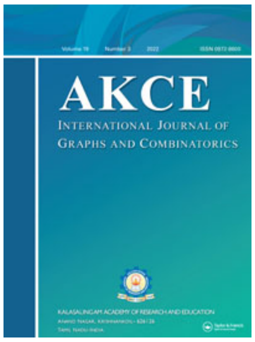

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Edge-maximal θ_{2k+1}-free non-bipartite Hamiltonian graphs of odd order

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ABSTRACT
Let $\mathcal{G}(n;\theta_{2k+1})$ denote the class of non-bipartite graphs on n vertices containing no θ_{2k+1}-graph and $f(n;\theta_{2k+1}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;\theta_{2k+1})\}$. Let $\mathcal{H}(n;\theta_{2k+1})$ denote the class of non-bipartite Hamiltonian graphs on n vertices containing no θ_{2k+1}-graph and $h(n;\theta_{2k+1}) = \max\{\mathcal{E}(G) : G \in \mathcal{H}(n;\theta_{2k+1})\}$. In this paper we determine $h(n;\theta_{2k+1})$ by proving that for sufficiently large odd n, $h(n;\theta_{2k+1}) \le \lfloor\frac{(n-2k+3)^2}{4}\rfloor + 2k - 3$. Furthermore, the bound is best possible. Our results confirm the conjecture made by Bataineh in 2007.

KEYWORDS
Ramsey number; theta graph; complete graph

MSC
05C55; 05C35

1. Introduction

We consider the graph H to be finite simple graph. The vertices and edges of H are denoted by $V(H)$ and $E(H)$, respectively. Moreover, $v(H)$ (the order of H, that is, the number of vertices) and $\mathcal{E}(H)$ (the size of H, that is, the number of edges) denote the cardinalities of these sets, respectively. We denote by the cycle C_n having n vertices $v_1, v_2, ..., v_n$ and the edges $v_1v_2, v_2v_3, ..., v_{n-1}v_n$ and v_nv_1. A cycle C is called even (or odd) if it has even (or odd) length. It is well known that a graph G is bipartite if and only if G contains no odd cycle. Let C be a cycle in a graph G. An edge joining two non adjacent vertices of C is called a chord of C. We say H has θ_k-graph if H has a cycle C of length k with a chord. The degree of a vertex $u \in H$, denoted by $d_H(u)$, is the number of vertices in H adjacent to u. The neighbor set of a vertex u of H in a subgraph A of H, denoted by $N_A(u)$, consists of the vertices of A adjacent to u; we write $d_A(u) = |N_A(u)|$. For vertex disjoint subgraphs H_1 and H_2 of H we let $E(H_1, H_2) = \{xy \in E(H) : x \in V(H_1), y \in V(H_2)\}$ and $\mathcal{E}(H_1, H_2) = |E(H_1, H_2)|$. For a subset Q of the vertex set of H, $H[Q]$ is defind to be the subgraph of H with vertex set Q and edge set consisting of all edges of H that joins vertices of Q, For a proper subgraph A of H we write $H[V(A)]$ and $H - V(A)$ simply as $H[A]$ and $H - A$, respectively.

Let n be a positive integer and $\mathcal{F}$ be a set of graphs. The Turán Type Extremal Problem is the problem of determine the maximum number of edges in an $\mathcal{F}$ -free graph on n vertices (see [19]). Furthermore, find the set consisting of all the so called extremal graphs, that is, the $\mathcal{F}$-free graphs on n vertices with the maximum number of edges is attained. When $\mathcal{F}$ consists of only one graph F we simply write F-free graph on n vertices

In this paper, the Turán-type extremal problem is considered, where the θ-graph is the forbidden subgraph in a Hamiltonian graphs. Also, we consider a non-bipartite Hamiltonian graphs because a bipartite graph does not contain odd θ-graph. The class of non-bipartite F-free graphs on n vertices is denoted by $\mathcal{G}(n;F)$ and $f(n;F) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;F)\}$. Moreover, $\mathcal{H}(n;F)$ denotes to the subclass of $\mathcal{G}(n;F)$ consisting of Hamiltonian graphs in $\mathcal{G}(n;F)$. We write $h(n;F) = \max\{\mathcal{E}(G) : G \in \mathcal{H}(n;F)\}$.

One of the important problems in extremal graph theory is to determine the values of $f(n;F)$ and $h(n;F)$. Also, characterize the extremal graphs of $\mathcal{G}(n;F)$ and $\mathcal{H}(n;F)$ where $f(n;F)$ and $h(n;F)$ are attained. A number of authors [2, 10, 14, 17] studied the edge maximal graphs of $\mathcal{G}(n;C_r)$. Bondy [7] and [8] proved that a Hamiltonian graph G on n vertices without a cycle of length r has at most $\frac{n^2}{4}$ edges with equality holding if and only if n is even and r is odd. Häggkvist et al. [13] proved that $f(n;C_r) \le \lfloor\frac{(n-1)^2}{4}\rfloor + 1$ for all r. This result is sharp only for $r=3$. Jia [17] conjectured that $f(n;C_{2r+1}) \le \lfloor\frac{(n-2)^2}{4}\rfloor + 3$ for $n \ge 4k+2$. Bataineh [2] positively confirmed the above conjecture for $n \ge 36k$. In the same work, he conjectured that for $k \ge 3$, $f(n;\theta_{2r+1}) \le \lfloor\frac{(n-2)^2}{4}\rfloor + 3$. Jaradat et al. [15] and Bataineh et al. [5] and [6] confirmed the conjecture for lage n. Further, Bataineh et al. [4] proved that $f(n;\theta_5) \le \lfloor\frac{(n-1)^2}{4}\rfloor + 1$.

Moon [18] proved that, if G contains no wheels, then $\mathcal{E}(G) \le \lfloor n^2/4\rfloor + \lfloor(n+1)/4\rfloor$. Furthermore, he characterized the extremal graphs. In recent years, several authors reported results on wheels see [1, 4, 11, 12].

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The problem of interest is the determination of $h(n;F)$ and the corresponding extremal graphs. In particular, the edge-maximal graphs of $\mathcal{H}(n;C_r)$ have been studied by a number of authors such as Hendry and Brandt [14] and Jia [17]. Jia, conjectured that for large odd n,

$$h(n;C_{2k+1}) \le \frac{(n-2k+1)^2}{4} + g(k),$$

where $g(k)$ is a polynomial of k. Recently, Bataineh (2007) positively confirmed the above conjecture for $n > (4k+2)(4k^2+10k)$, where $k \ge 3$. Moreover, Bataineh [2] post the following conjecture:

Conjecture 1. For odd $n \ge 4k+4$, $h(n,\theta_{2k+1}) \le \frac{(n-2k+3)^2}{4} + 2k-3$, where $k \ge 3$.

In [3] and [16], Bataineh et al. and Jaradat et al., respectively, confirmed the conjecture for $k=3$. In this paper we establish the above conjecture by proving that for sufficiently large odd n,

$$h(n;\theta_{2k+1}) \le \frac{(n-2k+3)^2}{4} + 2k-3.$$

Furthermore, the bound is best possible.

2. Main results

The following results will be used frequently in the sequel:

Theorem 1. *[15] For positive integers n and k, let G be a graph on $n \ge 6k+3$ vertices which has no θ_{2k+1} as a subgraph, then*

$$\mathcal{E}(G) \le \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Theorem 2. *[2] Let $\vartheta_k = \{\theta_4\} \cup \{\theta_5, \theta_7, ..., \theta_{2k+1}\}$. For $k \ge 5$ and large n, we have*

$$h(n,\vartheta_k) = \frac{(n-2k+3)^2}{4} + 2k-3.$$

Theorem 3. *[9] Let $F_k = \{C_{2i+1} : 1 \le i \le k$. For $k \ge 2$ and $n \ge 3k+1$, then*

$$f(n,F_k) = \frac{(n-2k+1)^2}{4} + 2k-1.$$

Theorem 4. *[2] Let $k \ge 3$ be a positive integer and $H \in \mathcal{H}(n;C_{2k+1})$. Then for $n > (4k+2)(4k^2+10k)$,*

$$h(n;C_{2k+1}) = \begin{cases} \dfrac{(n-2k+1)^2}{4} + 4k-3, & \text{for } n \text{odd} \\ \dfrac{(n-2k)^2}{4} + 4k+1, & \text{for } n \text{even} \end{cases}$$

Now, we are ready to prove our result which confirm the conjecture.

For n odd, let $\mathcal{G}_{n,k}$ be the graph obtained from $\bar{K}_{\frac{n-2k+3}{2}} \vee \bar{K}_{\frac{n-2k+3}{2}}$ by replacing the edge $y_1y_2 \in \bar{K}_{\frac{n-2k+3}{2}} \vee \bar{K}_{\frac{n-2k+3}{2}}$, by the path $y_1, w_2, ..., w_{2k-2}, y_2$ with the vertices $w_2, w_3, ..., w_{2k-2}$ being all new vertices. Observe that $\mathcal{G}_{n,k} \in \mathcal{H}(n;\vartheta_k)$ and it contains $\frac{(n-2k+3)^2}{4} + 2k-3$ edges. Thus, we have established that a graph $H \in \mathcal{H}(n;\vartheta_k)$ and

$$\mathcal{E}(H) \ge \frac{(n-2k+3)^2}{4} + 2k-3.$$

Theorem 5. *For positive integer $k \ge 5$ and for $n \ge 14k^2$, we have*

$$h(n;\theta_{2k+1}) = \left\lfloor \frac{(n-2k+3)^2}{4} \right\rfloor + 2k-3.$$

Proof. By the above inequality, it is sufficient to prove that

$$h(n;\theta_{2k+1}) \le \left\lfloor \frac{(n-2k+3)^2}{4} \right\rfloor + 2k-3.$$

Let $k \ge 5$ and let $H \in \mathcal{H}(n;\theta_{2k+1})$. Let A be the set of vertices in H such that the degree of each vertex less than or equal $14k-7$. Let $|A| = r$. We have,

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(H-A) + \mathcal{E}(H-A,A) + \mathcal{E}(A) \\ &\le \left\lfloor \frac{(n-r)^2}{4} \right\rfloor + r(14k-7).\end{aligned}$$

For $n \ge 14k^2$, the maximum for the right hand side when $r \ge 2k+4$ is at $r = 2k+4$. Thus,

$$\mathcal{E}(H) \le \left\lfloor \frac{(n-2k+3)^2}{4} \right\rfloor + 2k-3.$$

Now, we consider $r \le 2k+3$. If $H - A$ is a bipartite graph with bipartition X and Y, then $|X|, |Y| \ge 12k-9$ ($\delta(H-A) \ge 14k-6$) and $|X|+|Y| \ge n-(2k+4) \ge 40k$. Since H does not contain θ_{2k+1}, then any vertex in A is adjacent to vertices in A and one partition X or Y. Since H is Hamiltonian and $\Delta(A) \le 14k-7$, then there is a vertex u in A such that this vertex is adjacent to part of the vertices of X or Y. Therefor, $(H-A) \cup \{u\}$ is a non-bipartite graph. Moreover, this vertex does not effect the proof of the cases below. Thus, we consider $H - A$ is a non-bipartite graph. To complete the proof, we consider two cases as follows:

Case 1. $W = H - A$ contains θ-graph of order less than $2k+1$. Let $3 \le j < k$ be the maximum positive integer such that θ_{2j+1} is in W. The chord on θ_{2j+1} divided θ_{2j+1} onto two circles. Now, since $j \ge 3$, then we choose $\{x_1, x_2, x_3, x_4, x_5\}$ on one circle. Also, if $u \in N_{W-\theta_{2j+1}}(x_i) \cap N_{W-\theta_{2j+1}}(x_2)$, $i=2, 4$, assume that $i=2$, then $(N_{W-\theta_{2j+1}}(x_3) - \{u\}) \cap (N_{W-\theta_{2j+1}}(x_4) - \{u\}) = (N_{W-\theta_{2j+1}}(x_2) - \{u\}) \cap (N_{W-\theta_{2j+1}}(x_1) - \{u\}) = (N_{W-\theta_{2j+1}}(x_4) - \{u\}) \cap (N_{W-\theta_{2j+1}}(x_5) - \{u\}) = \phi$ as otherwise $\theta_{2(j+1)+1}$ is produced, a contradiction. Moreover, if $(N_{W-\theta_{2j+1}}(x_r)) \cap (N_{W-\theta_{2j+1}}(x_3)) = \phi$, $r=2, 3$ and $u \in N_{W-\theta_{2j+1}}(x_i) \cap N_{W-\theta_{2j+1}}(x_{i+1})$, $i=1, 4$, assume that $i=1$, then $(N_{W-\theta_{2j+1}}(x_4) - \{u\}) \cap (N_{W-\theta_{2j+1}}(x_5) - \{u\}) = \phi$ as otherwise $\theta_{2(j+1)+1}$ is produced, a contradiction. As a result, any vertex in $W - \theta_{2j+1}$ is adjacet to at most three vertice of $\{x_1, x_2, x_3, x_4, x_5\}$ except possibly for one vertex. Note that $\delta(W) \ge 14k-6$. For $s = 1, 2, 3, 4, 5$, let A_s be a set that consist of $4k-2$ neighbors of x_s in $W - \theta_{2j+1}$ selected so that $A_s \cap A_t = \phi$ for $t \ne s$. Figure 1.2 depict the situation. Let $T = W[x_1, x_2, ..., x_j, ..., x_{2j+1}, A_1, A_2, A_3, A_4, A_5]$ and $R = W - T$. Note that by Theorem 1 we have

$$\mathcal{E}(R) \leq \left\lfloor \frac{(n - r - (20k + 2j - 9))^2}{4} \right\rfloor$$

and

$$\mathcal{E}(T) \leq \left\lfloor \frac{(20k + 2j - 9)^2}{4} \right\rfloor.$$

So we want to find $\mathcal{E}(R, T)$. Let $u \in V(R)$. Then we have the following observations:

- If u is adjacent to a vertex in A_1 say a_1, then we have the following:
  1. If u is adjacent to a vertex in A_3 say a_3, then the trail $x_1x_{2j+1}x_{2j} \cdots x_4x_3a_3ua_1x_1x_{2j}$ forms $\theta_{2(j+1)+1}$, a contradiction since j is the maximum.
  2. If u is adjacent to x_2, then the trail $x_1x_{2j+1}x_{2j}x_{2j-1} \cdots x_3x_2ua_1x_1x_2$ forms $\theta_{2(j+1)+1}$, a contradiction since j is the maximum.
  3. If u is adjacent to x_{2j+1}, then the trail forms $\theta_{2(j+1)+1}$, a contradiction since j is the maximum.
  4. If u is adjacent to a vertex in A_2 say a_2 and to a vertex in A_5 say a_5, then any vertex $v \in V(R - \{u\})$ can not be adjacent to a_1 and a_2 or a_2 and a_5 at the same time as otherwise $\theta_{2(j+1)+1}$ is produced, a contradiction.
- If u is adjacent to a vertex in A_2 say a_2, then we can notice the following:
  1. u is not adjacent to any vertex in A_4.
  2. u is not adjacent to any vertex in $\{x_1, x_3\}$.
- If u is adjacent to a vertex in A_3 say a_3, then we can notice the following:
  1. u is not adjacent to any vertex in A_s, $s = 1, 5$.
  2. u is not adjacent to any vertex in $\{x_2, x_4\}$.
- If u is adjacent to a vertex in A_4 say a_4, then we can notice the following:
  1. u is not adjacent to any vertex in A_2.
  2. u is not adjacent to any vertex in $\{x_3, x_5\}$.
- If u is adjacent to a vertex in A_5 say a_5, then we can notice the following:
  1. u is not adjacent to any vertex in A_s, $s = 3$.
  2. u is not adjacent to any vertex in $\{x_4, x_6\}$.

Taking in consideration the above observations, we have

$$\mathcal{E}(R, T) \leq (8k + 2j - 5)(n - r - (20k + 2j - 9)) + 8k - 4.$$

Consequently, we have

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R) + \mathcal{E}(R, T) + \mathcal{E}(T) + r(14k - 7) \\ &\leq \left\lfloor \frac{(n - r - (20k + 2j - 9))^2}{4} \right\rfloor + (8k + 2j - 5)\times \\ &\quad (n - r - (20k + 2j - 9)) + 8k - 4 + r(14k - 7) + \left\lfloor \frac{(20k + 2j - 9)^2}{4} \right\rfloor \\ &\leq \left\lfloor \frac{-8j^2 + j(-64k + 4n + 40) + 160k^2 - 8kn + n^2 - 2n - 34}{4} \right\rfloor.\end{aligned}$$

Define

$$g(j) = \frac{1}{4}(-8j^2 + j(-64k + 4n + 40) + 160k^2 - 8kn + n^2 - 2n - 34),$$

for $3 \leq j < k$. Note that, g is an increasing function with respect to j. Therefore, g has its maximum at $j = k - 1$. Thus

$$\begin{aligned}\mathcal{E}(H) &\leq g(j) \\ &\leq \left\lfloor \frac{n^2 - 4kn - 6n + 88k^2 - 120k - 82}{4} \right\rfloor \\ &\leq \left\lfloor \frac{(n - 2k + 3)^2 - 12n + 84k^2 + 132k - 91}{4} \right\rfloor \\ &< \left\lfloor \frac{(n - 2k + 3)}{4} \right\rfloor + \alpha(n)\end{aligned}$$

where

$$\alpha(n) = \left\lfloor \frac{-12n + 84k^2 + 132k - 91}{4} \right\rfloor.$$

For $k \geq 5$ and $n \geq 9k^2$ odd, $\alpha(n)$ is negative and hence

$$\mathcal{E}(H) < \left\lfloor \frac{(n - 2k + 3)^2}{4} \right\rfloor + 2k - 3$$

as required. Now we need to consider that there exist no $3 \leq j < k$ such that θ_{2j+1} is in $H - A$ as a subgraph. So we have to consider two cases according the existing of θ_5-graph as a subgraph in $H - A$.

Subcase 1. $W = H - A$ has θ_5-graph as a subgraph.

Let $x_1x_2x_3x_4x_5x_1x_4$ be θ_5-graph subgraph in W. Note that $\delta(W) \geq 14k - 6$. For $i = 1, 2, 3$, let A_i be a set that consist of $4k - 4$ neighbors of x_i in W selected so that $A_i \cap A_j = \phi$ for $i \neq j$. Let $T_1 = W[x_1, x_2, x_3, x_4, x_5, A_1, A_2, A_3]$ and $R_1 = W - T_1$, Figure 1.3 depict the situation. Let $u \in V(R_1)$. If u is joined to a vertex in one of the sets A_1, A_2 and A_3, then u cannot be joined to a vertex in the other two sets as otherwise, H would have a θ_7-graph as a subgraph. Also, if u is joined to a vertex in A_i for some $i = 1, 2, 3$, then u cannot be joined to x_{i+1} and x_{i-1}, otherwise, H would have a θ_7-graph as a subgraph. Thus, $\mathcal{E}(\{u\}, T_1) \leq 4k - 1$. Consequently, we have

$$\mathcal{E}(R_1, T_1) \leq (4k - 1)(n - r - (12k - 7)).$$

Also, by Theorem 1 we have

$$\mathcal{E}(R_1) \leq \left\lfloor \frac{(n - r - (12k - 7))^2}{4} \right\rfloor$$

and

$$\mathcal{E}(T_1) \leq \left\lfloor \frac{(12k - 7)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R_1) + \mathcal{E}(R_1, T_1) + \mathcal{E}(T_1) + r(14k - 7) \\ &\leq \left\lfloor \frac{(n - r - (12k - 7))^2}{4} \right\rfloor + (4k - 1)(n - r - (12k - 7)) \\ &\quad + r(14k - 7) + \left\lfloor \frac{(12k - 7)^2}{4} \right\rfloor \\ &\leq \left\lfloor \frac{n^2 - 8nk + 10n + 96k^2 - 176k + 70}{4} \right\rfloor \\ &\leq \left\lfloor \frac{(n - 2k + 3)^2 - 4nk + 4n + 92k^2 - 164k + 61}{4} \right\rfloor \\ &< \left\lfloor \frac{(n - 2k + 3)^2}{4} \right\rfloor + \alpha(n)\end{aligned}$$

where

$$\alpha(n) = \left\lfloor \frac{(n(-4k+4)+92k^2-164k+61}{4} \right\rfloor.$$

For $k \geq 5$ and $n \geq 9k^2$ odd, $\alpha(n)$ is negative. Therefore,

$$\mathcal{E}(H) < \left\lfloor \frac{(n-2k+3)^2}{4} \right\rfloor + 2k - 3$$

as required.

Subcase 2. $W = H - A$ has no θ_5-graph as a subgraph. Consider the case that W contains θ_4-graph. Let $x_1x_2x_3x_4$ be θ_4 with x_2x_4 the chord. Note that $\delta(W) \geq 14k-6$. For $i = 1, 2, 3$, let A_i be a set that consist of $4k-4$ neighbors of x_i in W selected so that $A_i \cap A_j = \phi$ for $i \neq j$. Let $T_2 = W[x_1, x_2, x_3, x_4, A_1, A_2, A_3]$ and $R_2 = W - T_2$, Figure 1.4 depict the situation. Let $u \in V(R_2)$. If u is joined to a vertex in one of the sets A_1, A_2 and A_3, then u cannot be joined to a vertex in the other two sets as otherwise, H would have a θ_7-graph as a subgraph. Also, if u is joined to a vertex in A_i for some $i = 1, 2, 3$, then u cannot be joined to x_{i+1} and x_{i-1}, otherwise, W would have a θ_5-graph as a subgraph. Thus, $\mathcal{E}(\{u\}, T_2) \leq 4k-2$. Consequently, we have

$$\mathcal{E}(R_2, T_2) \leq (4k-2)(n-r-(12k-8)).$$

Also, by Theorem 1 we have

$$\mathcal{E}(R_2) \leq \left\lfloor \frac{(n-r-(12k-8))^2}{4} \right\rfloor$$

and

$$\mathcal{E}(T_2) \leq \left\lfloor \frac{(12k-8)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(R_2) + \mathcal{E}(R_2, T_2) + \mathcal{E}(T_2) + r(14k-7) \\
&\leq \left\lfloor \frac{(n-r-(12k-8))^2}{4} \right\rfloor \\
&\quad + (4k-2)(n-r-(12k-8)) + r(14k-7) \\
&\quad + \left\lfloor \frac{(12k-8)^2}{4} \right\rfloor \\
&\leq \left\lfloor \frac{n^2-8nk+8n+96k^2-160k+64}{4} \right\rfloor \\
&\leq \left\lfloor \frac{(n-2k+3)^2-4nk+2n+92k^2-148k+55}{4} \right\rfloor \\
&< \left\lfloor \frac{(n-2k+3)^2}{4} \right\rfloor + \alpha(n)
\end{aligned}$$

where

$$\alpha(n) = \left\lfloor \frac{(n(-4k+2)+92k^2-148k+55}{4} \right\rfloor.$$

For $k \geq 5$ and $n \geq 9k^2$ odd, $\alpha(n)$ is negative. Therefore,

$$\mathcal{E}(H) < \left\lfloor \frac{(n-2k+3)^2}{4} \right\rfloor + 2k - 3$$

as required.

Case 2. $H - A$ contains no θ_4 and θ_{2j+1}, $2 \leq j \leq k-1$.

Subcase 3. $H - A$ does not contain C_{2j+1} for some $1 \leq j \leq k$. Apply the same arguments in Case 1 on $C_{2(j-1)+1}$, we get the required result.

Subcase 4. $W = H - A$ does not contain C_{2k+1}. We consider W does not contain C_{2j+1} for all $1 \leq j \leq k-1$ as otherwise we get Case 3. Then by Theorem 4

$$\mathcal{E}(W) \leq \left\lfloor \frac{(n-r-2k+1)^2}{4} \right\rfloor + 2k - 1.$$

Therefore,

$$\begin{aligned}
\mathcal{E}(H) &\leq \left\lfloor \frac{(n-r-2k+1)^2}{4} \right\rfloor + 2k - 1 + r(14k-7) \\
&\leq \left\lfloor \frac{4k^2-4kn+60kr+4k+n^2-2nr+2n+r^2-30r-3}{4} \right\rfloor \\
&\leq \left\lfloor \frac{(n-2k+3)^2+16k-4n-12}{4} \right\rfloor \\
&< \left\lfloor \frac{(n-2k+3)^2}{4} \right\rfloor + \alpha(n)
\end{aligned}$$

where

$$\alpha(n) = \left\lfloor \frac{16k-4n-12}{4} \right\rfloor.$$

For $k \geq 5$ and odd $n \geq 9k^2$, $\alpha(n)$ is negative. Therefore,

$$\mathcal{E}(H) < \left\lfloor \frac{(n-2k+3)^2}{4} \right\rfloor + 2k - 3$$

as required.

Subcase 5. $W = H - A$ contains $C_{2k+1} = x_1x_2 \cdots x_{2k+1}x)1$. Let $R = W - C_{2k+1}$. We have the following observations.

1. If $u \in R$ is adjacent to x_s, x_t and x_w, then the number of vertices between any two vertices is even as otherwise θ_4 or θ_{2j+1}, $2 \leq j \leq k$ is produced.
2. If $u \in R$, then u is adjacent to at most three vertices of the vertices of C_{2k+1}.
3. If $u \in R$ is adjacent to three vertices of the vertices of C_{2k+1}, then v is adjacent to at most two vertices of the vertices of C_{2k+1} for any $v \in R - \{u\}$ as otherwise some θ-graph is produced.

As a result from the above observations, we have

$$\begin{aligned}
\mathcal{E}(H) &\leq \left\lfloor \frac{(n-r-2k-1-2)^2}{4} \right\rfloor + 2(n-r-2k-1) + 1 + r(14k-7) \\
&\leq \left\lfloor \frac{4k^2-4kn+60kr-12k+n^2-2nr+6n+r^2-34r-7}{4} \right\rfloor \\
&\leq \left\lfloor \frac{(n-2k+3)^2+16k-4n}{4} \right\rfloor \\
&< \left\lfloor \frac{(n-2k+3)^2}{4} \right\rfloor + \alpha(n)
\end{aligned}$$

where

$$\alpha(n) = \left\lfloor \frac{16k - 4n}{4} \right\rfloor.$$

For $k \geq 5$ and odd $n \geq 9k^2$, $\alpha(n)$ is negative. Therefore,

$$\mathcal{E}(H) < \left\lfloor \frac{(n - 2k + 3)^2}{4} \right\rfloor + 2k - 3$$

as required. □

For the completeness we repost the following unsettled conjecture made by Bataineh in his Ph.D Theses [2] for the even n:

Conjecture 2. For even $n \geq 4k + 4$, $h(n, \theta_{2k+1}) \leq \frac{(n-2k+2)^2}{4} + 2k$, where $k \geq 3$.

It worths mentioning that in 2019, Bataineh et al. [4] confirmed the above conjecture for $k = 3$ with some other constraints on graphs.

3. Conclusion

In this paper, we considered the Turán-type extremal problem, where the θ-graph is the forbidden subgraph in a Hamiltonian graphs. Also, we consider a non-bipartite Hamiltonian graphs because a bipartite graph does not contain odd θ-graph. In fact, confirmed a conjecture made by Bataineh in [2] by proving that for large odd integer n and $k \geq 5$, the maximum number of edges of a graph with n vertices that contains no $(2k + 1)$-theta subgraph is equal to $\lfloor \frac{(n-2k+3)^2}{4} \rfloor + 2k - 3$. Further, we restated the conjecture for even n.

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Authors contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conflicts of interest

The authors declare that they have no conflict of interest.

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Availability of data and material

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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