random variable of interest is known, rather than its exact value, is unknown. In survival analysis, the event of interest (death, illness recurrence, or distant metastases) is a random variable.

A form of censorship called "right censoring" occurs when the endpoint is known to be greater than some fixed threshold.

In addition to the three forms of censored data, there are two methods for grouping censored data: singly censored and multiply censored.

Censoring of type-I occurs when an event is recorded only if it occurs before a certain deadline, there may be some individual variation in the length of time until content is blocked.

When the experimental objects are monitored until a certain number of them fail, this is known as type II censoring. Such a layout is unusual in the field of biomedicine, although it could be use in commercial situations where the uptime of equipment is a top priority.

1.5 Progressive Type-II Censoring

Due to the quick progress in industry and technology, increasingly durable and dependable items are being introduced into everyday life.

Obtaining accurate information about the lifetime of a batch of products is becoming increasingly difficult. Consequently, censorship systems are implemented to combat this scenario.

To address this issue, Cohen and Clifford suggested the gradually censoring scheme, a broader method of censorship.

The most prevalent forms of censoring tests are types I and II. These experiments include subjecting n units to a test and stopping the experiment either after a predetermined length of time has passed or after m(n) units have failed the test. In modern industry, however, the test cost is high, and the product has a long life. Progressive censoring tests such as progressive type I censoring, type-II censoring and progressive random censoring, etc., have been used to make it easier to collect failed samples and enhance the precision of Procedures for making inferences in lifetime studies when compared to typical censoring schemes.

Many authors have done a lot of research on the effects of progressive censorship with different life distributions.

Rarely do lifetime studies observe whole samples. Most life-testing experiments are ended before monitoring the lifespan of all units being evaluated. It may occur owing to a lack of time, insufficient cash, or another unavoidable cause.

In recent decades, numerous censorship systems have been promoted in statistical literature. Like type-I and type-II censorship systems. The difference between two types is the former stops the test at a certain time, while the latter keeps going until a certain number of failures have occurred.

In Type-I censoring we stop the experiment in specific point of time, whereas type-II censoring ends the study once a particular number of failures have been recorded. Unfortunately, while testing, none of these censorship solutions allow for the removal of live components. Using progressive censoring, the researcher may remove working units without affecting the overall results of the study.

Combining elements of both type-II and progressive censorship, we get "progressive type-II censorship".

With m and n sample sizes and $R_1 \ldots\ldots R_m$ positive integers, find those such that:

$$R_1 + \cdots + R_m = n - m.$$

In order to increase the efficiency of the experiment, we propose using a broader definition of type-II censoring is progressive type-II censoring.

The following becomes an approach of enforcing progressive type-II censorship. Put n units through a life test and label their lifetimes($x_1, x_2 \ldots\ldots, x_n$).

Assume that x_i are uncorrelated random variables that independently satisfy the PDF and CDF.

We select the censoring procedure and the number of samples to be observed (m) just before the experiment begins, where ($R = R_1 , R_2 \ldots\ldots, R_m$)

$$where\ R_i > 0\ \ and\ i = 1,2, \ldots\ldots, m\ \ and\ \ \sum_{i=1}^{m} R_i\ +\ m =\ n\ .$$

The other R_i units have been removed from the experiment. at random after every ith failure. Following this rule, the experiment will continue until m failures are recorded, at which point it will end.

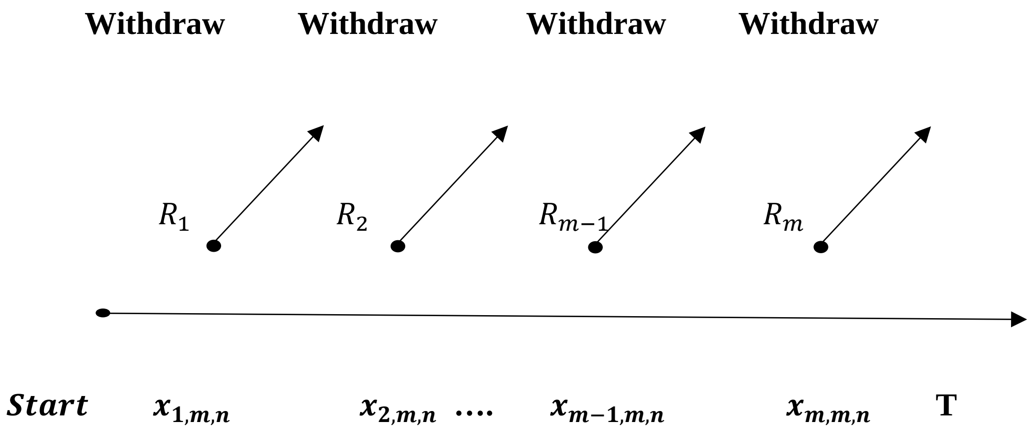

Figure 2. Generation of Progressive Type-II Censored Data

To further improve the efficiency and cost-effectiveness of data collection in experiments, progressive censoring is put into practice. This enables engineers to withdraw components from the experiment at various stages. Type II progressive censorship has become the standard in the field of progressive censorship.

Progressive censorship has various advantages over the conventional type I and type II censorship techniques. Over the past few years, substantial effort devoted to the creation of the progressive censorship procedure.

CHAPTER 2: LITREATURE REVIEW

The objective of this chapter is to acquire a better grasp of the recent research that is pertinent to our study. Life testing and reliability are important in medical, industry, and many other fields of research to predict future life and evaluate product quality.

Due to the importance of the stress-strength model in theory and applications, several authors have contributed excellent works to the literature .

Reliability parameter is the most essential model for establishing product reliability. It is important to note that a capacitor's break-even voltage (X) must be higher than its power supply voltage (Y) for the device to operate properly. Related examples of rocket engines were described by Kotz et al. (2003).

For example, there are several studies focused on estimating the stress-strength parameter . Censored data is used to shorten the time required to test the reliability. Asgharzadeh et al. (2011) discussed inference for the SSR parameter R using progressive type-II censored data when the random variables follow the Weibull distribution.

Furthermore, Saraçoglu et al, (2012) estimated reliability for stress strength under exponential distribution by using progressive type-II censored data. Tomer and Chaudhary. (2018) estimated the reliability parameter using Maxwell distributions and when the information was gathered using a Type-II progressively censored design.

Additional articles are available in Aban (2016), Bai (2019), Wang et al. (2018), and Zhang and Xie (2011).

In recent years, many of papers proposed novel uses for the SSR. Medicare-related data was used as an example in the discussion of stress-strength reliability estimation by Jose et al (2022). The number of patients who do not have health insurance coverage represents stress Y, whereas the number of patients who do have insurance represents strength X.

Using a real data set related to excessive drought afterward, Saini et al. (2021) estimated the SSR regarding the generalized Maxwell distribution, with the progressive censoring of early failures. If the amount of water achieved in August over the last 20 years (Y) is less than the water capacity of a reservoir in an area for the next 15 years in August, then there will be no excessive drought afterward (X).

Data from a clinical trial of an antibiotic ointment for pain treatment was analyzed by Liu et al. (2022), who studied at the SSR and a model of the distribution of power functions with two randomly censored parameters.

In recent years several authors, including Balakrishnan (2007), Jaheem and Mousa (2002), Krishna and Kumar (2002), Ng et al. (2000), Childs and Balakrishnan (2000), researchers have examined distribution characteristics based on progressively suppressed samples.

Cramer and Iliopoulos (2010) had already put forward a type-II adaptive progressive censoring approach that addresses both preset censoring schemes and random censoring based on a probability distribution.

Singh et al. (2013) used the binomial probability law for the generalized Lindley distribution to investigate the predicted total test time when subjected to progressive type-II censoring.

Pareto first proposed the Pareto distribution as a tool for analyzing income inequality, it has since found application in fields as varied as insurance, business, economics, engineering, hydrology, and reliability. Interested readers may learn more about the history and other features of this distribution in Johnson et al.(1994) .

Gupta et al. (1999) proposed the exponentiated Pareto distribution that can be employed well with assessing a wide range of lifetime data.

Both Amin (2008) and Soliman (2008) used a subjective Bayesian strategy to estimate and predict data under the Pareto distribution exposed to progressive Type-II censoring, the key difference between the two methods is their choice of priors for the parameters.

CHAPTER 3: METHODOLOGY

3.1 Introduction

The SSR parameter is expressed in this chapter using a truncated lower Pareto distribution. In addition, the likelihood function of the SSR parameter R is established for three cases: the arbitrary parameter case, the common truncated parameter, and the common resilience parameter.

The double truncated distribution of lifetime studies has PDF and CDF functions Wang,Yu,and Jones (2010) gives the double truncated function as:

$$F_{DT}\ (t) = \frac{F\ (t)-F(\mu)}{F\ (v)-F(\mu)} \qquad and \qquad f_{DT}(t) = \frac{f(t)}{F(v)-F(\mu)}, where\ \ \mu \le t \le v\ \ . \ (4)$$

with an upper truncated parameter μ and a lower truncated parameter v.

Suppose a random variable T is drawn under the generalized distribution with the CDF function, and PDF:

$$F(t;\theta) = 1-[\bar{G}(t)]^{\theta}\ \ and\ \ f(t;\theta) = \theta g(t)[\bar{G}(t)]^{\theta-1}, where\ t > 0 \quad . \quad (5)$$

where $\theta > 0,$ is the resiliance parameter,

$\bar{G}(t) = 1 - G(t)$ is the survival function stisfiying $\bar{G}(0) = 1$ and

$$\bar{G}(cc) = 1.$$

For lower truncated distribution of proportional hazard rate, can be determined from (1) and (2) with the following PDF and CDF:

$$F_{LP}(t;\theta,\mu) = 1-\left[\frac{\bar{G}(t)}{\bar{G}(\mu)}\right]^{\theta}\ \ and\ \ f_{LP}(t;\theta,\mu) = \frac{\theta g(t)}{\bar{G}(\mu)}\left[\frac{\bar{G}(t)}{\bar{G}(\mu)}\right]^{\theta-1} \quad . \quad (6)$$

where $\theta > 0$ and $t \ge \mu$.

The lower truncated pareto distribution LTP can be obtained from equation (6):

by putting $\overline{G}(t) = \frac{1}{t}$,were

$\overline{G}(t) = 1 - G(t)$, $G(t) = 1 - \frac{1}{t}$, and $g(t) = \frac{1}{t^2}$. (7)

3.2 SSR Under Truncated Pareto Distribution

In this section, we will provide the formulas for the stress strength model's reliability under truncated Pareto distribution in three cases.

Let X and Y be two independent random variables with X being a random variable of strength with parameters θ_1, μ_1 following LTP and Y being a random variable of stress with parameters θ_2, μ_2 following LTP :

-under an arbitrary parameter case for a lower truncated distribution, the SSR can be formulated as follows:

When $\mu_1 > \mu_2$,

$$R = P(X > Y) = \int_{\mu_1}^{\infty} \left[\int_{\mu_2}^{x} f_{LP}(y; \theta_2, \mu_2)\, dy\right] f_{LP}(x_; \theta_1, \mu_1)\, dx = 1 - \frac{\theta_1}{\theta_1 + \theta_2}\left[\frac{\overline{G}(\mu_1)}{\overline{G}(\mu_2)}\right]^{\theta_2}$$

$$=\left(1 - \frac{\theta_1}{\theta_1+\theta_2}\left[\frac{\mu_2}{\mu_1}\right]^{\theta_2}\right) \quad . \qquad (8)$$

When $\mu_1 < \mu_2$

$R= P(X > Y) = 1 - P(Y > X) =$

$$\int_{\mu_1}^{\infty}\left[\int_{\mu_1}^{y} f_{LP}(x; \theta_1, \mu_1)\, dx\right] f_{LP}(y, \theta_2, \mu_2)\, dy = \left(\frac{\theta_2}{\theta_1+\theta_2}\left[\frac{\bar{G}(\mu_2)}{\bar{G}(\mu_1)}\right]^{\theta_1}\right)$$

$$=\left(\frac{\theta_2}{\theta_1+\theta_2}\left[\frac{\mu_1}{\mu_2)}\right]^{\theta_1}\right) \quad . \qquad (9)$$

Therefore:

$$R = \left(1 - \frac{\theta_1}{\theta_1+\theta_2}\left[\frac{\bar{G}(,\mu_1)}{\bar{G}(,\mu_2)}\right]^{\theta_2}\right) I_{(\mu_1>\mu_2)} + \left(\frac{\theta_2}{\theta_1+\theta_2}\left[\frac{\bar{G}(\mu_2)}{\bar{G}(\mu_1)}\right]^{\theta_1}\right) I_{(\mu_1\leq\mu_2)} \quad . \tag{10}$$

Using equation (7) we get Pareto lower truncated distribution.

$$R = \left(1 - \frac{\theta_1}{\theta_1+\theta_2}\left[\frac{\mu_2}{\mu_1}\right]^{\theta_2}\right) 1_{(\mu_1>\mu_2)} + \left(\frac{\theta_2}{\theta_1+\theta_2}\left[\frac{\mu_1}{\mu_2}\right]^{\theta_1}\right) 1_{(\mu_1\leq\mu_2)} \quad . \tag{11}$$

-under common truncated parameters, where $\mu_1 = \mu_2 = \mu$,

$$R = \frac{\theta_2}{\theta_1+\theta_2} \quad . \tag{12}$$

-Under common resilience case when $\theta_1 = \theta_2 = \theta$

$$R = \left(1 - \frac{1}{2}\left[\frac{\bar{G}(\mu_1)}{\bar{G}(\mu_2)}\right]^{\theta}\right) I_{(\mu_1>\mu_2)} + \left(\frac{1}{2}\left[\frac{\bar{G}(\mu_1)}{\bar{G}(\mu_2)}\right]^{\theta}\right) I_{(\mu_1\leq\mu_2)} \quad . \tag{13}$$

From (4), then R under Pareto distribution has the following form*:*

$$R = \left(1 - \frac{1}{2}\left[\frac{\frac{1}{\mu_1}}{\frac{1}{\mu_2}}\right]^{\theta}\right) I_{(\mu_1>\mu_2)} + \left(\frac{1}{2}\left[\frac{\frac{1}{\mu_1}}{\frac{1}{\mu_2}}\right]^{\theta}\right) I_{(\mu_1\leq\mu_2)}$$

$$R = \left(1 - \frac{1}{2}\left[\frac{\mu_2}{\mu_1}\right]^{\theta}\right) I_{(\mu_1>\mu_2)} + \left(\frac{1}{2}\left[\frac{\mu_2}{\mu_1}\right]^{\theta}\right) I_{(\mu_1\leq\mu_2)} \quad . \tag{14}$$

3.3 Maximum Likelihood Estimation

Assume X is the variable representing strength $=(x_1, x_2, \ldots\ldots, x_n)$ from progressive Type-II censored data under truncated Pareto distribution and have parameters (θ_1, μ_1) ,and assume Y is the variable stress random variable = $(y_1, y_2, \ldots\ldots, y_n)$from progressive Type-II censored data under truncated Pareto distribution and have parameters (θ_2, μ_2).

The likelihood function of strength random variable X. Balakrishnan and Cramer(2014):

$L(\theta_1, \mu_1) = c_1 \prod_{i=1}^{m1} f_{LP}(x_i; \theta_1, \mu_1)[1 - F_{LP}(x_i; \theta_1, \mu_1)]^{r_i^x} =$

$$c_1 \theta_1^{m_1} \prod_{i=1}^{m_1} \frac{g(x_i)}{\bar{G}(x_i)} \left[\frac{\bar{G}(x_i)}{\bar{G}(\mu_1)}\right]^{(r_i^x+1)\theta_1} . \quad (15)$$

where $x = (x_1, x_2, \ldots, x_n)$ is the progressive typ − IIcensored data with $R_x = (r_1^*, r_2^*, \ldots, r_{m_1}^*)$ and

$$c_1 = n_1 \prod_{i=1}^{m_1} \left(n_1 - i - \sum_{j=1}^{i} r_i^x\right) .$$

-The likelihood function of stress random variable *Y:*

$$L(\theta_2, \mu_2) = c_2 \prod_{j=1}^{m2} f_{LP}(y_j; \theta_2, \mu_2)[1 - F_{LP}(y_j; \theta_2, \mu_2)]^{r_j^y}$$

$$= c_2 \theta_2^{m_2} \prod_{j=1}^{m_2} \frac{g(y_i)}{\bar{G}(y_i)} \left[\frac{\bar{G}(y_i)}{\bar{G}(\mu_2)}\right]^{(r_j^y+1)\theta_2} . \quad (16)$$

$(y_1, y_2, \ldots, y_n)$ is the progressive typeII censored data with censoring scheme $R_y = (r_1^y, r_2^y, \ldots, r_{m_2}^y)$

and $c_2 = n_2 \prod_{i=1}^{m_2-1}\left(n_2 - i - \sum_{j=1}^{i} r_i^y\right)$.

We can calculate MLEs of the SSR parameter R for both the arbitrary parameter case and the common truncated parameter case.

3.3.1 Estimation under arbitrary parameter case

From equations (15) and (16) by applying the logarithm and differentiating with respect to the parameters $\theta_1, \mu_1, \theta_2, \mu_2$ the MLEs will be given by (Wang, 2020):

$$\hat{\mu}_1 = X_1, \quad \hat{\theta}_1 = \frac{m_1}{\omega_1(X_1)} \quad and \quad \hat{\mu}_2 = Y_1, \quad \hat{\theta}_2 = \frac{m_2}{\omega_2(Y_1)} \quad . \tag{17}$$

$$\omega_1(t) = \sum_{i=1}^{m_1} (r_i^X + 1)\, ln\left[\frac{\frac{1}{t}}{\frac{1}{xi}}\right], \quad \omega_2(t) = \sum_{j=1}^{m_2} \left(r_j^y + 1\right) ln\left[\frac{\frac{1}{t}}{\frac{1}{yi}}\right]$$

$$\omega_1(t) = \sum_{i=1}^{m_1} (r_i^X + 1)\, ln\left[\frac{xi}{t}\right], \quad \omega_2(t) = \sum_{j=1}^{m_2} \left(r_j^y + 1\right) ln\left[\frac{yi}{t}\right] \quad . \tag{18}$$

By using the invariance property, the MLE of SSR in this case is:

$$\hat{R} = \left(1 - \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}\left[\frac{\hat{\mu}_2)}{\hat{\mu}_1}\right]^{\hat{\theta}_2}\right) I_{(\hat{\mu}_1 > \hat{\mu}_2)} + \left(\frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}\left[\frac{\hat{\mu}_1)}{\hat{\mu}_2}\right]^{\hat{\theta}_1}\right) I_{(\hat{\mu}_1 \leq \hat{\mu}_2)} \quad . \tag{19}$$

3.3.2 Inference about common truncated parameters

Suppose X strength variable $=(x_1, x_{2,} \dots\dots, x_n)$ from progressive Type-II censored data with truncated Pareto distribution with parameters (θ_1, μ_1) ,and suppose Y stress random variable = $(y_1, y_{2,} \dots\dots, y_n)$from progressive Type-II censored data with truncated Pareto distribution and have parameters (θ_2, μ_2) .

When the truncated parameter $\mu_1 = \mu_2 = \mu$,then the joint likelihood function can be reformulated as follows:

$$\mathrm{L}(\theta_1, \theta_2, \mu) = \theta_1^{m_1} \theta_2^{m_2} \prod_{i=1}^{m_1} \left[\frac{\bar{G}(x_i)}{\bar{G}(\mu)}\right]^{(r_i^x+1)\theta_1} \prod_{j=1}^{m_2} \left[\frac{\bar{G}(y_i)}{\bar{G}(\mu)}\right]^{\left(r_j^y+1\right)\theta_2} . \quad (20)$$

By taking the logarithm for equation (17) and getting the first partial derivation with respect to θ_1, θ_2 and μ ,

The MLE for parameters are:

$$\acute{\theta}_1 = \frac{m_1}{\omega_{1(}(\acute{\mu})} , \qquad \acute{\theta}_2 = \frac{m_2}{\omega_{2(}(\acute{\mu})} \qquad \text{and } \acute{\mu} = \min\{X_1, Y_1\} . \quad (21)$$

The MLE of SSR parameters R:

$$\acute{R} = \frac{\acute{\theta}_2}{\acute{\theta}_1 + \acute{\theta}_2} \quad ,\text{Where } \mu_1 = \mu_2 = \mu . \quad (22)$$

3.3.3 Estimation about common resilience parameters when

$\boldsymbol{\theta_1 = \theta_2 = \theta.}$

The joint likelihood function under this case can be obtained as follows

$$L\,(\mu_1, \mu_2, \theta) \propto \theta^{m_1}\theta^{m_2} \prod_{i=1}^{m_1} \left[\frac{\bar{G}(x_i)}{\bar{G}(\mu_1)}\right]^{(r_1^x+1)\theta} \prod_{j=1}^{m_2} \left[\frac{\bar{G}(y_i)}{\bar{G}(\mu_2)}\right]^{\left(r_j^y+1\right)\theta} . \qquad (23)$$

The MLE of parameters after applying the logarithm for equation (20) and differentiating it with respect to $\mu_1, \mu_2,$ and θ are *:*

$$\grave{\mu}_1 = X_1, \quad \grave{\mu}_2 = Y_1 \quad \text{and} \quad \grave{\theta} = \frac{\grave{m}_1 + m_2}{\omega_1(X_1) + \omega_2(Y_1)} . \qquad (24)$$

The MLE of SSR parameter R can be obtained as follows:

$$\grave{R} = \left(1 - \frac{1}{2}\left[\frac{\grave{\mu}_2}{\grave{\mu}_1}\right]\tilde{\theta}\right) 1_{(\grave{\mu}_1 > \grave{\mu}_2)} + \left(\frac{1}{2}\left[\frac{\grave{\mu}_1}{\grave{\mu}_2}\right]\tilde{\theta}\right) 1_{(\grave{\mu}_1 \leq \grave{\mu}_2)} . \qquad (25)$$

3.4 Generalized Confidence Interval (GCI)

Theorem l. let X and Y are a progressively type-II censored samples, where *x=*$(x_1 x_2, \ldots x.)$*, y=* $(y_{1,,} y_2, \ldots . y)$, be stress strength samples with parameters (θ_1, μ_1) *and* (θ_2, μ_2)

Respectively denote the pivotal quantities (Weerahandi,1993):

$$\xi_1 = 2n_1\theta_1 ln\left[\frac{X_1}{\mu_1}\right], \quad where\ \eta_1 = 2\theta_1\omega_1(X_1) .$$

$$\xi_2 = 2n_2\theta_2 ln\left[\frac{Y_1}{\mu_2}\right], \quad where\ \eta_2 = 2\theta_2\omega_2(Y_1).$$

$$\omega_1(t) = \sum_{i=1}^{m_1} (r_i^X + 1)\, ln\left[\frac{X_i}{X_1}\right] , \quad \omega_2(t) = \sum_{j=1}^{m_2} \left(r_j^y + 1\right) ln\left[\frac{Y_i}{Y_1}\right],$$

where ξ_i and η_i , $i = 1,2$,have *2* and $2(m_i - 1)$ degrees of freedom, according to chi-square distribution.

3.4.1 GCI under arbitrary parameters case

The pivotal variables of *(* θ_i, μ_i) can be displayed as*:*

$$s_{\mu_i} = [1 - \bar{A}_i], i = 1,2 \quad ,$$

$$G^{-1}(t) = \frac{1}{1-t} = [1-t]^{-1}$$

,were $A_i = 1 - \frac{1}{\hat{\mu}_i \theta} exp\left[\frac{C_i m_i}{U_i n_i \hat{\theta}_{i\theta}}\right] \quad ; \quad i = 1,2$

$S_{\theta_i} = \frac{2m_i\theta_i}{\hat{\theta}_i}\frac{\theta_{io}}{2m_i} = \frac{\hat{\theta}_{i\circ}}{2m_i} x_{i(m_i-1)} = \frac{\hat{\theta}_{io}}{2m_i} U_i \qquad ,i=1,2.$

where *ci* and *vi*, *i=1,2* have chi-square distribution with *2*, *2(m-1)* degree of freedom respectively.

Based on s_{θ_i} and s_{μ_i} *,i=1,2* and utilizing the alternative methodologies of Weerhandi(1993),a generalized pivotal quantities of Reliability can be construct as

$$s_R = \left(\left(1 - \frac{s_{\theta_1}}{s_{\theta_1}+s_{\theta_2}}\left[\frac{s_{\mu_1}}{s_{\mu_2}}\right]^{s_{\theta_2}}\right) I s_{\mu_1} > s_{u_2}\right) + \left(\frac{s_{\theta_2}}{s_{\theta_1}+S_{\theta_2}}\left[\frac{s_{u_1}}{s\mu_2}\right]^{s_{\theta_1}}\right) I\left(s\mu_1 \le s_{u_2}\right) \;. \quad (26)$$

3.4.2 Generalized confidence interval with common truncated parameters

let similar truncated parameters exist $\mu_1 = \mu_2 = \mu$, and depending on pivotal factors and put forward on Section arbitrary parameter case one may create the generalization pivotal variables using *R* as*:*

$$\acute{S}_R = \frac{s_{\theta 2}}{s_{\theta_1} + s_{\theta_2}} \quad . \qquad (27)$$

3.4.3 *GCI* under common resilience parameters*:*

Let $\theta_1 = \theta_2 = \theta$ then the generalization pivoptal quantities for μ_1 can be constructed as:

$$S_{\mu_1} = \left[1 - \left[1 - \frac{1}{\grave{\mu}_{1o}} exp\left[\frac{m_1 + m_2}{n_1 \theta_0'} \frac{c_1}{U}\right]\right]\right]^{-1}$$

$$U \sim X^2_{2(m_1+m_2-2)}$$

for μ_2,

$$S_{\mu_2} = \left[1 - \left[1 - \frac{1}{\grave{\mu}_{2o}} exp\left[\frac{m_1 + m_2}{n_2 \theta_0'} \frac{c_2}{U}\right]\right]\right]^{-1}$$

A generalized pivotal variable for can be constructed as

$$s_{\theta} = \frac{\grave{\theta}_o}{2(m_1 + m_2)} U$$

Therefore, a generalized pivotal variable for SSR under this case can be provided as

$$s_R = \left(1 - \frac{1}{2}\left(\frac{s_{m_2}}{s_{m_1}}\right)^{s_\theta}\right) I\left(s_{\mu_1} > s_{\mu_2}\right) + \left(\frac{1}{2}\left(\frac{s\mu_1}{s\mu_2}\right)^{s_\theta}\right) I\left(s_{\mu_1} \leq s_{\mu_2}\right) \quad . \qquad (28)$$

3.5 Bootstrap confidence interval

Using the bootstrap method by Ephron (1987) the bootstrap is a technique for calculating confidence intervals and estimating the variance of an estimator. As a result of Bradley Ephron's 1970 invention of bootstrapping, which has now been around for more than 40 years, numerous types and techniques have been created.

Bootstrapped approaches often involve simulating the unknown population and replacing the genuine population with the estimated population to estimate the characteristic.

-The algorithm of bootstrapping can be calculated as follows:

- **Step 1**. Estimate parameters θ_1, μ_1 and θ_2, μ_2 say $\widehat{\theta_1}, \widehat{\mu_1}$ and $\widehat{\theta_2}, \widehat{\mu_2}$.
- **Step 2**. Generate censored bootstrap samples $x_1^*, x_2^*, \dots\dots, x_{m_1}^*$ and $y_1^*, y_2^*, \dots\dots, y_{m_2}^*$based on $\widehat{\theta_1}, \widehat{\mu_1}$ and $\widehat{\theta_2}, \widehat{\mu_2}$.

- **Step 3**. Compute MLEs of θ_1, θ_2 and μ_1, μ_2, and obtain bootstrap estimate $\tilde{r}^*$ of R.

- **Step 4**. Repeat Step 2 and Step 3 N times.

- **Step 5**. Rearrange the N values of $\tilde{r}^*$ in ascending order and obtain $\tilde{r}^*$ lower is ($(\tilde{r}_{boot}\left(\frac{\gamma}{2}\right)$N) and the r upper=($\tilde{r}_{boot}\left(1-\frac{\gamma}{2}\right)$N).

3.6 Testing hypothesis

The question of whether the parameters stress and stress variable are equivalent or not, therefore one might be interesting in conducting a test to determine whether the resilience and truncated parameters are the same.

We can use the likelihood ratio test be presented to evaluate the stress strength character.

The joint likelihood function of truncated and resilience parameters can be written as follows:

$$L_1(\theta_1, \mu_1, \theta_2, \mu_2) = c_1 c_2 {\theta_1}^{m1} {\theta_2}^{m2} \prod_{i=1}^{m1} \frac{1}{xi} [\frac{\mu_1}{xi}]^{(r_i+1)\theta_1} \cdot \prod_{j=1}^{m2} \frac{1}{yi} [\frac{\mu_2}{yi}]^{(r_j+1)\theta_2}$$

the log likelihood function calculate as follows:

$l_1(\theta_1, \mu_1, \theta_2, \mu_2) \ = m_1 log\theta_1 + m_2 log\theta_2 + \sum_{i=1}^{m1}[-\log(x_i) + (r_i + 1)\theta_1 \log(\frac{\mu_1}{xi})]+$

$\sum_{j=1}^{m2}[-\log(y_i) + (r_j + 1)\theta_2 \log(\frac{\mu_2}{yi})].$

The testing hypothesis is examined as follows:

$$H_0: \quad \mu_1 = \mu_2 \ = \mu \qquad Vs \qquad H_1: \ \ \mu_1 \neq \mu_2 \ .$$

The likelihood function under common truncated parameter cases:

$$L_2(\theta_1, \theta_2, \mu) = {\theta_1}^{m1} {\theta_2}^{m2} \prod_{i=1}^{m1} \ [\frac{\mu}{xi}]^{(r_i+1)\theta_1} \cdot \prod_{j=1}^{m2} \ [\frac{\mu}{yj}]^{(r_j+1)\theta_2}$$

The loglikelihood function with common truncated parameter can be calculated as follows:

$l_2(\theta_1, \theta_2, \mu) = m_1 log\theta_1 + m_2 log\theta_2 + \sum_{i=1}^{m1}[\ (r_i + 1)\theta_1 \log(\mu/xi)] + \sum_{j=1}^{m2}[\ (r_j +$

$1)\theta_2 \log(\mu/yj)]$.

Note that according to asymptotic property of the LRT statistics for large n we have,

$$-2\{l_1(\widehat{\theta_1}, \widehat{\theta_2}, \widehat{\mu_1}, \widehat{\mu_2}) - l_2(\acute{\theta_1}, \acute{\theta_{1,}}, \acute{\mu})\} \ \rightarrow {\chi_1}^2.$$

Similarly for testing,

$$H_0: \quad \theta_1 = \theta_2 \ = \theta \qquad Vs \qquad H_1: \ \ \theta_1 \neq \theta_2 \, .$$

The likelihood function under common resilience parameter cases is given by:

$$L_3(\mu_1, \mu_2, \theta) \propto \theta^{m1+m2} \prod_{i=1}^{m1} \ [\frac{\mu_1}{xi}]^{(r_i+1)\theta_1} \cdot \prod_{j=1}^{m2} \ [\frac{\mu_2}{yj}]^{(r_j+1)\theta_2}$$

The loglikelihood statistics can be calculated as follows:

$$l_3(\mu_1, \mu_2, \theta) = (m_1 + m_2)log\theta + \sum_{i=1}^{m1}[\,(r_i + 1)\theta\log(\frac{\mu_1}{xi})] + \sum_{j=1}^{m2}[\,(r_j + 1)\theta\log(\frac{\mu_2}{yj})]$$

Note that for large n we have:

$$-2\{l_1(\widehat{\theta_1}, \widehat{\theta_2}, \widehat{\mu_1}, \widehat{\mu_2}) - l_3(\acute{\theta_1}, \theta_{1,}^{\acute{}}, \acute{\mu})\} \rightarrow {\chi_1}^2.$$

Chapter4: simulation study

To further explain the procedures described in the previous chapter and to assess the relative merits of various point and interval estimates, simulation studies were developed to test the following quantities:

1) (AB) the absolute bias of the reliability estimator R can be defined as follows: $1/_N \sum_{n=1}^{N} |\widehat{R} - R|$ where N is the number of simulations runs.

2) (MSE) mean square of the point estimator $\hat{R}$ can be obtained by $1/_N \sum_{n=1}^{N} (\widehat{R} - R)^2$, where N is the number of simulations runs.

3) (CP) coverage probability of (1-α) % for a confidence interval of R, it can be thought of as the likelihood that the predicted confidence interval includes the actual parameter.

4) Average width (AW) of confidence interval of R.

Steps for simulation procedure:

Step1: In the numerical simulations, we simplify matters by assuming, n_1=n_2 and $m_1 = m_2$ the same censoring method is used for the variables of strength and stress, Various censoring strategies ($R_1, R_2 \ldots\ldots R_m$) and sample sizes (n, m) are taken into account. The study covers twelve different case studies, each with its own set of simulation parameters layouts are shown in Table 1 below.

Tables 2 and 3 display the criteria quantities for point and interval estimations of SSR based on 5000 iterations. The level of significance for interval estimations is set at 0.05.

Step2: Given μ_1 ,θ_1, μ_2, θ_2 we can calculate the true value of the reliability parameter.

Step3: According to the simulation algorithm provided by Balakrishnan and Sandhu, we can create Type-II censored samples U_1, U_2,…. U_m from a uniform distribution U.

Step4: Compute the MLE of μ_1 ,θ_1, μ_2, θ_2 based on the generated progressive type two samples.

Step5: Calculate the MLE of parameter R based on $\hat{\theta}_1, \hat{\theta}_2, \hat{\mu}_1, \hat{\mu}_2$.

Step6: Calculate generalized confidence interval based on pivotal quantities and bootstrap confidence intervals.

Table 1. Design simulation schemes**.**

cs	n	m	$(r_1 \dots\dots\dots\dots\dots\dots\dots\dots\dots, r_m)$
1	16	8	(8,0,0,0,0,0,0,0)
2	16	8	(0,0,0,0,0,0,0,8)
3	16	8	(2,0,2,0,2,0,2,0)
4	16	8	(1,1,1,1,1,1,1,1)
5	26	12	(14,0,0,0,0,0,0,0,0,0,0,0)
6	26	12	(0,0,0,0,0,0,0,0,0,0,0,14)
7	26	12	(2,0,2,0,2,0,2,0,2,0,2,0,2)
8	26	12	(1,1,1,1,1,1,1,1,1,1,1,1,1)
9	36	20	(16,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
10	36	20	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,16)
11	36	20	(2,0,2,0,2,0,2,0,2,0,2,0,2,0,2,0,2,)
12	36	20	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)

Table.2 Arbitrary parameter case, average bias (ABs) of the points estimates of SSR parameter and the mean square error.

$(\theta_{1,}, \mu_{1,}\theta_2, \mu_2) = (0.7.1.4,1.3,0.6)$

cs	ABs (maximum likelihood estimates)	MSE (MLE)	ABs (generalized point estimate)	MSE(GPE)
1	0.0655	0.0061	0.0701	0.0080
2	0.0594	0.0051	0.0618	0.0066
3	0.0643	0.0637	0.0701	0.0080
4	0.0556	0.0045	0.0591	0.0055
5	0.0472	0.0034	0.0515	0.0046
6	0.0480	0.0035	0.0478	0.0040
7	0.0534	0.0039	0.0515	0.0042
8	0.0455	0.0034	0.0457	0.0035
9	0.0399	0.0025	0.0424	0.0030
10	0.0420	0.0024	0.0417	0.0025
11	0.0411	0.0028	0.0419	0.0031
12	0.0365	0.0020	0.0362	0.0020

Table.3 Arbitrary parameter case, average width (AW) of bootstrap and generalized confidence intervals and the coverage probability(cp).

cs	AW (bootstrap CI)	CP (coverage probability for BCI)	AW (generalized confidence interval)	CP (coverage probability for GCI)
1	0.2622	0.9000	0.3191	0.9500
2	0.2434	0.9000	0.3046	0.9700
3	0.2410	0.8600	0.3006	0.9200
4	0.2286	0.8200	0.2912	0.9400
5	0.2260	0.9500	0.2557	0.9900
6	0.2089	0.9300	0.2415	0.9600
7	0.2002	0.8400	0.2318	0.9300
8	0.2015	0.8600	0.2340	0.9400
9	0.1812	0.9400	0.1944	0.9300
10	0.1658	0.8300	0.1801	0.8900
11	0.1732	0.8800	0.1871	0.9200
12	0.1671	0.9400	0.1812	0.9400

Table .4 Common truncated parameter case, bias (ABs) of the points estimates of SSR parameter and the mean square error.

	$(\theta_{1,}, \theta_2, \mu\ ) = (1.8{,}0.9{,}2)$

CS	ABs (maximum likelihood estimates	MSE(MLE)	RGP ABs (generalized point estimate)	MSE(GPE)
1	0.0999	0.0155	0.0948	0.0134
2	0.0919	0.0126	0.0869	0.0115
3	0.0998	0.0152	0.0950	0.0139
4	0.0924	0.0121	0.0856	0.0107
5	0.0709	0.0072	0.0681	0.0067
6	0.0757	0.0097	0.0733	0.0091
7	0.0758	0.0083	0.0724	0.0077
8	0.0707	0.0079	0.0690	0.0075
9	0.0530	0.0046	0.0515	0.0044
10	0.0543	0.0044	0.0535	0.0042
11	0.0587	0.0054	0.0577	0.0053
12	0.0614	0.0065	0.0600	0.0062

Table5. Common truncated parameter case, average width (AW) of bootstrap and generalized confidence intervals and the coverage probability(cp).

CS	AW (bootstrap CI)	CP (coverage probability for BCI)	AW (generalized confidence interval)	CP (coverage probability for GCI)
1	0.4053	0.9200	0.4103	0.9100
2	0.4111	0.9100	0.4171	0.9100
3	0.4166	0.9100	0.4164	0.9100
4	0.3996	0.9200	0.4061	0.9100
5	0.3420	0.9700	0.3458	0.9700
6	0.3376	0.8900	0.3404	0.8900
7	0.3397	0.9500	0.3387	0.9300
8	0.3465	0.9500	0.3467	0.9500
9	0.2674	0.9400	0.2691	0.9200
10	0.2682	0.9700	0.2697	0.9700
11	0.2721	0.9000	0.2708	0.9000
12	0.2672	0.9700	0.2675	0.9700

Table .6 Common resilience parameter case, bias (ABs) of the points estimates of SSR parameter and the mean square error.

$(\theta, \mu_1, \mu_2) = (1,0.9,1.2)$

CS	ABs (maximum likelihood estimates	MSE(MLE)	RGP ABs (generalized point estimate)	MSE(GPE)
1	0.1094	0.0238	0.0947	0.0159
2	0.1407	0.0408	0.1134	0.0254
3	0.1022	0.0187	0.0887	0.0126
4	0.1102	0.0217	0.0925	0.0136
5	0.0745	0.0098	0.0690	0.0076
6	0.0813	0.0158	0.0745	0.0116
7	0.0711	0.0093	0.0643	0.0070
8	0.0875	0.0133	0.0717	0.0086
9	0.0690	0.0077	0.0642	0.0064
10	0.0604	0.0066	0.0533	0.0052
11	0.0554	0.0052	0.0509	0.0043
12	0.0614	0.0056	0.0561	0.0047

Table7. Common resilience parameter case, Average width (AW) of bootstrap and generalized confidence intervals and the coverage probability(cp).

CS	AW (bootstrap CI)	CP (coverage probability for BCI)	AW (generalized confidence interval)	CP (coverage probability for GCI)
1	0.2388	0.8700	0.3897	0.8700
2	0.2358	0.8400	0.4206	0.8400
3	0.2413	0.6500	0.3873	0.8800
4	0.2390	0.6400	0.4015	0.9100
5	0.1742	0.8900	0.3067	0.8900
6	0.1736	0.7000	0.3180	0.8900
7	0.1738	0.6900	0.3086	0.9300
8	0.1737	0.6600	0.3291	0.9300
9	0.1149	0.6100	0.2398	0.8700
10	0.1150	0.9100	0.2430	0.9100
11	0.1150	0.9100	0.2374	0.9200
12	0.1144	0.9100	0.2368	0.9200

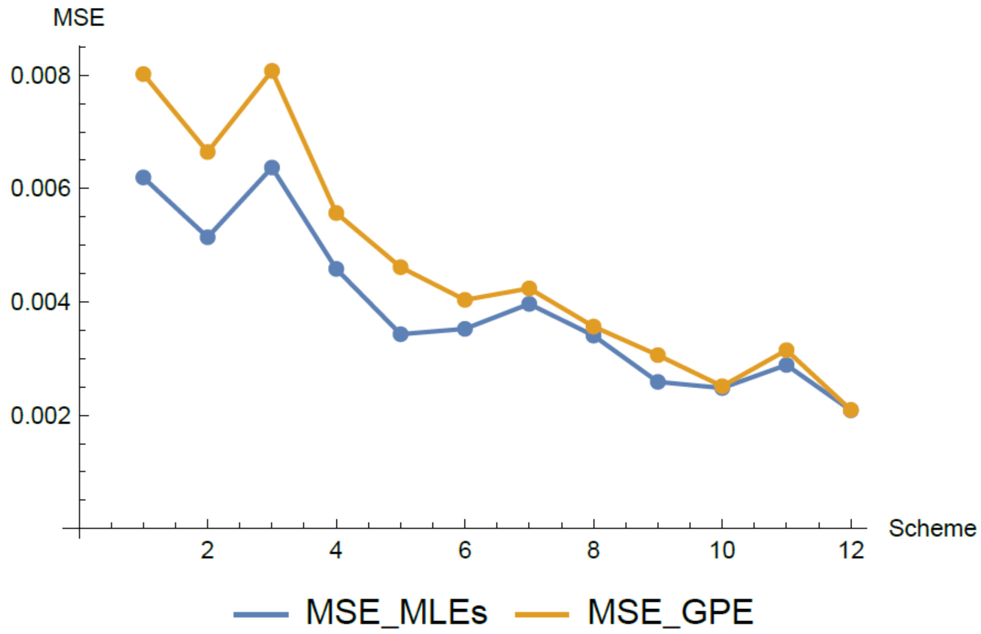

Figure 3. Plot of mean square error for arbitrary parameter case when $\theta_1 = 0.7, \mu_1 = 1.7, \theta_2 = 1.3, \mu_2 = 0.6$.

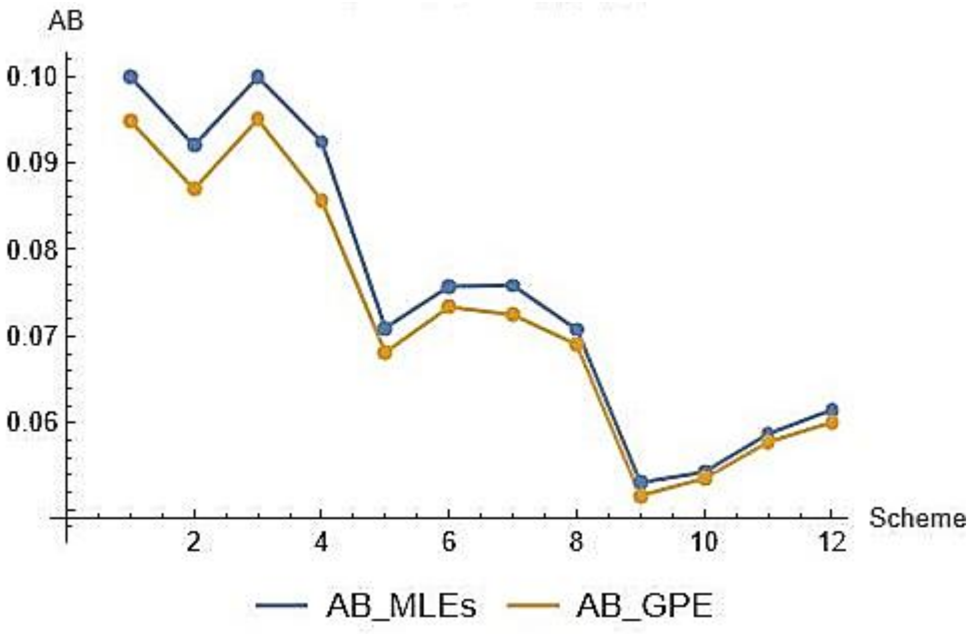

Figure 4. Plot of average bias for arbitrary parameter case when $\theta_1 = 0.7, \mu_1 = 1.7, \theta_2 = 1.3, \mu_2 = 0.6$.

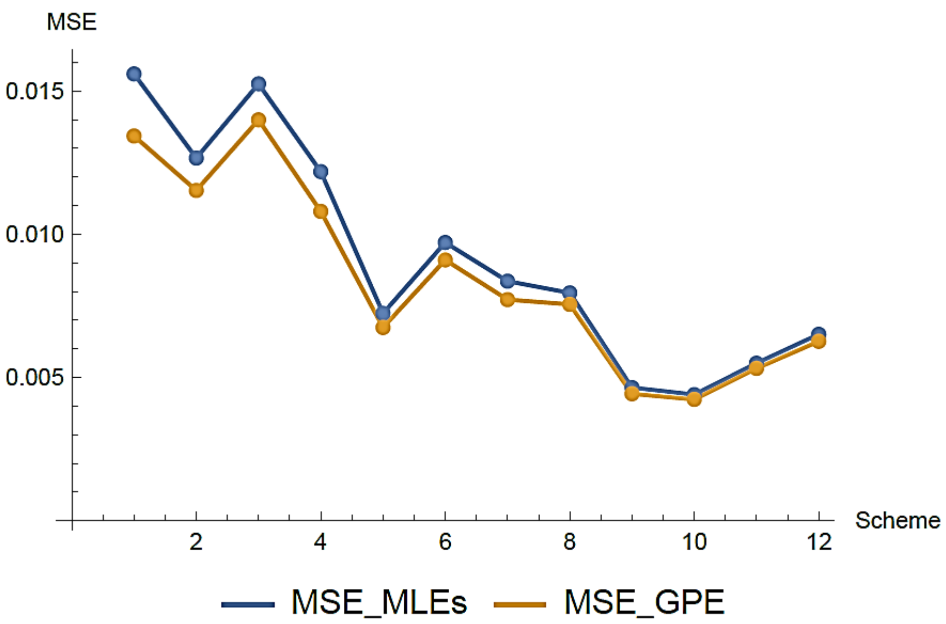

Figure 5.Plot of mean square error for common truncated parameter case when $\theta_1 = 1.8, \theta_2 = 0.9, \mu = 2$.

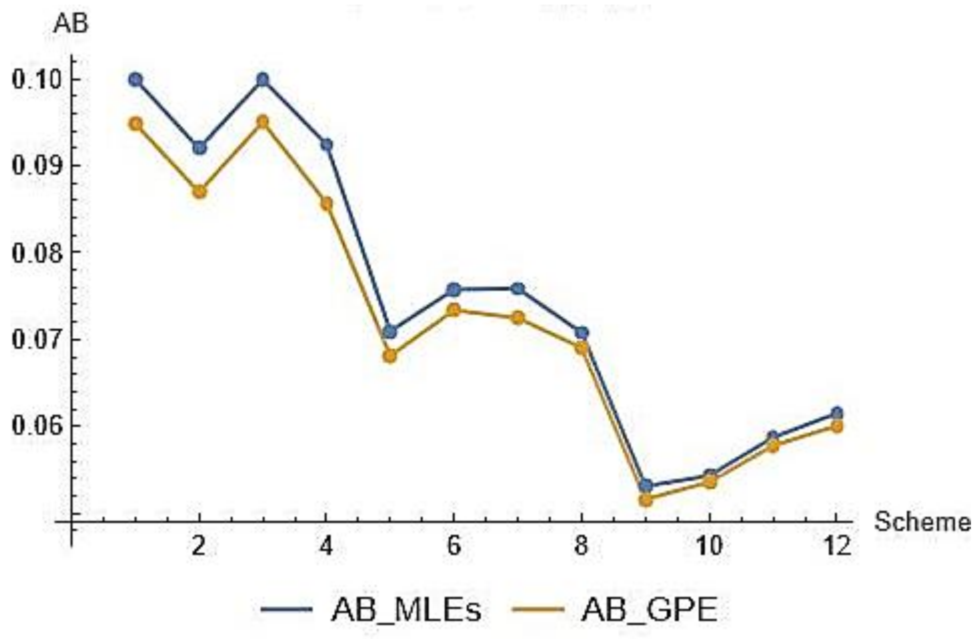

Figure 6. Plot of average bias for common truncated parameter case when $\theta_1 = 1.8, \theta_2 = 0.9, \mu = 2$.

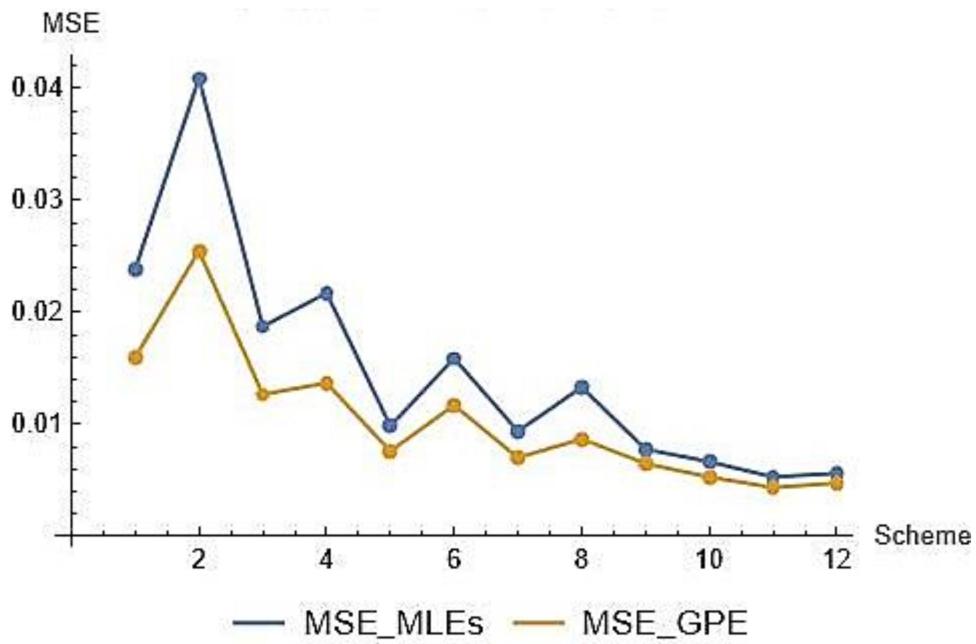

Figure 7. Plot of common residence parameter case when $\theta = 1, \mu_1 = 0.9, \mu_2 = 0.2$.

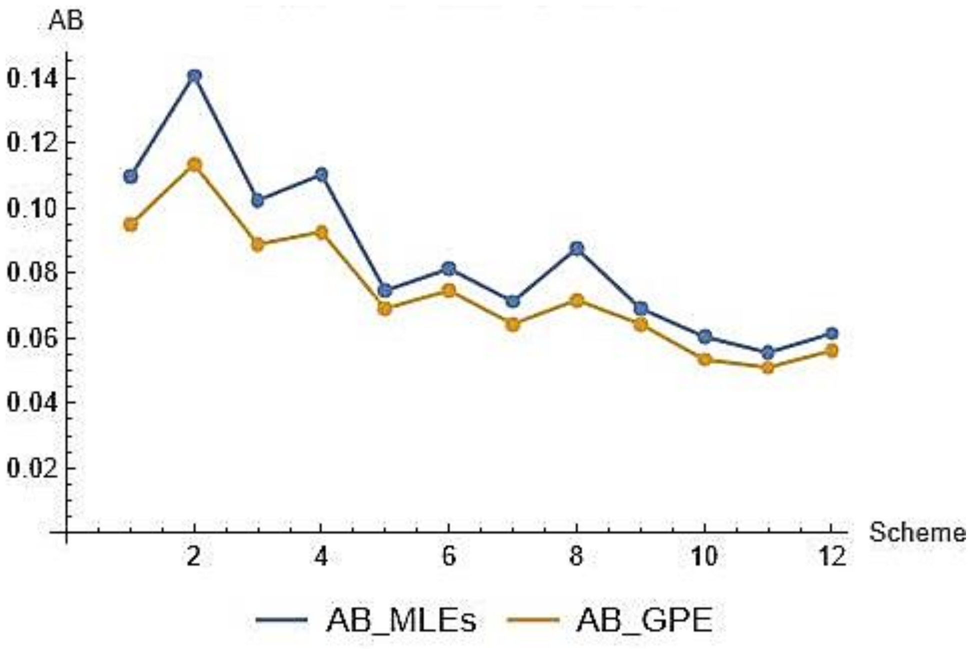

Figure 8. Plot of average bias for common resilience case when when $\theta = 1, \mu_1 = 0.9, \mu_2 = 0.2$.

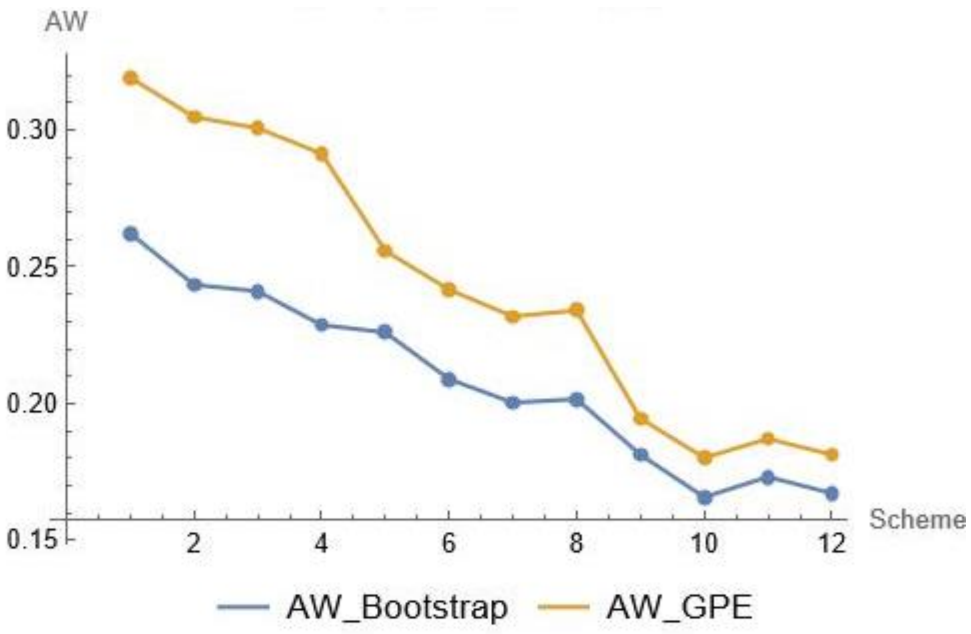

Figure 9. Plot of average widths for bootstrap and generalized confidence interval for arbitrary and parameter case when $\theta_1 = 0.7, \mu_1 = 1.7, \theta_2 = 1.3, \mu_2 = 0.6$.

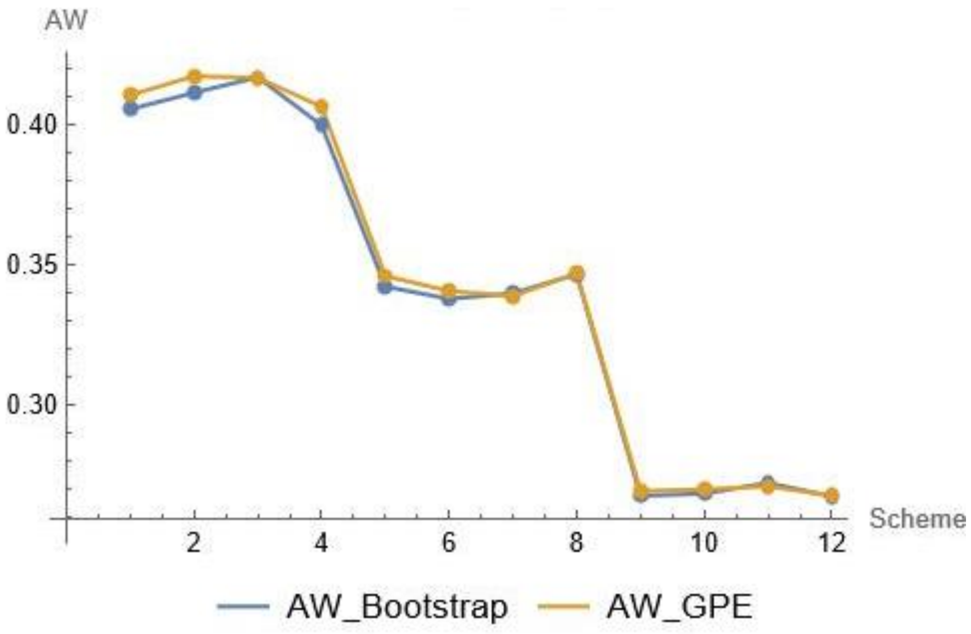

Figure 10. Plot of average widths for bootstrap and generalized confidence interval for common truncated parameter case when $\theta_1 = 1.8, \theta_2 = 0.9, \mu = 2$.

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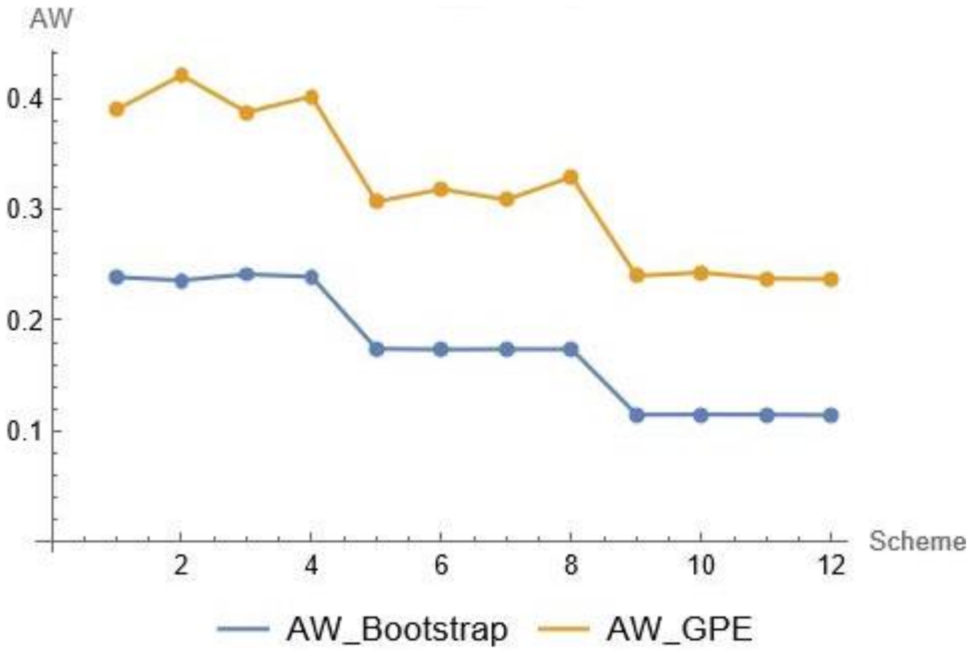

Figure 11. Plot of average widths for bootstrap and generalized confidence interval for common resilience parameter case when $\theta = 1, \mu_1 = 0.9, \mu_2 = 0.2$.

According to figures 3,5 and 7 it is evident that the mean square error decrease when the actual sample size increases by either n or m or both n and m for all the censoring schemes.

According to figures 4,6 and 8 it is evident that the average bias (AB) decreases for generalized point estimate and maximum likelihood estimate when the actual sample size increases by either n or m or both n and m for all the censoring schemes.

It is found that the GPEs of SSR parameters outperform the MLEs in terms of MSEs under the specified CSs.

The bootstrap confidence intervals (BCIs) have a marginal advantage over the generalized confidence intervals (GCIs) with regards to the size of the confidence intervals under SSR scenarios, both BCI and GCI show that increasing effective sample sizes increase coverage probability (CPs) while decreasing related average widths (AWs).

Ch.5 Real data analysis

In this chapter, we will study two real examples to explain the procedures in Chapter 3 . Note that the reliability parameter $R = P(X > Y)$, has many applications in the context of survival analysis. Our first example, illustrate an application in medical study.

For each set of data, we will use the truncated Pareto model and find the estimates $\hat{\theta}_1, \hat{\theta}_2, \hat{\mu}_1, \hat{\mu}_2$ for stress strength model. we will find the confidence intervals for generalized point estimation and bootstrap confidence intervals.

Example1 head and neck cancer patient from (Efron,1988)

A real data set was discussed by Ephron (1988). The research was carried out by the Northern California Oncology Group. It is about a head and neck cancer study and represents the survival times of patients treated with radiotherapy with N_1 = 34 and another group of head and neck cancer patients treated with radiotherapy and chemotherapy with N_2 = 29.

The first group data set, which will refer to as X, contains information about the lengths of 34 head and neck cancer patients who had radiation treatment, in contrast with another group of datasets, designated Y, which represents 29 patients who were treated for head and neck cancer with a combination of radiation and chemotherapy.

Table. 8 Kolmogorov-Smirnov distances and p-values of truncated Pareto model.

i	dataset	μ_i	θ_i	KS-distance	p-value
1	X	108	1.2314	0.1319	0.5502
2	Y	92	0.8129	0.1050	0.8731

Table .9 Estimates of truncated Pareto distribution.

	MLE (maximum likelihood estimates)	RGP (generalized point estimate)
Arbitrary parameter case	0.4712	0.4753
Common truncated parameter case	0.4557	0.4525
Common resilience parameter case	0.5742	0.5722

Table .10 Estimates of bootstrap confidence intervals.

	LBC (lower bound)	UBC (upper bound)	AWB (average width)
Arbitrary parameter case	0.3252	0.6346	0.3094
Common truncated parameter case	0.3206	0.6139	0.2933
Common resilience parameter case	0.5189	0.6240	0.1050

Table.11 Estimates of generalized confidence intervals.

	LGCI (lower bound)	UGC (upper bound)	AWG (average width)
Arbitrary parameter case	0.3232	0.6174	0.2942
Common truncated parameter case	0.3167	0.5949	0.2782
Common resilience parameter case	0.5329	0.6157	0.0827

We select the progressive type-II censored data from the complete samples based on the given censoring scheme R_x= (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0), where $m_1 \leq n_1$ depending on a certain algorithm.

Step1: set the complete sample in an ascending order with the complete data with length of strength n= 34, X= (108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417) .

Step2: Remove the first observation where $x[1] = \min\{x_1\}$.

Step2: Remove $x[1]$from the complete sample, where length $x = n - 1$.

Step3 Remove randomly R_x values from X where length of $x = n - 1 - R_m$.

Step4: Repeat step 3 m_1 times to get samples x =(108, 112, 129, 133, 133, 139, 140, 146, 149, 154, 157, 160, 160, 165, 173, 218, 241, 273, 277, 405, 417, 420, 583, 594, 1101).

With $m_1 = 25$ where m_1 is fauiler time of strength variable X.

For stress variable Y with n=29, where y= (92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 459, 519, 633, 725, 817, 1557, 1776) .

We repeat the similar algorithm to with censoring scheme $R_y(1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,0,0)$,we get y= {92, 94, 110, 119, 127, 130, 140, 146, 155, 159, 179, 194, 209, 249, 432, 459, 519, 725, 817, 1776}, with $m_2 = 20$, where m_2 is fauiler time of stress variable y.

First, we put our strategies to the test using these datasets. The Kolmogorov-Smirnov test was used to determine whether the data set X and Y was fit by the truncated Pareto model. The MLE and p-values are shown in Table 8. The large p-values for the Kolmogorov-Smirnov test (0.55024) and (0.873156) for the X and Y data sets, respectively, indicate that a truncated Pareto distribution can be used to fit these datasets.

The following figures show the empirical distributions of X and Y datasets obtained from the empirical distribution function of data x and y most closely fitted by the Pareto distribution.

From quantile plots, it is seen that the Pareto distribution is the most fit distribution for these survival data.

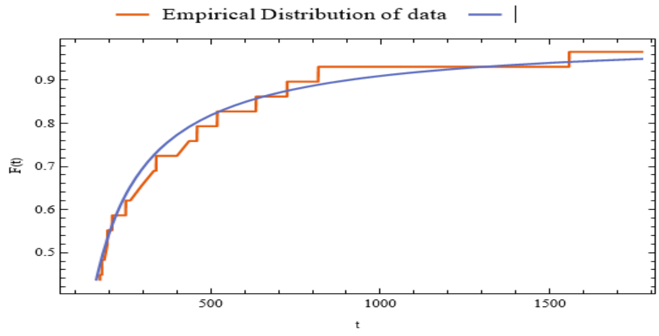

Figure **12**. Plot of empirical Distribution of X Datasets.

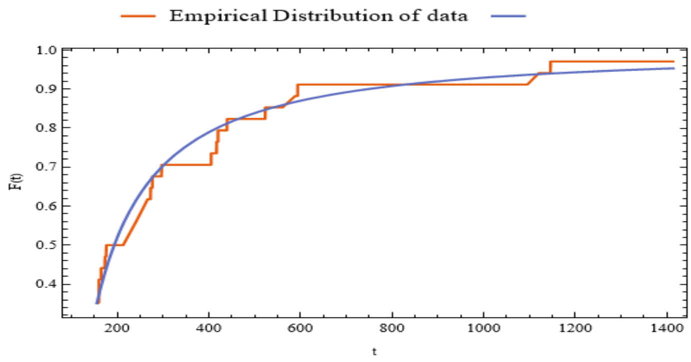

Figure **13**. Plot of empirical Distribution of Y Datasets.

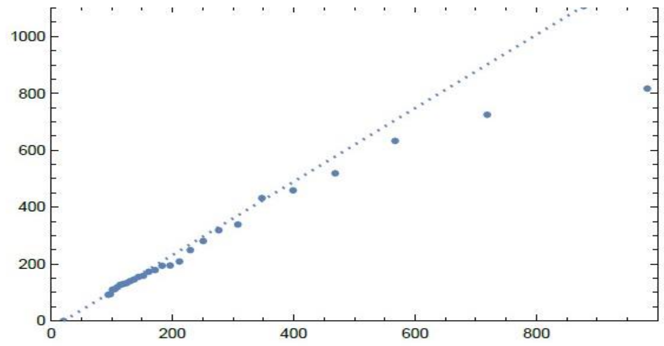

Figure.**14** Plot of quantiles Plots for X Data Sets.

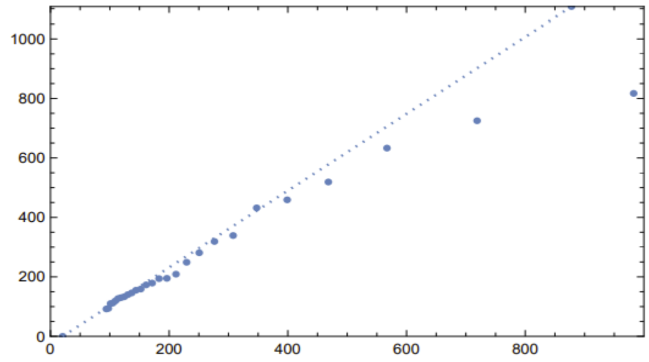

Figure.**15** Plot of quantiles Plots for y Data Sets.

Example2 average of sulfur-dioxide from (Priest and Bader,1982)

A real data It was first published by Preist and Badr (1982). The data consist of two groups of datasets: the first group is the strength variable was mentioned as the average of 1-hour levels of sulfur-dioxide in parts per million on the beach in Los Angeles, California, over the period of nineteen years in May, and the second group is the stress variable dataset, which obtained the average of sulfur dioxide in October.

The first group data set, which will refer to as X, contains information about the lengths of 17 levels of sulfur-dioxide in May, in contrast with another group of datasets, designated Y, which represents 19 levels of sulfur-dioxide in October.

This led them to their conclusion, sulfur-dioxide levels in May are fewer than they were of average in October. We check the reading from May to October to support this conclusion.

First, we put our strategies to the test using these datasets. The Kolmogorov-Smirnov test was used to determine whether the data set X and Y was fit by the truncated Pareto model. The MLE and p-values are shown in Table 12. The large p-values for the Kolmogorov-Smirnov test (0.259811) and (0.679038) for the X and Y data sets, respectively, indicate that a truncated Pareto distribution can be used to fit these datasets.

Table. **12** Kolmogorov-Smirnov distances and p-values of truncated Pareto model.

i	dataset	μ_i	θ_i	KS-distance	p-value
1	X	8	2.0973	0.2352	0.2598
2	Y	10	1.3439	0.1571	0.6790

Table .**13** Estimates of truncated Pareto distribution.

	MLE (maximum likelihood estimates)	RGP (generalized point estimate)
Arbitrary parameter case	0.7931	0.7702
Common truncated parameter case	0.7348	0.7284
Common resilience parameter case	0.7090	0.6978

Table .**14** Estimates of bootstrap confidence intervals.

	LBC (lower bound)	UBC (upper bound)	AWB (average width)
Arbitrary parameter case	0.6556	0.9403	0.2847
Common truncated parameter case	0.5711	0.8645	0.2933
Common resilience parameter case	0.6646	0.7898	0.1251

Table.**15** Estimates of generalized confidence intervals.

	LGCI (lower bound)	UGC (upper bound)	AWG (average width)
Arbitrary parameter case	0.6226	0.9118	0.2891
Common truncated parameter case	0.5957	0.8727	0.2782
Common resilience parameter case	0.6281	0.7641	0.1359

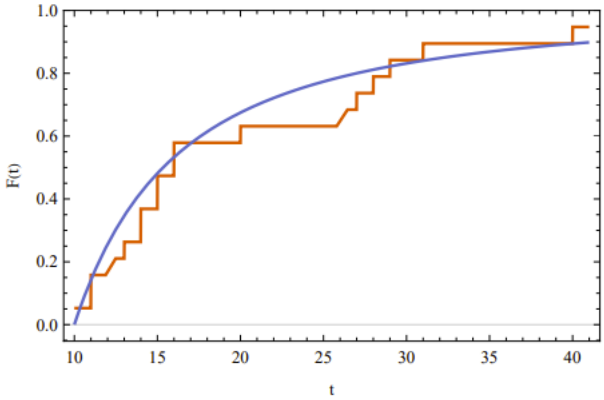

Figure.**16** plot of empirical distribution of x datasets.

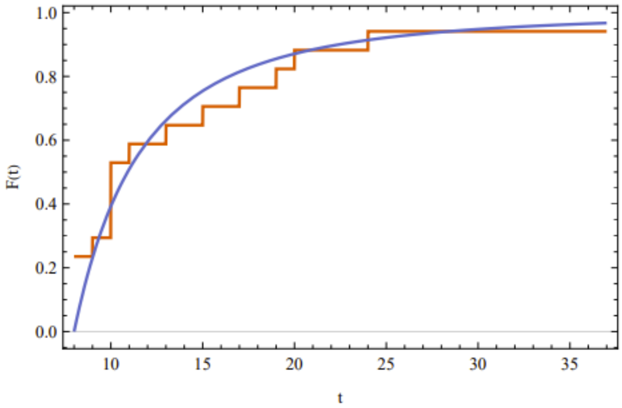

Figure.**17** Plot of empirical distribution of y datasets.

The following figures 7&8 show the empirical distributions of X and Y datasets obtained from the empirical distribution function of data x and y most closely fitted by the Pareto distribution.

We select the progressive type-II censored data from the complete samples based on the given censoring scheme $R_x =$ (15, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,0), where $m_1 \leq n_1$ depending on a certain algorithm.

We select the progressive type-II censored data from the complete samples based on the given censoring scheme $R_y =$ (16, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,0), where $m_2 \leq n_2$ depending on a certain algorithm.

With $m_1 = 15$ where m_1 is fauiler time of strength variable X and with $m_2 = 16$ where m_2 is fauiler time of strength variable Y.

For stress variable Y with n=18, where y= (2, 3, 8, 8, 8, 8, 9, 10, 10, 10, 10, 11, 13, 15, 17, 19, 20, 24, 37) .

We repeat the similar algorithm was mentioned in example 1we get y= { 8, 8, 8, 8, 9, 10, 10, 10, 10, 11, 13, 15, 17, 19,20}, with $m_2 = 15$.

For strength variable X with n=19, where X= (10, 11, 11, 12, 13, 14, 14, 15, 15, 16, 16, 20, 26, 27, 28, 29, 31, 40, 41)we repeat the similar algorithm was mentioned in example 1we get x= { 10, 11, 11, 12, 13, 14, 14, 15, 15, 16, 16, 20, 26, 27, 28,29},

with $m_1 = 16$.

Conclusion remarks

We have demonstrated the usefulness of the proposed inferential technique by carrying out some simulation experiments and real data examples. From real examples, we can note that the confidence interval is wider in the arbitrary parameter case, and that there is no large difference between estimates of reliability using different methods.

We can say that the stress strength parameter can be estimated depending on progressive type-II censored samples obtained from a truncated Pareto distribution.

To obtain point estimation under the arbitrary parameter case and common truncated parameter cases, maximum likelihood estimation is established. We obtained a generalized point estimator and confidence intervals based on pivotal quantities. Also, we obtained a bootstrap confidence interval based on an algorithm presented by Balakrishnan and Sandhu (1995).

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Appendix

Table of notations

Notation	***Description***
CS	*Censoring Scheme*
ABS	*Absolute Average Bias*
LGC	*Lower Bound Generalized Confidence Interval*
LBC	*Lower Bound Bootstrap Confidence Interval*
UBC	*Upper Bound Bootstrap Confidence Interval*
UGC	*Upper Bound of Confidence Interval*
RGP	*Reliability of the Generalized Estimator*