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Research Article

A Class of Koszul Algebra and Some Homological Invariants through Circulant Matrices and Cycles

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Recent advances in graph theory, linear algebra, and commutative algebra render us to tackle problems in one bough of mathematics with assistance and guidance from others. We will elaborate foremost and conceptually fathomless homological invariants inextricably linked with circulant matrices and cycles through various path lengths in this article, as well as a class of Koszul algebra, which portrays combinatorial correlation, in the end.

1. Introduction and Definitions

Over the last few years, scholars have become increasingly reliant on computers for the majority of their research work, and the second fashion is graph theory, an extremely popular bough of mathematics. By interacting with several graphs, people can grasp numerous practical applications. In essence, the problem of Konigsberg bridge, happened in 1735, was the genesis of graph theory, and the researchers later worked significantly and intensively on the complete graph, bipartite graph, and Eulerian graph. Cauchy and L'Huilier were influential in introducing a strong area of mathematics, topology, succeeding Leonhard Euler's work. In the field of theoretical chemistry, Arthur Cayley was the very first scientist to analyze trees to predict chemical composition. Sylvester initially introduced the word graph, a mathematical structure that can be used to model the relationship between objects, in his work, and Frank Harary published a heroic book on graph theory in 1969 to unify mathematicians, chemists, engineers, biologists, social scientists, and computer scientists. Under the shadow of basic graphs, now, we can understand yeast two-hybrid problem [1], microarrays and RNA-seq [2–4], major problems of discrete mathematics, and protein-protein interaction problem [5–9]. When dealing with chemical reaction networks (CRNs) [10], graph theory is a valuable, prolific, versatile, and companionable tool. It has unquestionably become an essential academic discipline in a variety of domains, including computation flow, GPS (route, track, and waypoint), communication networks, computer science, Google Maps, computational devices, Simulink [11, 12], and so on. It appears hard in today's reality to describe characteristics of classical random graphs in relevance to representations of meaningful complex networks so bipartite graphs can be leveraged to solve this complicated problem [13] and assisting in the advance coding theory, database management, document/word problem, optimal assignment problem, communication network addressing, radar system, query log analysis, missile guidance, astronomy, personnel assignment problem, circuit design, crystallography, projective geometry [14], and x-ray [15, 16] (see Figure 1).

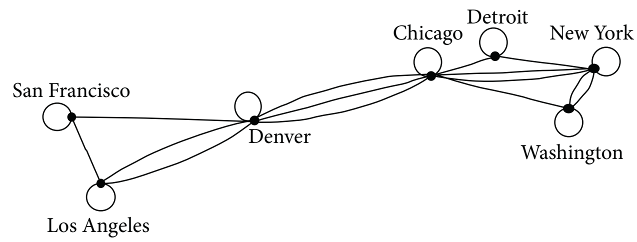

Figure 1: A computer network with diagnostic links.

The relationship in the form of turbo codes and low-density parity-check probabilistic decoding between a factor graph, a particular graph, and belief network are extremely close (see [17]). The captivating part of mathematics, chemical graph theory, by interrelating chemistry, is a new direction of modern research. In the form of a molecular graph, the molecules from chemistry are modelled mathematically. Vertices represent atoms in a molecular graph, whereas edges represent chemical bonds. These molecular structures are subjected to a variety of graph theory approaches in order to determine their topological and structural properties. The boiling point of a chemical compound, which is a physical entity, can be approximated using the degree and distance between the chemical compound's vertices, for example. Thus, topology of the molecular structure plays a crucial role in predicting the compelling advantages of the accompanying chemical compound through mathematics [18]. In 1988, it was reported that a few hundred specialist analysts worked on delivering roughly 500 research publications each year, looking at various aspects of chemical structures, including Gutman's two-volume fastidious material [19]. In [20–29], you can find more applications of this intriguing field of research, including discussions of topological indices. The study of chemical compounds in terms of mathematical modelling is one of the most recent research directions among scientists [27, 28].

Chemical compounds with distinctive mathematical structures and a diverse variety of uses in industrial, medicinal, research, and commercial chemistry occur in large numbers. A chemical compound's atom arrangements follow specific structural laws that have beneficial anomalous behavior. Thus, in applied research, adopting mathematical methods such as combinatorics and topology to investigate these attributes play a key role. It is fair to say that the discipline of chemical graph theory makes valuable strides to mathematical chemistry [20, 21]. Many chemical graph theory invariants, such as indices or descriptors, are applied in other sciences, particularly in the pharmaceutical and chemical industries [24, 25]. The research of distance-based and degree-based indices, in particular, plays an important role in the development of related subjects [30]. It aids in the collection of large amounts of data in the form of numerical values associated with chemical structures and the comparison of those values utilizing modern computer systems [31]. Many topological descriptors were introduced in the latter decade of the nineteenth century to meet the needs of chemists [32, 33].

Mathematicians have made substantial progress in the study of Koszul algebras and their representations in the recent few decades. In commutative algebra, topology, algebraic geometry, and representation theory, they exploited it extensively. In the form of linear minimal graded free resolution, Jean-Louis Koszul, a French mathematician, introduced Koszul algebra. This resolution yields graded Betti numbers, which can be used to really need homological invariants of a module. Stewart explored Koszul resolutions for enormous classes of algebras, Steenrod algebra, and universal enveloping algebra in depth in 1970 (see [34]). Conca introduced Koszul filtrations in [35] as a result of enthusiastic work on strongly Koszul algebras in [36]. The Koszulness of the ring [37] is determined by the quadratic Gröbner basis of an ideal of the residue class ring; however, there are some Koszul algebras whose defining ideals are not generated quadratically under some monomial ordering. A toric ideal, a particular form of binomial ideal, combines combinatorics, geometry, and algebra and has a variety of applications, including contingency tables, integer programming, triangulations of convex polytopes, and algebraic geometry. The quadratic binomials generate a toric ideal with a finite graph combinatorially, and this leads us to believe that a Koszul algebra is normal if squarefree quadratic monomials generate the toric ideal but not the other way around [38]. Numerous classical algebra questions can be answered by connecting the three well-known boughs of mathematics, commutative algebra, graph theory, and linear algebra. The cyclotomic polynomial is connected to the circulant matrix, which is a specific type of Toeplitz matrix named after Otto Toeplitz. These special matrices have applications in linear algebra and graph theory. With the help of circulant matrices, the system of linear equations can be converted into circular convolution, and by using the circular convolution theorem, we can use the discrete Fourier transform to transform the cyclic convolution into componentwise multiplication. Researchers can call a graph is a circulant if the adjacency matrix of a simple finite graph is circulant. In other words, we can state that a graph is a circulant if its group of automorphisms comprises a full-length cycle. Möbius ladders, for example, are circulant graphs.

A graph $\Gamma = (\Sigma, \Upsilon)$ is made up of two sets, edge set Υ and vertex set Σ. If both these sets have a finite number of items, then Γ is the finite graph. Otherwise, Γ is a graph that is infinite. A simple graph is one that has no loops and multiple edges and is undirected if the edges do not reflect directions. If there is an edge between two vertices x_1 and x_2 in the set Σ

of the graph Γ, they are considered neighbours, and the edge is said to be an incident in this situation. A simple graph is connected if there is a path between any two vertices. A cycle is a simple graph with $n \geq 3$ vertices in which the sequence of edges is built by connecting the end vertex of one edge to the starting vertex of the next edge, denoted by C_n.

An infinite ring containing members of the type $\mathfrak{a}_0 + \mathfrak{a}_1 x + \mathfrak{a}_2 x^2 + \cdots + \mathfrak{a}_n x^n$ in one variable x is referred to as a polynomial ring, denoted as $\mathfrak{K}[x]$, in which all the coefficients $\mathfrak{a}_i$ are drawn from the field $\mathfrak{K}$ of characteristic 0 and powers are nonnegative integers. Similarly, $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_n]$ can be used to define and indicate a polynomial ring in n variables over the field $\mathfrak{K}$. $\mathfrak{K} \subset Z(\mathfrak{R})$. A graded ring $\mathfrak{R}$ is a ring that can be written as a direct sum $\mathfrak{K}$ under the condition $\mathfrak{R}$, where the additive subgroups $\mathfrak{R}^{(0)} \oplus \mathfrak{R}^{(1)} \oplus \mathfrak{R}^{(2)} \cdots$ and elements of $\mathfrak{R}^{(\chi_1+\chi_2)} \supseteq \mathfrak{R}^{(\chi_1)} \mathfrak{R}^{(\chi_2)}$ $\forall \chi_1, \chi_2 > 0$ are known as homogeneous components of $\mathfrak{R}^{(\chi)}$ of degree $\mathfrak{R}^{(\chi)}$ and homogeneous of degree $\mathfrak{R}$, respectively. If an ideal χ, then χ is called a graded ideal.

A graded rings homomorphism is a ring homomorphism $\phi\colon \mathfrak{R}_1 \to \mathfrak{R}_2$ of two graded rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$ with $\phi(\mathfrak{R}_1^{(\chi)}) \subseteq \mathfrak{R}_2^{(\chi)}$ for $\chi = 0, 1, 2, \ldots$, for $\mathfrak{R}_2$. It can be shown easily that $\mathfrak{R}$, $\mathfrak{R}^{(0)}$-module, and $\mathfrak{R}^{(0)}$-algebra have the subring $\mathfrak{R}^{(0)}$. is called $\mathfrak{R}$-algebra, if both $\mathfrak{R}$ and $\mathfrak{K} \subset Z(\mathfrak{R})$ (inclusion) have the same identity. The residue class ring $\mathfrak{K}[x_1, \ldots, x_n]/I_{\mathfrak{R}}$ is isomorphic to any standard graded $\mathfrak{K}$-algebra where $I_{\mathfrak{R}}$ is graded ideal. For example, With $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_n]$ and $\deg(x_i) = 1$ for $\mathfrak{K}$, we say the prototype of a standard graded $\mathfrak{K}$-algebra is the polynomial ring of n variables. (1) Let the set $\mathfrak{S}(\mathfrak{R})$ contains all the chains $i = 1, \ldots, n$ of prime ideals in $\mathfrak{R}$, mathematically, in set builder form, this set is equal to $\{\mathfrak{p}_{\text{seq}} = (p_0 \subset \cdots \subset p_m \subset \mathfrak{R}) | p_i$ prime ideal$\}$. (2) If $i = 1, 2, \ldots, n$, then length $(\mathfrak{p}_{\text{seq}}) = m$. (3) The numerical value dim $(\mathfrak{R}) = \sup\{\text{length}(\mathfrak{p}_{\text{seq}}) | \mathfrak{p}_{\text{seq}} \in \mathfrak{S}(\mathfrak{R})\}$ is Krull dimension of $\mathfrak{R}$.

The chain $(0) \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, x_2, \ldots, x_n)$ convinces us to say that the Krull dimension of $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_n]$ is n. Let / be any ideal of $\mathfrak{R}$ then this notion, Krull dimension, based on prime ideals is the same $\mathfrak{R}/I$ and I. $\mathfrak{R}/I$; its ideal is the same. The largest number $\gimel$ associated with the chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_\gimel = \mathfrak{p}$ of prime ideals is taken as the height of $\mathfrak{p}$, and this number can be denoted by ht $(\mathfrak{p}) = \gimel$. Let us suppose $\mathfrak{K}$, a base field; $\mathfrak{R}$, a standard graded $\mathfrak{K}$-algebra; and $\mathfrak{N}$, a nonzero finitely generated graded $\mathfrak{R}$-module of dimension $\mathfrak{d}$ with the graded components $\mathfrak{N}_i$ (finite dimensional $\mathfrak{K}$-vector spaces of $\mathfrak{N}$). Any member of $\mathfrak{N}_i$ is said to be homogeneous of degree i, and any member of $\mathfrak{N}$ is the unique finite sum of homogeneous members.

A function of numerical values $h\colon \mathbb{Z} \to \mathbb{Z}$, defined by $i \mapsto \dim_{\mathfrak{K}} \mathfrak{N}_i$, is known as the Hilbert function of $\mathfrak{N}$. From this function, we can derive the series (Tex translation failed) $\dim_{\mathfrak{K}} \mathfrak{N}_i\, t^i$ known as Hilbert series of $\mathfrak{N}$, there is a polynomial, Laurent-polynomial $\mathfrak{Q}_{\mathfrak{N}}(t)$ in $\mathbb{Z}[t, t^{-1}]$ with $\mathfrak{Q}_{\mathfrak{N}}(1) > 0$ and $h_{\mathfrak{N}}(t) = \mathfrak{Q}_{\mathfrak{N}}(t)/(1-t)^{\mathfrak{d}}$ and $\mathfrak{d} - 1$ degree polynomial $P_{\mathfrak{N}}(x)$, Hilbert polynomial, in $\mathbb{Q}[x]$ with $\dim_{\mathfrak{K}} \mathfrak{N}_i = P_{\mathfrak{N}}(i)$ for all $i > \deg(\mathfrak{Q}_{\mathfrak{N}}) - \mathfrak{d}$. The pole order of $h_{\mathfrak{N}}(t)$ at $t = 1$ is said to be Krull dimension $\mathfrak{d}$ of $\mathfrak{N}$ at $t = 1$. The multiplicity $e(\mathfrak{N})$ of $\mathfrak{N}$ is the positive number $\mathfrak{Q}_{\mathfrak{N}}(1)$, and $e(\mathfrak{N})/(\mathfrak{d}-1)!$ is the leading coefficient of $P_{\mathfrak{N}}(x)$ of $\mathfrak{N}$. The number $\deg(\mathfrak{Q}_{\mathfrak{N}}(t)) - \mathfrak{d}$, degree of the Hilbert series $h_{\mathfrak{N}}(t)$, is called a-invariant. Let $\mathfrak{Q}_{\mathfrak{N}}(t) = \sum_{i=r_1}^{r_2} h_i t^i$.

The coefficient vector $(h_{r_1}, h_{r_1+1}, \ldots, h_{r_2})$ of $\mathfrak{Q}_{\mathfrak{N}}(t)$ is said to be the h-vector of $\mathfrak{N}$. Let us consider an ideal I of the ring $\mathfrak{R}$. The maximal length $\mathfrak{n}$ of an $\mathfrak{N}$-sequence $b_1, \ldots, b_{\mathfrak{n}} \in I$ is said to be the I-depth of $\mathfrak{N}$ with the condition $\mathfrak{N} \neq I\mathfrak{N}$ and can be written as depth$(I, \mathfrak{N})$. If $\mathfrak{N} = I\mathfrak{N}$ then I-depth of $\mathfrak{N}$ is by convention ∞. A pair $(\mathfrak{m}, \mathfrak{R})$ of unique maximal ideal and commutative ring is taken to be the local ring. In the case of the local ring $\mathfrak{R}$, we can say depth of $\mathfrak{N}$, depth$(\mathfrak{N})$ = depth$((\mathfrak{m}, \mathfrak{N}))$, is simply $\mathfrak{m}$-depth of $\mathfrak{N}$. The following Auslander–Buchsbaum theorem shows the relation between depth and projective dimension. In the presence of $\mathfrak{R}$, Noetherian local ring, and $\mathfrak{N}$, finitely generated $\mathfrak{R}$-module of finite projective dimension, the statement of Auslander–Buchsbaum theorem is depth$(\mathfrak{N}) + pd_{\mathfrak{R}}(\mathfrak{N})$ = depth$(\mathfrak{R})$ where the projective dimension of $\mathfrak{N}$ in $\mathfrak{R}$ is $pd_{\mathfrak{R}}(\mathfrak{N})$. $\mathfrak{N}$ is Cohen–Macaulay if $\mathfrak{N}$ is trivial and in case of nontrivial it must have depth$(\mathfrak{N})$ = dim $(\mathfrak{N})$.

An exact sequence $0 \leftarrow \mathfrak{N} \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$, defined by $\varphi_i\colon F_i \to F_{i-1}$ and $\epsilon\colon F_0 \to \mathfrak{N}$, is called graded $\mathfrak{R}$-resolution of $\mathfrak{N}$ where all F_i are graded free $\mathfrak{R}$-modules generated by finite sets, and if $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_n]$, this sequence $\mathbb{G}_{\min}$ (say) is known as minimal graded free $\mathfrak{R}$-resolution if $\varphi_i(F_i) \subset \mathfrak{m}F_{i-1}\ \forall i$, where $\mathfrak{m}$ is the graded maximal ideal $(x_1, \ldots, x_n)$ in $\mathfrak{R}$. The dimension of the final free module in $\mathbb{G}_{\min}$ is known as the type of Cohen–Macaulay $\mathfrak{R}$-module $\mathfrak{N}$. A Cohen–Macaulay $\mathfrak{R}$-module $\mathfrak{N}$ with type 1 is called Gorenstein. Let us consider above-mentioned $\mathbb{G}_{\min}$ with $F_a = \oplus_b \mathfrak{R}(-b)^{\beta_{a,b}(\mathfrak{N})}$. The uniquely determined numbers $\beta_{a,b}(\mathfrak{N})$ or $\beta_{a,b}$ by $\mathfrak{N}$ are said to be graded Betti numbers of $\mathfrak{N}$. The graded Betti numbers play a vital role to deduce other important homological invariants of $\mathfrak{N}$. The projective dimension of $\mathfrak{N}$ is the number $pd_{\mathfrak{R}}(\mathfrak{N}) = \max\{a\colon \beta_{a,b} \neq 0, \text{for some } b\}$, and reg $(\mathfrak{N}) = \max\{b\colon \beta_{a,a+b} \neq 0, \text{for some } a\}$ is said to be the regularity of $\mathfrak{N}$, and depth$(\mathfrak{N}) = \min\{a\colon \beta_{n-a,b} \neq 0, \text{for some } b\}$ is known as the depth of $\mathfrak{N}$. We say $\mathbb{G}_{\min}$ is linear if reg$(\mathfrak{N}) = \alpha(\mathfrak{N})$, where $\alpha(\mathfrak{N})$ indicates least degree of a generator of $\mathfrak{N}$. $\mathfrak{N}$ is said to be Koszul algebra if $\mathbb{G}_{\min}$ is linear.

A nonvacuous set $\mathfrak{A} = \{v_1, \ldots, v_m\}$ of the monomials in $\mathfrak{R} = \mathfrak{K}[t_1, \ldots, t_n]$ is called monomial configuration of $\mathfrak{R}$. Then the toric ring $\mathfrak{K}[v_1, \ldots, v_m]$ is the subring of $\mathfrak{R}$ denoted by $\mathfrak{K}[\mathfrak{A}]$. The toric ideal, defining ideal of $\mathfrak{K}[\mathfrak{A}]$, $I_{\mathfrak{A}}$ of $\mathfrak{A}$ is the kernel of epimorphism $\phi\colon \mathfrak{R}' \to \mathfrak{K}[\mathfrak{A}]$ defined by $\phi(x_i) = v_i$ for $i = 1, \ldots, m$, where $\mathfrak{R}'$ is the polynomial ring $\mathfrak{K}[x_1, \ldots, x_m]$. Every toric ideal $I_{\mathfrak{A}}$ is a prime ideal. A polynomial $v_1 - v_2$ in $\mathfrak{R}$ is called a binomial where v_1 and v_2 are monomials of $\mathfrak{R}$ and binomially generated ideal is called binomial ideal. $I_{\mathfrak{A}}$ is the binomial ideal generated by $v_1 - v_2$ with $\phi(v_1) = \phi(v_2)$. A primitive binomial is the binomial $f = v_1 - v_2 \in I_{\mathfrak{A}}$ if there $\nexists\, g = u_1 - u_2 \in I_{\mathfrak{A}}$ with the inequality $g \neq f$ such that $u_1 | v_1$ and $u_2 | v_2$. If A_1 and A_2 are the sets of primitive binomials and irreducible binomials, respectively, in $\mathfrak{R}$, then $A_1 \subset A_2$. If we have initial ideal $in<(I)$, generated by the monomials $in<(v_1), in<(v_2), \ldots, in<(v_s)$, of I, nonzero ideal of $\mathfrak{K}[x_1, \ldots, x_n]$, then a finite subset of nonzero polynomials $\{v_1, v_2, \ldots, v_s\} \subset I$ is called the Gröbner basis or standard basis of I under any ordering $<$. With respect to monomial ordering $<$, Gröbner basis of I always exist, and every finite superset $\mathfrak{G}'$ of $\mathfrak{G}$, Gröbner basis of I, is also Gröbner basis of I under $<$. Nonzero polynomials $u_1, \ldots, u_s$ form Gröbner basis if $in<(u_i) = in<(v_i)$, where $\mathfrak{G} = \{v_1, v_2, \ldots, v_s\}$ is a Gröbner basis of I. A standard basis $u_1 u_2, \ldots, u_s$ is considered to be reduced if leading coefficients of $in<(u_i) = in<(v_i)$ for $\mathfrak{G} = \{v_1, v_2, \ldots, v_s\}$ are unity I I, and in case of $\forall 1 \leq i \leq s$, $\{v_1, v_2, \ldots, v_s\}$ does not divide any supp $in<(v_i)$.

Uniquely determined reduced (standard) Gröbner basis always exists. A reduced standard basis of toric ideal $I_{\mathfrak{A}}$ consists of primitive binomials. Let us consider a finite connected graph, $\Gamma = (\Sigma, \Upsilon)$ without loops and multiple edges, on the vertex set $\Sigma = \{t_1, t_2, \ldots, t_q\}$, the edge set $\Upsilon = \{e_1, e_2, \ldots, e_n\}$ and $\mathfrak{K}[\mathbf{t}]$ is the polynomial ring $\mathfrak{K}[t_1, t_2, \ldots, t_q]$. Now we can attach the squarefree quadratic monomial $t_i t_j \in \mathfrak{K}[\mathbf{t}]$ with an edge $e = \{t_i, t_j\} \in \Upsilon$ where t_i, t_j are the members of Σ. The $\mathfrak{K}[\Gamma]$, edge ring of Γ, is considered to be the toric subring of $\mathfrak{K}[\mathbf{t}]$, which is generated by $\mathbf{t}e_1, \mathbf{t}e_2, \ldots, \mathbf{t}e_n$. The toric ideal, defining ideal of $\mathfrak{K}[\Gamma]$, I_Γ is the kernel of epimorphism $\phi\colon \mathfrak{R} \to \mathfrak{K}[\Gamma]$ defined by $\phi(x_i) = \mathbf{t}e_i$ for $i = 1, \ldots, n$, where $\mathfrak{R}$ is the polynomial ring $\mathfrak{K}[x_1, \ldots, x_n]$.

2. Main Results

Let $\mathscr{A} = \{a_1, \ldots, a_n\} \subset \mathbb{Z}_2^n$ be the set of vectors such that each a_i has at least one nonzero entry. Let $\mathfrak{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field $\mathfrak{K}$ of characteristic 0. Consider the semigroup homomorphism $\varphi\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ defined by $u = (u_1, u_2, \ldots, u_n) \mapsto u_1 a_1 + u_2 a_2 + \cdots + u_n a_n$. The image of φ is the semigroup $\mathbb{Z}_2\mathscr{A} = \{\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n\colon \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}_2\}$. The map φ lifts to a surjective homomorphism $\phi\colon \mathfrak{K}[x_1, \ldots, x_n] \to \mathfrak{K}[t_1, \ldots, t_n]$. The toric ideal $I_{\mathscr{A}}$ associated with $\mathscr{A}$ is the kernel of the epimorphism $\phi\colon \mathfrak{K}[x_1, \ldots, x_n] \to \mathfrak{K}[t_1, \ldots, t_n]$, given by $\phi(x_i) = \mathbf{t}^{\mathbf{a}_i} = t_1^{a_{1,i}} t_2^{a_{2,i}}, \ldots, t_n^{a_{n,i}}$, where $\mathbf{a}_i = (a_{1,i}, \ldots, a_{n,i})$. The image of ϕ is called toric ring denoted by $\mathfrak{K}[\mathscr{A}]$. Here is a well-known classic result.

Lemma 1. *If C_n be the cycle of length $n > 2$, then for $\mathbf{a}_i \in \mathbb{Z}_2^n$, we have $\dim_{\mathfrak{K}} \mathfrak{K}[\mathscr{A}] = \operatorname{rank} \mathscr{A}$, where $\dim_{\mathfrak{K}} \mathfrak{K}[\mathscr{A}]$ denotes the vector space dimension.*

Proof. If w_1 and w_2 are two nodes of C_n, a path of length m from w_1 to w_2 is a sequence of nodes $w_1 = t_{i_1}, \ldots, t_{i_{m+1}} = w_2$ of C_n such that $\{t_{i_j}, t_{i_{j+1}}\}$ is any link in C_n for all $j = 1, 2, \ldots, m$. We define the path ideal of C_n, denoted by $I_m(C_n)$ to be the ideal of $\mathfrak{K}[t_1, \ldots, t_n]$ generated by the monomials of the form $t_{i_1} t_{i_2}, \ldots, t_{i_{m+1}}$ where $t_{i_1}, t_{i_2}, \ldots, t_{i_{m+1}}$ is a path in C_n.

To each path $p_i(C_n, m) = t_{i_1}, t_{i_2}, \ldots, t_{i_{m+1}}$ of length m in C_n, we associate a vector $a_i(C_n, m) \in \mathbb{Z}_2^n$ such that

$$a_{i,j}(C_n, m) = \begin{cases} 1, & \text{if } t_{i_j} \in p_i(C_n, m), \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Since there are $p_i(C_n, m) = t_{i_1}, t_{i_2}, \ldots, t_{i_{m+1}}$ paths of length m in C_n, we set $a_i(C_n, m) \in \mathbb{Z}_2^n$, and thus, we obtain required result.

Theorem 1. *Let C_n be a cycle of length n and $I_{\mathscr{A}(C_n,m)}$ be the toric ideal associated to the path ideal $I_m(C_n)$. Then we have the following:*

(a) If gcd $(n, m+1) = 1$, then $I_{\mathscr{A}(C_n,m)} = 0$.

(b) If gcd $(n, m+1) = d$ with $d \neq 1$, then $I_{\mathscr{A}(C_n,m)} = (f_i)$ where $f_i = x_d x_{2d} \cdots x_{kd} - x_i x_{i+d} \cdots x_{i+(k-1)d} = \prod_{\theta=1}^{k} (x_{\theta d} - x_{i+(\theta-1)d})$ for $i = 1, \ldots, d-1$. So, here, we have $d-1$ generators, and each of the generator has degree $k = n/d$.

Proof. Obviously, there are n paths of length m in C_n. Thus, the matrix $[a_1(C_n, m), a_2(C_n, m), \ldots, a_n(C_n, m)]$ is a circulant matrix of order n given by

$$\mathscr{A} = \begin{pmatrix} \kappa_0 & \kappa_{n-1} & \cdots & \kappa_2 & \kappa_1 \\ \kappa_1 & \kappa_0 & \cdots & \kappa_3 & \kappa_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \kappa_{n-2} & \kappa_{n-3} & \cdots & \kappa_0 & \kappa_{n-1} \\ \kappa_{n-1} & \kappa_{n-2} & \cdots & \kappa_1 & \kappa_0 \end{pmatrix}, \tag{2}$$

where

$$\kappa_i = \begin{cases} 1, & \text{if } 0 \leq i \leq m; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The rank of circulant matrix $\mathscr{A}$ is equal to $n - d$ (see [39]), where d is the degree of gcd $(f(t), t^n - 1)$, where $f(t) = 1 + t + t^2 + \cdots + t^m$ is the associated polynomial to the matrix $\mathscr{A}$. Since $t^{m+1} - 1 = (t-1) f(t)$, it is enough to consider gcd $(t^{m+1} - 1, t^n - 1)$.

Now, $t^n - 1 = \prod_{d_1/n} \Phi_{d_1}(t)$, where $\Phi_{d_1}(t)$ is the minimal polynomial of a primitive d_1^{th} root of unity called cyclotomic polynomial. It is an irreducible polynomial in $\mathbb{Z}[t]$ with root $a \in \mathbb{C}$ with $a^{d_1} = 1$ but $a^i \neq 1$ for $i = 1, \ldots, d_1 - 1$.

(a) If $m+1$ and n are relatively prime, then $t^{m+1} - 1$ and $t^n - 1$ have only $a = 1$ as a common root. Hence, rank of $\mathscr{A}$ is n, thus by previous result, $I_{\mathscr{A}(C_n,m)} = 0$.

(b) If gcd $(n, m+1) = d$ with $d \neq 1$, then gcd $(t^{m+1} - 1, t^n - 1) = d$ so by the previous lemma $\dim_{\mathfrak{K}} \mathfrak{K}[\mathscr{A}] = n - d$. Let $A = \{f_i\}$ and monomial ordering $<$ is lex with $x_1 > x_2 > \cdots > x_n$. The image of every element of A is zero under ϕ so $(A) \subseteq I_{\mathscr{A}(C_n,m)}$ is obvious. It therefore suffices to show that each polynomial in $I_{\mathscr{A}(C_n,m)}$ is a $\mathfrak{K}$-linear combination of these binomials f_i. Suppose $g \in I_{\mathscr{A}(C_n,m)}$ cannot be written as a $\mathfrak{K}$-linear combination of binomials. We choose a polynomial g with this property such that the initial term $in_<(g) = \prod_{\theta=1}^{k} x_{i+(\theta-1)d}$ for $i = 1, \ldots, d-1$, is minimal with respect to the term order $<$ among all the elements of $I_{\mathscr{A}(C_n,m)}$.

When expanding $\phi(g(x_1, \ldots, x_n)) = g(\mathbf{t}^{a_1}, \mathbf{t}^{a_2}, \ldots, \mathbf{t}^{a_n}) = g(t_1 t_2, \ldots, t_{m+1}, t_2 t_3, \ldots, t_{m+2}, \ldots, t_n t_1, \ldots, t_m)$, we get zero because $g \in I_{\mathscr{A}(C_n,m)}$. In particular, the term $\phi(\prod_{\theta=1}^{k} x_{i+(\theta-1)d}) = \mathbf{t}^{\varphi(u_\theta)}$ must cancel during this expansion. Hence, there is some other monomial $\prod_{\theta=1}^{k} x_{\theta d}$ appearing in g such that $\varphi(u_\theta) = \varphi(v)$. Also, the polynomial $g' = g + \prod_{\theta=1}^{k} x_{i+(\theta-1)d} - \prod_{\theta=1}^{k} x_{\theta d}$ cannot be written as a $\mathfrak{K}$-linear combination of binomials in $I_{\mathscr{A}(C_n,m)}$ because g cannot be written as a $\mathfrak{K}$-linear combination of binomials. Hence, $in_<(g') < in_<(g)$. This is the contradiction to the fact that $in_<(g) = \prod_{\theta=1}^{k} x_{i+(\theta-1)d}$ is minimal with respect to the term order $<$ among all the elements of $I_{\mathscr{A}(C_n,m)}$. □

Now our next notion is the primary decomposition of the leading ideal of the toric ideal. First, we see an example in this case and then give a proposition.

Example 1. Consider the polynomial ring $\mathfrak{K}[x_1, \ldots, x_{15}]$ and $m+1=6$. Then the primary decomposition of $L(I_{\mathscr{A}(C_{15},5)})$ is given by

$$\begin{aligned} L\Big(I_{\mathscr{A}(C_{15},5)}\Big) &= (x_1x_4x_7x_{10}x_{13}, x_2x_5x_8x_{11}x_{14}), \\ L\Big(I_{\mathscr{A}(C_{15},5)}\Big) &= (x_1,x_2)\cap(x_2,x_4)\cap(x_1,x_5)\cap(x_{10},x_{14})\cap(x_{13},x_{14})\cap(x_4,x_5)\cap(x_2,x_7)\cap(x_5,x_7) \\ &\quad \cap(x_1,x_8)\cap(x_4,x_8)\cap(x_7,x_8)\cap(x_2,x_{10})\cap(x_5,x_{10})\cap(x_8,x_{10})\cap(x_1,x_{11})\cap(x_4,x_{11})\cap(x_7,x_{11}) \\ &\quad \cap(x_{10},x_{11})\cap(x_2,x_{13})\cap(x_5,x_{13})\cap(x_8,x_{13})\cap(x_{11},x_{13})\cap(x_1,x_{14})\cap(x_4,x_{14})\cap(x_7,x_{14}). \end{aligned} \tag{4}$$

where $L(I)$ is the leading ideal of ideal I under the lex monomial ordering $<$ with $x_1 > x_2 \ldots x_n$.

Proposition 1. *By the same notations, the primary decomposition of the leading ideal* $L(I_{\mathscr{A}(C_n,m)}) = \cap_{\lambda=1}^{k^{d-1}} \mathfrak{p}_\lambda$ *consists of the following characteristics:*

(1) Total number of primary components is k^{d-1}

(2) Number of variables (generators) in each component is $d-1$

(3) $x_d, x_{2d}, \ldots, x_{kd}$ *are the missing variables in this presentation*

(4) This presentation is irredundant so it is unique

(5) Total number of existing variables $n-k$

(6) Total number of generators in the primary decomposition is $(d-1)k^{d-1}$

(7) Each variable is appearing $(d-1)k^{d-1}/n-k$ *times in the primary decomposition*

In the remaining portion of this chapter, we shall use $\mathfrak{N}$ for the residue ring $\mathfrak{K}[x_1, \ldots, x_n]/I_{\mathscr{A}(C_n,m)}$.

Lemma 2. *By the same notations,* $\dim \mathfrak{N} = n-d+1$.

Proof. Let monomial ordering $<$ is lex with $x_1 > x_2 > \cdots > x_n$. By primary decomposition $L(I_{\mathscr{A}(C_n,m)}) = \cap_{\lambda=1}^{k^{d-1}} \mathfrak{p}_\lambda$, we can easily verify that the cardinality of $A = \{f_i\}$ for $i = 1, \ldots, d-1$ is equal to the height of $L(I_{\mathscr{A}(C_n,m)})$. That is, $htL(I_{\mathscr{A}(C_n,m)}) = d-1$. So Krull dimension of $\mathfrak{K}[x_1, \ldots, x_n]/L(I_{\mathscr{A}(C_n,m)})$ is $n-(d-1) = n-d+1$. Thus, by Corollary 5.3.14. of [40], we have the result. □

Lemma 3. *With the previous notations, the generators* $\prod_{\theta=1}^{k}(x_{\theta d} - x_{i+(\theta-1)d})$ *for* $i = 1, \ldots, d-1$ *of* $I_{\mathscr{A}(C_n,m)}$ *are the primitive binomials with lex monomial ordering and* $x_1 > x_2 > \cdots > x_n$.

Proof. By Buchberger algorithm, it is easy to find S-polynomial $S(f_p, f_q)$ of any two generators $f_p = \prod_{\theta=1}^{k}(x_{\theta d} - x_{p+(\theta-1)d})$ and $f_q = \prod_{\theta=1}^{k}(x_{\theta d} - x_{q+(\theta-1)d})$ for $p \neq q$, $1 \leq p, q \leq d-1$, can be written as $u_p f_p + u_q f_q$, where $u_p = -\prod_{\theta=1}^{k} x_{\theta d} = -u_q$. In other words, this S-polynomial reduces to zero through the set of generators $A = \{f_i\}$ for $i = 1, \ldots, d-1$. Thus, A is the Gröbner basis of the toric ideal $I_{\mathscr{A}(C_n,m)}$ with lex monomial ordering and $x_1 > x_2 > \cdots > x_n$. It is obvious to say that elements of A are the reduced Gröbner basis, so by proposition 10.1.2 of [41], we can say $I_{\mathscr{A}(C_n,m)}$ is generated by primitive binomials. □

Proposition 2. *With the same notations, the minimal graded free resolution of* $\mathfrak{N} = \mathfrak{K}[x_1, \ldots, x_n]$*-module* $\mathfrak{N}$ *is*

$$\mathfrak{N} \leftarrow \mathfrak{N}^{C_0^{d-1}}(0) \leftarrow \mathfrak{N}^{C_1^{d-1}}(1-k) \leftarrow \mathfrak{N}^{C_2^{d-1}}(2-2k) \leftarrow \cdots \leftarrow \mathfrak{N}^{C_{d-1}^{d-1}}((d-1)(1-k)), \tag{5}$$

with the graded Betti numbers

$$\beta_{i,j} = \begin{cases} \beta_{i,ki} = C_i^{d-1}, & \text{if } i = 0, \ldots, d-1; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

3. Numerical Data Obtained from Graded Betti Numbers

In this section, we shall deduce the most important homological invariants of the finitely generated module from the graded Betti numbers.

Let $\mathfrak{N}$ be a finitely generated graded $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_n]$-module with Betti numbers $\beta_{a,b} = \beta_{a,b}(\mathfrak{N})$ where

$$\beta_{a,b} = \begin{cases} C_a^{d-1}, & \text{if } a = 0, \ldots, d-1; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

The numbers

$$pd_{\mathfrak{R}}(\mathfrak{N}) = \max\{a\colon \beta_{a,b} \neq 0 \text{ for some } b\} = d-1, \tag{8}$$

$$\operatorname{reg}(\mathfrak{N}) = \max\{b\colon \beta_{a,a+b} \neq 0 \text{ for some } a\} = (d-1)(k-1), \tag{9}$$

are projective dimension and regularity of $\mathfrak{N}$, respectively. It is obvious from (8) that $pd_{\mathfrak{R}}(\mathfrak{N}) < n$, so by the Auslander–Buchsbaum theorem

$$\operatorname{depth}(\mathfrak{N}) + pd_{\mathfrak{R}}(\mathfrak{N}) = n, \tag{10}$$

we can write

$$\operatorname{depth}(\mathfrak{N}) = \min\{a\colon \beta_{n-a,b} \neq 0 \text{ for some } b\} = n-d+1. \tag{11}$$

Due to Lemma 2 and equation (11), the $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_n]$-module $\mathfrak{N}$ is Cohen–Macaulay. By Auslander–Buchsbaum theorem, we can say this module as Cohen–Macaulay because its projective dimension and codimension are equal as $\dim(\mathfrak{K}[x_1, \ldots, x_n]) - \dim(\mathfrak{N}) = n - (n-d+1) = d-1$. Since $\mathfrak{N}$ is Cohen–Macaulay, and its type is 1 so it is Gorenstein.

Proposition 3. *Let $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_n]$ be the standard graded $\mathfrak{K}$-algebra and $\mathfrak{N}$ be a $\mathfrak{R}$-module with graded Betti numbers $\beta_{a,b}$. Then $h_{\mathfrak{N}}(t) = (1-t^k)^{d-1}/(1-t)^n$ with $(1-t^k)^{d-1} = \sum_a (-1)^a (\sum_b \beta_{a,b} t^b)$. After dividing $(1-t^k)^{d-1}$ by the maximum possible power of $1-t$, then this series will be $h_{\mathfrak{N}}(t) = (\sum_{\alpha=0}^{k-1} t^{\alpha})^{d-1}/(1-t)^{n-d+1}$.*

A positive integer $n-d+1$ in this proposition is the Krull dimension of $\mathfrak{N}$ as discussed in Lemma 2, and the number $(\sum_{\alpha=0}^{k-1}(1)^{\alpha})^{d-1} = k^{d-1}$ is multiplicity of $\mathfrak{N}$ denoted by $e(\mathfrak{N})$. Let $\mathfrak{Q}_{\mathfrak{N}}(t) = \sum_{i=0}^{(k-1)(d-1)} h_i t^i = (\sum_{\alpha=0}^{k-1} t^{\alpha})^{d-1}$. The h-vector $h = (h_0, h_1, h_2, \ldots, h_{(k-1)(d-1)})$ contains, respectively, ones, counting numbers, triangular numbers, ... when $d = 2, 3, 4, \ldots$ of the Pascal triangle. Note that $0 = 0(\mathfrak{N})$. The 0-invariant $0(\mathfrak{N})$ of $\mathfrak{N}$ is degree of $h_{\mathfrak{N}}(t)$. Mathematically, $0(\mathfrak{N}) = (k-1)(d-1) - (n-d+1) = -k = \deg((1-t^k)^{d-1}) - n$ is the upper bound of $\max\{b\colon \beta_{a,b} \neq 0 \text{ for some } i\} - n = k(d-1) - n$.

We can display the graded Betti numbers of $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_n]$-module $\mathfrak{N}$ by the diagram called Betti diagram. In Betti diagram 2, graded Betti number $\beta_{a,a+b}$ is at the position (a, b); any nonzero graded Betti number lies inside the bounded region; and corner points represent the extremal Betti numbers see (Figure 2).

4. A Class of Koszul Algebras

Koszul algebras, Artin–Schelter regular algebras of lower global dimension three [42], play a vital role in the study of combinatorics, algebra, topology, and mathematical physics initially defined by Priddy [34]. Generally, it is very difficult to detect whether a given algebra is Koszul or not. One approach is to compute the first few matrices in the resolution and to find if they are linear. If the matrices are not linear, then this algebra is not Koszul. A more efficient way to prove that an algebra is Koszul is given by various kinds of filtration arguments. A family F of ideals called a Koszul filtration of $\mathfrak{N}$ has the properties listed below:

(1) F contains only ideals I generated by linear forms

(2) F also contains (0), zero ideal, and $\mathfrak{m}$, maximal ideal

(3) I/J is a cyclic module whose annihilator belongs to F with $0 \neq I, J \in F$, and $J \subset I$

Koszul filtration and strongly Koszul algebra were discussed, in a good manner, in [35, 36], respectively. Any ideal of Koszul filtration has linear resolution [35] leads to Koszul algebra, but this is the one-way statement [43] and a residue class ring is this algebra if its ideal accepts the standard basis of degree 2 by [37]. However, the defining ideal of a Koszul algebra may not have a quadratic Gröbner basis with respect to any monomial order. Also, in [44], the writers give an example of a binomial link ideal whose residue class ring has a Koszul filtration, while the ideal has no quadratic Gröbner basis related with graphs. In [45], authors worked associated with four cycles for the Koszul filtration of edge ring. We denote a vector (n, d, k) to represent $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_n]$-module $\mathfrak{N} = \mathfrak{K}[x_1, \ldots, x_n]/I_{\mathscr{A}(C_n,m)}$ where $\gcd(n, m+1) = d$ and $k = n/d$. Then the vector (2d,d,2) shows a class of Kosul algebras due to the fact that $I_{\mathscr{A}(C_n,m)}$ is generated by the standard basis of degree 2.

Now, we conclude this article by giving a concrete example. Let us consider $\mathfrak{R} = \mathfrak{K}[x_1, \ldots, x_{15}]$-module $\mathfrak{N} = \mathfrak{K}[x_1, \ldots, x_{15}]/I_{\mathscr{A}(C_{15},5)}$. The rank of a circulant matrix

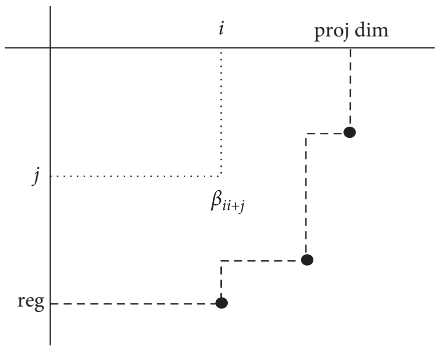

Figure 2: Betti diagram.

$$\mathscr{A} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\ 1&1&0&0&0&0&0&0&0&0&0&1&1&1&1\\ 1&1&1&0&0&0&0&0&0&0&0&0&1&1&1\\ 1&1&1&1&0&0&0&0&0&0&0&0&0&1&1\\ 1&1&1&1&1&0&0&0&0&0&0&0&0&0&1\\ 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 0&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\ 0&0&1&1&1&1&1&1&0&0&0&0&0&0&0\\ 0&0&0&1&1&1&1&1&1&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&1&1&1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&1&1&1&1&1&0&0&0\\ 0&0&0&0&0&0&0&1&1&1&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&0\\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1 \end{pmatrix}. \quad (12)$$

is $\dim_{\mathfrak{K}} \mathfrak{K}[\mathscr{A}] = \operatorname{rank} \mathscr{A} = n - d = 15 - 3 = 12$ Toric ideal $I_{\mathscr{A}(C_{15},5)}$ is generated by the reduced Gröbner basis $f_1 = x_3x_6x_9x_{12}x_{15} - x_1x_4x_7x_{10}x_{13}$ and $f_2 = x_3x_6x_9x_{12}x_{15} - x_2x_5x_8x_{11}x_{14}$ so the Krull dimension of $\mathfrak{N} = \mathfrak{K}[x_1, \ldots, x_{15}]/I_{\mathscr{A}(C_{15},5)}$ is $n - d + 1 = 13$. Minimal graded free resolution of $\mathfrak{N} = \mathfrak{K}[x_1, \ldots, x_{15}]/I_{\mathscr{A}(C_{15},5)}$ is

$$\mathfrak{N} \longleftarrow \mathfrak{N}^{C_0^2}(0) \longleftarrow \mathfrak{N}^{C_1^2}(-4) \longleftarrow \mathfrak{N}^{C_2^2}(-8), \quad (13)$$

with the graded Betti numbers $\beta_{0,0} = 1, \beta_{1,5} = 2$ and $\beta_{2,10} = 1$, $f_2 = x_3x_6x_9x_{12}x_{15} - x_2x_5x_8x_{11}x_{14}$, and $\mathfrak{N} = \mathfrak{K}[x_1, \ldots, x_{15}]/I_{\mathscr{A}(C_{15},5)}$ that can be represented by Table 1.

The projective dimension, regularity, and depth of $\mathfrak{N}$ are $d - 1 = 2$, $(d-1)(k-1) = 8$, and $n - d + 1 = 13$, respectively. Auslander–Buchsbaum theorem is verified for these numbers; $\mathfrak{N} = \mathfrak{K}[x_1, \ldots, x_{15}]/I_{\mathscr{A}(C_{15},5)}$ is Cohen–Macaulay and Gorenstein. The coefficients of h-vector $h = (1, 2, 3, 4, 5, 4, 3, 2, 1)$ are the counting numbers can be obtained from the Hilbert series $h_{\mathfrak{N}}(t) = (1 - t^5)^2/(1 - t)^{15} = (1 + 2t + 3t^2 + 4t^3 + 5t^4 + 4t^5 + 3t^6 + 2t^7 + t^8)/(1 - t)^{13}$ of degree $-k = -5$, and $k^{(d-1)} = 25$ is the multiplicity of $\mathfrak{N} = \mathfrak{K}[x_1, \ldots, x_{15}]/I_{\mathscr{A}(C_{15},5)}$.

Table 1: Graded Betti numbers of $\mathfrak{N}$.

$\mathfrak{N}$	0	1	2
0	1	0	0
1	0	0	0
2	0	0	0
3	0	0	0
4	0	2	0
5	0	0	0
6	0	0	0
7	0	0	0
8	0	0	1

5. Conclusion

This paper comprises the appositeness among linear algebra, commutative algebra, and graph theory in the form of some homological invariants, Hilbert series, Krull dimension, graded Betti numbers, depth, and so on, and a class of Koszul algebra by taking different path lengths of a cycle. We can explore this class of Koszul algebra more broadly in the future by correlating any suitable type of simple graphs.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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