The theta-complete graph Ramsey number $r(\theta_k, K_5); k = 7, 8, 9$

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Abstract. Finding the Ramsey number is an important problem of the well-known family of the combinatorial problems in Ramsey theory. In this work, we investigate the Ramsey number $r(\theta_s, K_5)$ for $s = 7, 8, 9$ where θ_s is the set of theta graphs of order s and K_5 is a complete graph of order 5. Our result closed the problem of finding $R(\theta_s, K_5)$ for each $s \geq 6$.

Keywords: Ramsey number, theta graph, complete graph.

1. Introduction

All graphs we consider are undirected, finite and simple. Let G be a graph with the *vertex set* $V(G)$ and the *edge set* $E(G)$. A subset $S \subset V(G)$ is an *independent set* if no two vertices of S are adjacent. The size of the largest

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independent set of the graph G is called *independent number* which is denoted by $\alpha(G)$. For a given vertex u of a graph G, the number of incident edges with u is called the *degree* of u, denoted by $d(u)$. $N(u)$ stands for the set of all vertices adjacent to u, and $N[u] = N(u) \cup \{u\}$. For a set of vertices $\{u_1, u_2, \ldots, u_t\}$, $N[u_1, u_2, \ldots, u_t] = N[u_1] \cup N[u_2] \cup \ldots \cup N[u_t]$ and the *induced subgraph* $G\langle u_1, u_2, \ldots, u_t\rangle_G$ of G consists of the vertex set $\{u_1, u_2, \ldots, u_t\}$ and all edges of G connecting two vertices in $\{u_1, u_2, \ldots, u_t\}$. $\Delta(G)$ and $\delta(G)$ stand for the *maximum* and *minimum* degree in the graph G, respectively. We denote C_s, P_s and K_s to be cycle, path and complete graph of order s, respectively. The *theta graph* of order s is a cycle C_s and an edge joining two non-adjacent vertices in C_s.

For a given set of graphs $\mathcal{S}$ and a graph H, the *Ramsey number* $R(\mathcal{S}, H)$ is defined to be the smallest positive integer N such that for every graph G with at least N vertices, either G contains a graph in $\mathcal{S}$ as a subgraph or the complement of G (i.,e. $\overline{G}$) contains H as a subgraph. If $\mathcal{S}$ contains only one graph, say H_1, we simply write $R(H_1, H)$.

Ramsey numbers for graphs of small orders have been studied for long time (see, [5, 8, 9, 10, 12, 20, 22]) and it was noted that computing the exact Ramsey numbers of such graphs become more challenging and their difficulties increase with the increasing of the number of edges of the given graphs, but it is more interesting for spare graphs.

One of the Ramsey numbers problems is the Erdös et al. [10] conjecture which states that $r(C_s, K_t) = (s-1)(t-1)+1$, for all $s \geq t \geq 3$ except $r(C_3, K_3) = 6$. The conjecture attracted the attention of many authors and confirmed only on few cases (see, [1, 3, 4, 7, 13, 14, 15, 16, 21, 24, 25, 26, 27, 28]).

In this paper, we end the remaining open values of the Ramsey numbers of theta graphs verses complete graphs of order 5 that initiated in [14] see Theorem 1.2. In fact, we prove that $r(\theta_7, K_5) = 25, r(\theta_8, K_5) = 29$, and $r(\theta_9, K_5) = 33$.

Let us present some results concerning the Ramsey number of the theta graphs verses complete graph and complete deleting edge. Chvátal and Harary [8], determined that $r(\theta_4, K_4) = 11$. Bolze and Harborth [5] and Faudree et al. [11] proved that $r(\theta_4, K_5) = 16$ and $r(\theta_4, K_5 - e) = 13$, respectively. McNamara [19] showed that $r(\theta_4, K_6) = 21$ and McNamara and Radziszowski [20] obtained the following two results: $r(\theta_4, K_6 - e) = 17$ and $r(\theta_4, K_7 - e) = 28$. An upper bound for $r(\theta_4, K_7)$ and the exact number for $r(\theta_4, K_8)$ were investigated by Boza [6], in fact, he obtained that $r(\theta_4, K_7) \leq 31$ and $r(\theta_4, K_8) = 42$. Bataineh et al. [2], established the following result:

Theorem 1.1. *For $t = 2, 3, 4$ and $s > t$, $r(\theta_s, K_t) = (s-1)(t-1)+1$.*

Finally, Jaradat et. al, [14] proved the following result:

Theorem 1.2. *The Ramsey number $r(\theta_s, K_5) = 4s - 3$ for $s = 6$ and $s \geq 10$.*

For more results on Ramsey numbers, we direct the reader to the well known updated survey by Radziszowski [22].

2. Main results

The purpose of this section is to investigate the Ramsey number of theta graphs of order 7, 8, and 9 versus complete graphs of order 5 which are the remaining open value in Theorem 1.2. One can see that $G = (m-1)K_{n-1}$, contains neither θ_n nor m-element independent set. Hence, we obtained that $r(\theta_s, K_t) \geq (s-1)(t-1)+1$. Now, we present some results that will be used in the sequel. Clancy [9] proved that

$$r(C_4, K_5) = 14. \tag{1}$$

Hendrey [12], Jayawardene and Rousseau [17] determined that

$$r(C_5, K_5) = 17. \tag{2}$$

Further, Jayawardene and Rousseau [18] showed that

$$r(C_6, K_5) = 21. \tag{3}$$

Finally, for any graph H, it is clear that

$$r(H, K_0) = 1. \tag{4}$$

To achieve our goal, we first prove the following sequence of six Lemmas:

Lemma 2.1. *Let G be a graph of order more than or equal to $4(s-1)+1$, $s = 7, 8, 9$, that contains neither θ_s nor 5-element independent set. Then $\delta(G) \geq s-1$.*

Proof. Assume that G contains a vertex of degree less than $s-1$, say u. Then $| V(G - N[u]) | \geq 3(s-1)+1$. Since $r(\theta_s, K_4) = 3(s-1)+1$, $G - N[u]$ contains an independent set consisting of 4 vertices. This set with the vertex u is a 5-element independent set, a contradiction. □

Lemma 2.2. *Let G be a graph of order more than or equal to $4(s-1)+1$, $s = 7, 8, 9$, and contains neither θ_s nor 5-element independent set. If $\{u_1, u_2, \ldots, u_t\}$, $t \leq 4$, is an independent set of vertices, then $| N(u_1) \cup N(u_2) \cup \ldots \cup N(u_t) | \geq t(s-2)+1$.*

Proof. Suppose that $| N(u_1) \cup N(u_2) \cup \ldots \cup N(u_t) | < t(s-2)+1$. Then $| V(G - N[u_1, u_2, \ldots, u_t]) | \geq (4-t)(s-1)+1$. Since, by (4) and Theorems 1.1, 1.2, $r(\theta_s, K_{5-t}) = (4-t)(s-1)+1$, $G - N[u_1, u_2, \ldots, u_t]$ contains an independent set consisting of $5-t$ vertices. This set with the vertices u_1, u_2, $\ldots$, u_t is a 5-element independent set, a contradiction. □

Lemma 2.3. *Let G be a graph of order $4(s-1)+1$, $s = 7, 8, 9$. If G contains neither θ_s nor 5-element independent set, then G contains no K_{s-1}.*

Proof. Suppose that G contains K_{s-1}. Let $W = \{u_1, u_2, \ldots, u_{s-1}\}$ be the vertices of K_{s-1}, $R = G - W$ and $W_i = N(u_i) \cap V(R)$, $i = 1, 2, \ldots, s-1$. Since $\delta(G) \geq s-1$, $\mid W_i \mid \geq 1$. Also, since G does not contain θ_s, $W_i \cap W_j = \emptyset$, $1 \leq i < j \leq s-1$, and $xy \notin E(G)$ for any $x \in W_i$ and $y \in W_j$, $1 \leq i < j \leq s-1$. Note that, $\{v_1, v_2, \ldots, v_{s-1}\}$ is an independent set where $v_i \in W_i$, $i = 1, 2, \ldots, s-1$. Since $s - 1 \geq 6$, we conclude that G contains an independent set with at least 6 vertices, a contradiction. □

Lemma 2.4. *Let G be a graph of order $4(s-1)+1$, $s = 7, 8, 9$. If G contains neither θ_s nor 5- element independent set, then G contains no $K_1 + P_{s-2}$.*

Proof. Suppose that G contains $K_1 + P_{s-2}$. Let $W = \{u_1, u_2, \ldots, u_{s-2}\}$ and u be the vertices of P_{s-2} and K_1, respectively. Let $R = G - (W \cup \{u\})$ and $W_i = N(u_i) \cap V(R)$, $i = 1, 2, \ldots, s-2$. Since $\delta(G) \geq s-1$, $\mid W_i \mid \geq 1$, say $v_i \in W_i$, $i = 1, 2, \ldots, s-2$. Note that no vertex of $\{v_1, v_2, v_3, v_{s-2}\}$ is adjacent to two vertices of $\{u_1, u_2, u_3, u_{s-2}\}$ as otherwise θ_s is produced. Also, there is no edge between any two vertices of $\{v_1, v_2, v_3, v_{s-2}\}$ as otherwise θ_s is produced. Note that $uv_{s-2} \notin E(G)$ and $u_i v_{s-2} \notin E(G)$, $i = 1, 2, \ldots, u_{s-3}$ as otherwise θ_s is produced. Since G contains no θ_s, $v_i w \notin E(G)$ for any $w \in N(v_{s-2}) \cap V(R)$, $i = 1, 2, 3$. Therefore, $\mid N(v_{s-2}) \cap V(R) \mid \geq s-2$. Since by Lemma 2.3 G contains no K_{s-1}, as a result $N[v_{s-2}]$ contains two independent vertices, say $\{w_1, w_2\}$. Hence, $\{v_1, v_2, v_3, w_1, w_2\}$ is a $5-$element independent set, a contradiction. □

Lemma 2.5. *Let G be a graph of order $4(s-1)+1$, $s = 7, 8, 9$. If G contains neither θ_s nor 5-element independent set, then G contains no $K_1 + C_{s-3}$.*

Proof. Suppose that G contains $K_1 + C_{s-3}$. Let $W = \{u_1, u_2, \ldots, u_{s-3}\}$ and u be the vertices of C_{s-2} and K_1, respectively. Let $R = G - (W \cup \{u\})$ and $W_i = N(u_i) \cap V(R)$, $i = 1, 2, \ldots, s-3$. Since $\delta(G) \geq s-1$, $\mid W_i \mid \geq 2$, $i = 1, 2, \ldots, s-3$. By Lemma 2.4, $uv \notin E(G)$ for any $v \in W_i$, $i = 1, 2, \ldots, s-3$. Since G contains no θ_s, as a result for any $x \in W_i$ and $y \in W_j$, $1 \leq i < j \leq s-3$, we have $xy \notin E(G)$. Now we consider two cases according to $W_i \cap W_j$:

Case 2.0.1. $W_i \cap W_j = \emptyset$, $1 \leq i < j \leq 4$.

In this case, $\{u, v_1, v_2, v_3, v_4\}$ is a 5-element independent set where $v_i \in W_i$, $i = 1, 2, 3, 4$. This is a contradiction.

Case 2.0.2. $W_i \cap W_j \neq \emptyset$, for some $1 \leq i < j \leq 4$.

Let $v \in W_a \cap W_b$ where $1 \leq a < b \leq 4$. since $W_i \geq 2$, then $\mid W_a \cup W_b \mid \geq 2$. Let $w \in W_a \cup W_b$. Moreover, since G contains no θ_s, $(W_i - \{v\}) \cap (W_j - \{v\}) = \emptyset$, $1 \leq i < j \leq 4$ and $\{i, j\} \neq \{a, b\}$. Therefore, $\{v, w, f, g, u\}$ is a 5-element independent set where $f \in W_c$ and $g \in W_d$, $1 \leq c < d \leq 4$ and $\{c, d\} \neq \{a, b\}$, a contradiction.

□

Lemma 2.6. *Let G be a graph of order equal $4(s-1)+1$, $s=7,8,9$. If G contains neither θ_s nor 5-element independent set, then $\langle N(N(u))-\{u\}\rangle_G$ does not contain C_{s-3}.*

Proof. Suppose that G contains C_{s-3}. By Lemma 2.5, the vertices of C_{s-3} are not adjacent to the same vertex in $N(u)$. Therefore, there are two adjacent vertices in the vertex set of C_{s-3} that adjacent to two different vertices in the vertex set of $N(u)$. The vertex set $V(C_{s-3})$, two vertices of the vertex set of $N(u)$ and u generate θ_s, a contradiction. □

Theorem 2.1. $r(\theta_s, K_5) = 4(s-1)+1$, $s=7,8,9$.

Proof. By the inequality, $r(\theta_n, K_m) \geq (n-1)(m-1)+1$, it suffices to prove that $r(\theta_s, K_5) \leq 4(s-1)+1$. Let G be a graph on $4(s-1)+1$ vertices. Suppose that G contains no θ_s as a subgraph and $\alpha(G) \leq 4$, then $\delta(G) \geq s-1$. Let u be a vertex of $v(G)$, $H = \langle N(u)\rangle_G$ and S be the remaining vertices. We consider a number of cases according to $\alpha(H)$.

Case 2.1.1. $\alpha(H) = 1$. So, obviously G contains θ_s as a subgraph, a contradiction.

Case 2.1.2. $\alpha(H) = 2$. Then H is a union of two complete disjoint components, say $H = K_r \cup K_t$, with $\mid V(K_r \cup K_t) \mid \geq s-1$. Since G does not contain θ_s, $N(V(K_r)) \cap N(V(K_t)) \cap S = \emptyset$ and $xy \notin E(G)$ for any $x \in V(K_r)$ and $y \in V(K_t)$. Since G does not contain θ_s, as a result $\langle N(V(K_r)) \cap S\rangle_G$ and $\langle N(V(K_t)) \cap S\rangle_G$ are not complete. Therefore, $\{u_1, u_2, v_1, v_2, u\}$ is a 5-element independent set, where $\{u_1, u_2\} \in V(K_r)$ and $\{v_1, v_2\} \in V(K_t)$, a contradiction.

Case 2.1.3. $3 \leq \alpha(H) \leq 4$, say $\{u_1, \ldots, u_t\}$, $3 \leq t \leq 4$. Then by Lemma 2.2, $\mid V(N(\{u_1, \ldots, u_t\}) - \{u\}) \mid \geq 3(s-2)$. By Lemma 2.6, $\langle N(\{u_1, \ldots, u_t\}) - \{u\}\rangle_G$ does not contain C_{s-3}. Therefore, since by (1), (2) and (3), $r(C_{s-3}, K_5) \leq 3(s-2)$, then $\langle N(\{u_1, \ldots, u_t\}) - \{u\}\rangle_G$ contains a 5-element independent set, a contradiction.

□

The following result follows from Theorems 1.2 and 2.1.

Theorem 2.2. *The Ramsey number $r(\theta_s, K_5) = 4(s-1)+1$, for each $s \geq 6$.*

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