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BOUNDEDNESS OF ROUGH INTEGRAL OPERATORS ON TRIEBEL-LIZORKIN SPACES

H. M. Al-Qassem, L. C. Cheng, and Y. Pan

Abstract: We prove the boundedness of several classes of rough integral operators on Triebel-Lizorkin spaces. Our results represent improvements as well as natural extensions of many previously known results.

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1. Introduction

Throughout this paper, we let p' denote the dual exponent to p defined by $1/p+1/p'=1$. Let $n\geq 2$ and $\mathbf{S}^{n-1}$ represent the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma=d\sigma(\cdot)$. Let K be a kernel of Calderón-Zygmund type on $\mathbf{R}^n$ given by

$$K(x)=\frac{\Omega(x)}{|x|^n},$$

where Ω is a homogeneous function of degree 0, integrable over $\mathbf{S}^{n-1}$, and satisfies

$$\int_{\mathbf{S}^{n-1}}\Omega(u)\,d\sigma(u)=0. \tag{1.1}$$

It is well-known that the Triebel-Lizorkin space $\dot{F}_p^{\beta,q}(\mathbf{R}^n)$ is a unified setting of many well-known function spaces including Lebesgue spaces $L^p(\mathbf{R}^n)$, the Hardy spaces $H^p(\mathbf{R}^n)$ and the Sobolev spaces $L_p^\beta(\mathbf{R}^n)$.

Our main focus as the title of the paper suggests will be on studying the boundedness of three types of rough integral operators on Triebel-Lizorkin spaces. We start with the first type which concerns the homogeneous Calderón-Zygmund singular integral operator T_Ω given by

$$T_\Omega f(x)=\text{p.v.}\int_{\mathbf{R}^n} f(x-y)\,K(y)\,dy,$$

where $f\in\mathcal{S}(\mathbf{R}^n)$, the space of Schwartz functions.

The investigation of the L^p boundedness of T_Ω was pioneered by Calderón and Zygmund in [**3**] and then continued by many authors. In [**3**], Calderón and Zygmund showed that the L^p $(1 < p < \infty)$ boundedness of T_Ω holds if $\Omega \in L\log L\left(\mathbf{S}^{n-1}\right)$ and that this condition is essentially the weakest possible size condition on Ω for the L^p $(1 < p < \infty)$ boundedness of T_Ω to hold. For endpoint results, A. Seeger in [**20**] proved that T_Ω is of weak-type $(1,1)$ under the same $L\log L\left(\mathbf{S}^{n-1}\right)$ condition, but in general T_Ω with such an Ω is not bounded on $H^1\left(\mathbf{R}^n\right)$, as pointed out by M. Christ (see [**21**]). On the other hand, Connett [**10**] and Coifman and Weiss [**9**] independently showed that T_Ω is bounded on L^p $(1 < p < \infty)$ for $\Omega \in H^1\left(\mathbf{S}^{n-1}\right)$. Here, $H^1\left(\mathbf{S}^{n-1}\right)$ is the Hardy space on the unit sphere which contains the space $L\log L\left(\mathbf{S}^{n-1}\right)$ as a proper space. In [**13**], Grafakos and Stefanov introduced the following condition:

$$\sup_{\xi\in\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \left(\log|\xi\cdot y|^{-1}\right)^{1+\alpha} d\sigma(y) < \infty \tag{1.2}$$

and showed that it implies the L^p boundedness of T_Ω for p in a range dependent on the positive exponent α. For any $\alpha > 0$, we let $\mathbf{G}_\alpha(\mathbf{S}^{n-1})$ denote the family of Ω's which are integrable over $\mathbf{S}^{n-1}$ and satisfy (1.2).

Theorem A ([**13**]). *Let* $\Omega\in\mathbf{G}_\alpha(\mathbf{S}^{n-1})$ *for some* $\alpha > 0$ *and satisfy* (1.1). *Then if* $\alpha > 0$, T_Ω *is bounded on* $L^p(\mathbf{R}^n)$ *for* $p \in (\frac{2+\alpha}{1+\alpha}, 2+\alpha)$.

This range of p was later improved to be $(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ (see [**11**]). However, it is still unknown whether the latter range of p is sharp. We point out that Grafakos and Stefanov in [**13**] showed that

$$\bigcup_{q>1} L^q(\mathbf{S}^{n-1}) \subsetneqq \mathbf{G}_\alpha(\mathbf{S}^{n-1}) \text{ for any } \alpha > 0, \tag{1.3}$$

$$\bigcap_{\alpha>0} \mathbf{G}_\alpha(\mathbf{S}^{n-1}) \nsubseteq H^1\left(\mathbf{S}^{n-1}\right) \nsubseteq \bigcup_{\alpha>0} \mathbf{G}_\alpha(\mathbf{S}^{n-1}). \tag{1.4}$$

In recent years, the investigation of boundedness of T_Ω on Triebel-Lizorkin space $\dot{F}_p^{\beta,q}(\mathbf{R}^n)$ has attracted the attention of many authors. For relevant results one may consult [**5**], [**16**], [**6**], [**7**], among others. For example, J. Chen and C. Zhang in [**7**] (see also [**24**]) proved the following:

Theorem B. *Suppose that* Ω *satisfies* (1.1) *and* $\Omega \in \mathbf{G}_\alpha(\mathbf{S}^{n-1})$ *for all* $\alpha > 1$. *Then the operator* T_Ω *is bounded on* $\dot{F}_p^{\beta,q}(\mathbf{R}^n)$ *for all* $1 < p, q < \infty$ *and* $\beta \in R$.

We notice that the condition imposed on Ω is that $\Omega \in \mathbf{G}_\alpha(\mathbf{S}^{n-1})$ for all $\alpha > 1$. The question that arises naturally is whether the operator T_Ω is bounded on $\dot{F}_p^{\beta,q}(\mathbf{R}^n)$ if $\Omega \in \mathbf{G}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$. We shall show that the condition $\Omega \in \mathbf{G}_\alpha(\mathbf{S}^{n-1})$ for all $\alpha > 1$ is not necessary and we just need $\Omega \in \mathbf{G}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$. The condition that $\alpha > 1$ is not necessary as described in the following theorem.

Theorem 1.1. *Let $\Omega \in \mathbf{G}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$ and satisfy* (1.1). *Then if $\alpha > 0$, T_Ω is bounded on $\dot{F}_p^{\beta,q}(\mathbf{R}^n)$ for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, $q \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $\beta \in \mathbf{R}$.*

The question concerning the boundedness of T_Ω when $\Omega \in H^1\left(\mathbf{S}^{n-1}\right)$, which is separate from the problem addressed in Theorems A and B in light of (1.4), had been answered by Y. Chen and Y. Ding in [**8**].

The second type of our operators concerns a certain class of oscillatory singular integral operators. To state our second result, we need some preparation. Let $\mathcal{P}(n;m)$ denote the set of polynomials on $\mathbf{R}^n$ which have real coefficients and degrees not exceeding m, and let $\mathcal{H}(n;m)$ denote the collection of polynomials in $\mathcal{P}(n;m)$ which are homogeneous of degree m. Also, let $\mathcal{P}(n;m,0)$ be the class of all $P \in \mathcal{P}(n;m)$ with $\nabla P(0) = 0$. For $P(x) = \sum_{|\eta|\le m} a_\eta x^\eta$, we set $\|P\| = \sum_{|\eta|\le m} |a_\eta|$. Let $n \ge 2$, $m \in \mathbf{N}$ and $\alpha > 0$. An integrable function Ω on $\mathbf{S}^{n-1}$ is said to be in the space $A(n;m;\alpha)$ if

$$\sup_{P\in\mathcal{H}(n;m),\, \|P\|=1} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \left(\log \frac{1}{|P(y)|}\right)^{1+\alpha} d\sigma(y) < \infty. \tag{1.5}$$

It was noted in [**2**] that $A(n;1;\alpha) = \mathbf{G}_\alpha(\mathbf{S}^{n-1})$ and in the case $n = 2$,

$$\bigcap_{m=1}^{\infty} A(2;m;\alpha) = \mathbf{G}_\alpha(\mathbf{S}^1).$$

For $P \in \mathcal{P}(n;d)$, let $T_{\Omega,P}$ be the oscillatory singular integral operator defined by

$$T_{\Omega,P} f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{iP(y)} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy. \tag{1.6}$$

We have the following:

Theorem 1.2. *Let $n \geq 2$, $d \in \mathbf{N}$. Let Ω satisfy* (1.1) *and $\Omega \in \bigcap_{m=1}^{d} A(n;m;\alpha)$ for some $\alpha > 0$. Then*

$$\sup_{P \in \mathcal{P}(n;d,0)} \|T_{\Omega,P}(f)\|_{\dot{F}_p^{\beta,q}(\mathbf{R}^n)} \leq C(\log d + 1) \|f\|_{\dot{F}_p^{\beta,q}(\mathbf{R}^n)}, \tag{1.7}$$

for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, $q \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $\beta \in \mathbf{R}$, where C is a positive constant that depends on β, p, q, but not on d.

While technically Theorem 1.1 can be subsumed in Theorem 1.2, we will first prove Theorem 1.1 and then use it in the proof of Theorem 1.2.

Corollary 1.3. *Let $n = 2$, $d \in \mathbf{N}$. Let Ω satisfy* (1.1) *and $\Omega \in \mathbf{G}_\alpha(\mathbf{S}^1)$ for some $\alpha > 0$. Then*

$$\sup_{P \in \mathcal{P}(2;d,0)} \|T_{\Omega,P}(f)\|_{\dot{F}_p^{\beta,q}(\mathbf{R}^n)} \leq C(\log d + 1) \|f\|_{\dot{F}_p^{\beta,q}(\mathbf{R}^n)}, \tag{1.8}$$

for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, $q \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $\beta \in \mathbf{R}$, where C is a positive constant that depends on β, p, q, but not on d.

We remark that the $L^p(\mathbf{R}^n)$ $(\frac{2+2\alpha}{1+2\alpha} < p < 2+2\alpha)$ boundedness of $T_{\Omega,P}$ can be obtained by using Theorem 2 in [**2**] and employing an argument in [**12**].

The third type of our operators concerns Marcinkiewicz integral operators $\mathcal{M}_{\Omega,q}$ defined by

$$\mathcal{M}_{\Omega,q}f(x) = \left(\int_0^\infty |F_\Omega f(t,x)|^q \, \frac{dt}{t}\right)^{1/q},$$

where

$$F_\Omega f(t,x) = \frac{1}{t} \int_{|u| \leq t} f(x-u) \frac{\Omega(u')}{|u|^{n-1}} \, du.$$

We notice that $\mathcal{M}_{\Omega,2}(f)$ is the classical Marcinkiewicz integral defined by Stein in [**22**]. Our result concerns $\mathcal{M}_{\Omega,q}$ is the following:

Theorem 1.4. *Let $\Omega \in \mathbf{G}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$ and satisfy* (1.1). *Then*

$$\|\mathcal{M}_{\Omega,q}(f)\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}, \tag{1.9}$$

for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $q \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$.

We point out that Theorem 1.4 represents a generalization of Theorem 1 in [**4**]. Earlier results concerning the operator $\mathcal{M}_{\Omega,q}$ can be found in [**5**] and [**16**], among others.

Throughout this paper, the letter C will stand for a positive constant that may vary at each occurrence. However, C does not depend on any of the essential variables.

2. Some definitions and lemmas

Now we recall the definition of the Triebel-Lizorkin spaces.

Fix a radial Schwartz function $\Psi \in \mathcal{S}(\mathbf{R}^n)$ such that $\operatorname{supp}(\hat{\Psi}) \subset \left\{\xi \in \mathbf{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}$, $0 \leq \hat{\Psi}(\xi) \leq 1$, $\hat{\Psi}(\xi) \geq c > 0$, if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$. Denote $\hat{\Psi}_t(\xi) = \hat{\Psi}(t\xi)$, $t \in \mathbf{R}$ so that $\Psi_t(x) = t^{-n}\Psi(x/t)$. For $1 < p, q < \infty$ and $\beta \in \mathbf{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\beta,q}(\mathbf{R}^n)$ is the space of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ satisfying

$$\|f\|_{\dot{F}_p^{\beta,q}(\mathbf{R}^n)} \cong \left\| \left(\int_0^\infty \left| t^{-\beta} \Psi_t * f \right|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)} < \infty.$$

It is well-known that $\mathcal{S}(\mathbf{R}^n)$ is dense in $\dot{F}_p^{\beta,q}(\mathbf{R}^n)$ and also the following hold:

(1) $L^p(\mathbf{R}^n) = \dot{F}_p^{0,2}(\mathbf{R}^n)$;

(2) $\left(\dot{F}_p^{\beta,q}(\mathbf{R}^n)\right)^* = \dot{F}_{p'}^{-\beta,q'}(\mathbf{R}^n)$;

(3) $\dot{F}_p^{\beta,q_1}(\mathbf{R}^n) \subset \dot{F}_p^{\beta,q_2}(\mathbf{R}^n)$ if $q_1 \leq q_2$.

We need the following result from [**1**].

Lemma 2.1. *Let $h(t) = b_0 + b_1 t + \cdots + b_d t^d$ be a real polynomial of degree at most d, and let $\psi \in \mathbf{C}^1[a,b]$. Then for any j_0 with $1 \leq j_0 \leq d$, there exists a positive constant C independent of a, b, the coefficients of $b_0, \ldots, b_d$ and also independent of d such that*

$$\left| \int_a^b e^{ih(t)} \psi(t)\, dt \right| \leq C \left| b_{j_0} \right|^{-\frac{1}{d}} \left\{ \sup_{a \leq t \leq b} |\psi(t)| + \int_a^b |\psi'(t)|\, dt \right\}$$

holds for $0 < a < b \leq 1$.

3. Proof of main results

Before we start proving our main results we need some preparation. By the translation invariance of T_Ω and $T_{\Omega,P}$, it suffices to establish their boundedness on the Triebel-Lizorkin spaces with $\beta = 0$. Choose a real valued, radial function $\phi \in \mathcal{S}(\mathbf{R}^n)$ such that $\operatorname{supp}\hat{\phi} \subset \left\{\xi \in \mathbf{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}$, $\hat{\phi}(\xi) \geq 0$, $\hat{\phi}(\xi) \geq c > 0$, if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$ and for all $\xi \neq 0$, $\int_{\mathbf{R}} \left| \hat{\phi}_{2^t}(\xi) \right|^2 dt = 1$, where $\hat{\phi}_{2^t}(\xi) = \hat{\phi}(2^t\xi)$, $t \in \mathbf{R}$. Note that $\phi_{2^t}(x) = 2^{-tn}\phi(x/2^t)$, $x \in \mathbf{R}^n$. Let $S_{2^t}f(x) = \phi_{2^t} * f(x)$. Then we have

$$f = \int_{\mathbf{R}} S_{2^t}(S_{2^t}f)\, dt \text{ for all } f \in \mathcal{S}(\mathbf{R}^n)$$

and

$$\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \sim \left\| \left(\int_0^\infty |\phi_t * f|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)} \sim \left\| \left(\int_{\mathbf{R}} |S_{2^t} f|^q \, dt \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)}.$$

For $t \in \mathbf{R}$, let

$$\sigma_{t,P}(x) = e^{iP(x)} \frac{\Omega(x')}{|x|^n} \chi_{[2^t, 2^{(t+1)})}(|x|),$$

$$\sigma_t(x) = \sigma_{t,0}(x),$$

$$|\sigma_t|(x) = \frac{|\Omega(x')|}{|x|^n} \chi_{[2^t, 2^{(t+1)})}(|x|),$$

and let

$$\sigma^* f(x) = \sup_{t \in \mathbf{R}} |\sigma_t| * |f|(x).$$

First, it is easy to see that

$$(3.1) \qquad |\sigma_t| * |f|(x) \leq C \int_{\mathbf{S}^{n-1}} |\Omega(y)| \, M_y f(x) \, d\sigma(y),$$

where

$$M_y f(x) = \sup_{\rho \in \mathbf{R}} \frac{1}{\rho} \int_0^\rho |f(x - sy)| \, ds$$

is the Hardy-Littlewood maximal function of f in the direction of y. Since M_y is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$ with bound independent of y, by Minkowski's inequality we get

$$(3.2) \qquad \|\sigma^*(f)\|_{L^p(\mathbf{R}^n)} \leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)} \text{ for } 1 < p < \infty.$$

Also, we shall need to study the boundedness of $|\sigma_t| * |f|$ on $\dot{F}_p^{0,q}(\mathbf{R}^n)$. Since

$$|\sigma_t| * |f|(x) = \int_{\mathbf{R}} S_{2^{(t+s)}} \left(|\sigma_t| * S_{2^{(t+s)}} |f|(x) \right) ds,$$

for any $t \in \mathbf{R}$ and for any $g \in \dot{F}_{p'}^{0,q'}(\mathbf{R}^n)$, by Hölder's inequality we have

$$|\langle |\sigma_t| * |f|), g\rangle| = \left| \int_{\mathbf{R}^n} \int_{\mathbf{R}} |\sigma_t| * S_{2^{(t+s)}} |f|(x) S^*_{2^{(t+s)}} g(x) \, ds \, dx \right|$$

$$\leq \left\| \left(\int_{\mathbf{R}} \left| |\sigma_t| * S_{2^{(t+s)}} |f| \right|^q ds \right)^{1/q} \right\|_p \left\| \left(\int_{\mathbf{R}} \left| S^*_{2^{(t+s)}} g \right|^{q'} ds \right)^{1/q'} \right\|_{p'}.$$

Taking supremum over g with $\|g\|_{\dot{F}_{p'}^{0,q'}(\mathbf{R}^n)} \leq 1$ and by Hölder's inequality we have

$$\text{(3.3)} \quad \||\sigma_t| * |f|\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \leq C \left\| \left(\int_{\mathbf{R}} ||\sigma_t| * S_{2^{(t+s)}}(|f|)|^q \, ds \right)^{1/q} \right\|_p$$

for any $t \in \mathbf{R}$.

Now, since $p > 1$, by duality there exists a nonnegative function $h \in L^{p'}(\mathbf{R}^n)$ with $\|h\|_{p'} = 1$ such that

$$\begin{aligned}\left\| \int_{\mathbf{R}} ||\sigma_t| * S_{2^{(t+s)}}(|f|)| \, dt \right\|_p &= \int_{\mathbf{R}} \langle ||\sigma_t| * S_{2^{(t+s)}}(|f|)| \, , h \rangle \, dt \\ &\leq \int_{\mathbf{R}} \langle |S_{2^{(t+s)}} \, |f| \, (x)| \, , \sigma^*(\tilde{h})(-x) \rangle \, dt \\ &\leq \left\| \int_{\mathbf{R}} |S_{2^{(t+s)}}(|f|)| \, dt \right\|_p \left\| \sigma^*(\tilde{h}) \right\|_{p'} .\end{aligned}$$

By the last inequality and (3.2) we have

$$\text{(3.4)} \quad \left\| \int_{\mathbf{R}} ||\sigma_t| * S_{2^{(t+s)}}(|f|)| \, dt \right\|_p \leq C \, \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \left\| \int_{\mathbf{R}} |S_{2^{(t+s)}}(|f|)| \, dt \right\|_p .$$

Also, by a similar argument as in the the proof of (3.2) we have

$$\text{(3.5)} \quad \left\| \sup_{t \in \mathbf{R}} ||\sigma_t| * S_{2^{(t+s)}}(|f|)| \right\|_p \leq C \, \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \left\| \sup_{t \in \mathbf{R}} |S_{2^{(t+s)}}(|f|)| \right\|_p .$$

By interpolation between (3.4) and (3.5) we get

$$\text{(3.6)} \quad \left\| \left(\int_{\mathbf{R}} ||\sigma_t| * S_{2^{(t+s)}}(|f|)|^q \, ds \right)^{1/q} \right\|_p \leq C \, \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \, \|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} ,$$

which when combined with (3.3) implies

$$\text{(3.7)} \quad \||\sigma_t| * |f|\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \leq C \, \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \, \|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}$$

for any $t \in \mathbf{R}$ and $1 < p, q < \infty$.

Proof of Theorem 1.1: It is easy to see that

$$T_\Omega f(x) = \int_{\mathbf{R}} \sigma_t * f(x) \, dt$$

and hence we have

$$T_\Omega(f) = \int_{\mathbf{R}} H_s(f) \, ds,$$

where

$$H_s(f)=\int_{\mathbf{R}} S_{2^{t+s}}\left(\sigma_t * S_{2^{t+s}} f\right) dt.$$

Since

$$\|T_\Omega(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \leq \int_{\mathbf{R}} \|H_s(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}\, ds, \tag{3.8}$$

we just need to estimate $\|H_s(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}$. By the same argument as proving (3.3) we get

$$\|H_s(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \leq C\left\|\left(\int_{\mathbf{R}} \left|\sigma_t * S_{2^{(t+s)}}(f)\right|^q dt\right)^{1/q}\right\|_{L^p(\mathbf{R}^n)}. \tag{3.9}$$

We need now to consider three cases:

Case 1. $p=q=2$. By (3.9) and Plancherel's theorem we obtain

$$\begin{aligned}\|H_s(f)\|^2_{\dot{F}_2^{0,2}(\mathbf{R}^n)} &\leq C\int_{\mathbf{R}}\int_{\mathbf{R}^n}\left|\hat{\phi}(2^{t+s}\xi)\hat{\sigma}_t(\xi)\hat{f}(\xi)\right|^2 d\xi\, dt\\ &\leq C\int_{\mathbf{R}}\int_{\mathbf{\Delta}_{t+s}}\left(\hat{\phi}(2^{t+s}\xi)\right)^2\left|\hat{\sigma}_t(\xi)\right|^2\left|\hat{f}(\xi)\right|^2 d\xi\, dt,\end{aligned} \tag{3.10}$$

where $\mathbf{\Delta}_{t+s}=\left\{\xi\in\mathbf{R}^n : \frac{1}{2}\leq\left|2^{(t+s)}\xi\right|\leq 2\right\}$. By (3.10), invoking the following estimate from [**13**]

$$|\hat{\sigma}_t(\xi)| \leq \min\left\{\left|2^t\xi\right|, \left(\log\left|2^t\xi\right|\right)^{-1-\alpha}\right\}, \tag{3.11}$$

the choice of ϕ and Plancherel's theorem along with the fact $\dot{F}_2^{0,2}(\mathbf{R}^n) = L^2(\mathbf{R}^n)$ we get

$$\|H_s(f)\|_{\dot{F}_2^{0,2}(\mathbf{R}^n)} \leq C\left(1+|s|\right)^{-(\alpha+1)}\|f\|_{\dot{F}_2^{0,2}(\mathbf{R}^n)}. \tag{3.12}$$

Case 2. $p=q$. By (3.3) and the $L^p(1<p<\infty)$ boundedness of M_y with bound independent of y we get

$$\begin{aligned}&\|H_s(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}\\ &\leq C\left(\int_{\mathbf{R}}\int_{\mathbf{R}^n}\left(\int_{\mathbf{S}^{n-1}}|\Omega(y)|M_y S_{2^{(t+s)}}f(x)\,d\sigma(y)\right)^q dx\, dt\right)^{1/q}\\ &\leq C\left(\int_{\mathbf{R}}\left(\int_{\mathbf{S}^{n-1}}|\Omega(y)|\left\|M_y S_{2^{(t+s)}}(f)\right\|_q d\sigma(y)\right)^q dx\, dt\right)^{1/q}\\ &\leq C\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}.\end{aligned} \tag{3.13}$$

Case 3. $p > q$. Let $\lambda = (p/q)'$. By (3.9) and duality, there exists a nonnegative function $g \in L^{\lambda}(\mathbf{R}^n)$ with $\|g\|_{\lambda} = 1$ such that

$$\|H_s(f)\|^q_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \leq C \int_{\mathbf{R}} \int_{\mathbf{R}^n} \left| \int_{2^t \leq |y| < 2^{(t+1)}} \frac{\Omega(y')}{|y|^n} S_{2^{(t+s)}} f(x-y)\, dy \right|^q g(x)\, dx\, dt.$$

Therefore, by Hölder's inequality, the choice of g and (3.2) we have

$$\begin{aligned}
&\|H_s(f)\|^q_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \\
&\leq C \|\Omega\|^{\frac{q}{q'}}_{L^1(\mathbf{S}^{n-1})} \int_{\mathbf{R}} \int_{\mathbf{R}^n} \int_{2^t \leq |y| \leq 2^{(t+1)}} \frac{|\Omega(y')|}{|y|^n} |S_{2^{(t+s)}} f(x-y)|^q g(x)\, dy\, dx\, dt \\
&\leq C \|\Omega\|^{\frac{q}{q'}}_{L^1(\mathbf{S}^{n-1})} \int_{\mathbf{R}^n} \sigma^*(\tilde{g})(-x) \left(\int_{\mathbf{R}} |S_{2^{(t+s)}} f(x)|^q \, dt \right) dx \\
&\leq C \|\Omega\|^{\frac{q}{q'}}_{L^1(\mathbf{S}^{n-1})} \left\| \left(\int_{\mathbf{R}} |S_{2^{(t+s)}} f|^q \, dt \right)^{1/q} \right\|^q_p \|\sigma^*(\tilde{g})\|_{\lambda} \\
&\leq C \|\Omega\|^{\frac{q}{q'}+1}_{L^1(\mathbf{S}^{n-1})} \|f\|^q_{\dot{F}^{0,q}_p(\mathbf{R}^n)} .
\end{aligned}$$

By the last inequality and (3.13) we get

$$\|H_s(f)\|_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \tag{3.14}$$

for $p \geq q$. By duality and interpolation we get

$$\|H_s(f)\|_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \tag{3.15}$$

for all $1 < p < \infty$ and $1 < q < \infty$. By interpolation between (3.12) and (3.15) we get

$$\|H_s(f)\|_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \leq C (1 + |s|)^{-(1+\alpha)\theta} \|f\|_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \tag{3.16}$$

for all $0 \leq \theta < 1$, $\dfrac{\theta}{2} < \dfrac{1}{p} < 1 - \dfrac{\theta}{2}$, and $\dfrac{\theta}{2} < \dfrac{1}{q} < 1 - \dfrac{\theta}{2}$. Assuming $\theta > \frac{1}{\alpha+1}$, by (3.16) and (3.8) we obtain

$$\|T_{\Omega}(f)\|_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \leq C \|f\|_{\dot{F}^{0,q}_p(\mathbf{R}^n)} \tag{3.17}$$

for $p, q \in (\frac{2+2\alpha}{1+2\alpha}, 2 + 2\alpha)$, which in turn completes the proof of Theorem 1.1. □

Proof of Theorem 1.2: Let $P(x) = \sum_{|\eta|\le m} a_\eta x^\eta$ with $\nabla P(0) = 0$. Since the constant term in $P(tx)$, if any, can be assimilated in the function f, we may assume without loss of generality that $P(tx)$ does not have a constant term. Write $P(rx) = \sum_{j=2}^{d} P_j(x) r^j$, where $P_s(x) = \sum_{|\eta|=s} a_\eta x^\eta$. We shall first consider the case $d = 2^k$ for some $k \ge 1$. The general case will be an easy consequence of this special case $d = 2^k$. Let $m_j = \|P_j\|$ and Q be given by $Q(rx) = \sum_{j=2}^{d/2} P_j(x) r^j$. By a dilation in r we may assume, without loss of generality, that $\max_{\frac{d}{2}<j\le d} m_j = 1$ (see [**6**, p. 392]). Also, there is a j_0, $\frac{d}{2} < j_0 \le d$, such that $m_{j_0} = 1$. It is easy to see that

$$T_{\Omega,P} f(x) = \int_{\mathbf{R}} \sigma_{t,P} * f(x)\, dt. \tag{3.18}$$

Decompose $T_{\Omega,P} f(x)$ as

$$\begin{aligned} T_{\Omega,P} f(x) &= \int_{t\le t_0} \sigma_{t,P} * f(x)\, dt + \int_{t>t_0} \sigma_{t,P} * f(x)\, dt \\ &:= T^0_{\Omega,P} f(x) + T^\infty_{\Omega,P} f(x), \end{aligned} \tag{3.19}$$

where $t_0 \in \mathbf{R}$ is to be chosen later. We start with $T^0_{\Omega,P}(f)$. We write

$$\begin{aligned} T^0_{\Omega,P} f(x) &= \int_{t\le t_0-1} (\sigma_{t,P} * f(x) - \sigma_{t,Q} * f(x))\, dt + \int_{t\le t_0} \sigma_{t,Q} * f(x)\, dt \\ &:= I_1 f(x) + I_2 f(x). \end{aligned} \tag{3.20}$$

Let

$$E(d) = \sup_{P\in\mathcal{P}(n;m,0)} \left\| T^0_{\Omega,P}(f) \right\|_{\dot F_p^{0,q}(\mathbf{R}^n)}.$$

Since $\nabla P(0) = 0$, we have $\nabla Q(0) = 0$ and since $\deg(Q) \le \frac{d}{2}$, by induction we get

$$\|I_2(f)\|_{\dot F_p^{0,q}(\mathbf{R}^n)} \le E\left(\frac{d}{2}\right) \quad \text{for } p \in \left(\tfrac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right) \text{ and } q \in \left(\tfrac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right). \tag{3.21}$$

Choose t_0 so that $\frac{d}{2} 2^{(t+1)(\frac{d}{2}-1)} \le 1$ for all $t \le t_0 - 1$. Therefore, for all $t \le t_0 - 1$ and $2^t \le r \le 2^{t+1}$ we have

$$\left| e^{iP(ry)} - e^{iQ(ry)} \right| \le \sum_{\frac{d}{2}<j\le d} m_j r^j \le \sum_{\frac{d}{2}<j\le d} 2^{(t+1)j} \le 2^{t+1}.$$

Hence

$$|I_1 f(x)| \le \int_{t\le t_0-1} 2^{t+1} \left(|\sigma_t| * |f|(x)\right) dt,$$

which easily implies

$$\|I_1(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \le \int_{t\le t_0-1} 2^{t+1} \,\||\sigma_t| * |f|\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}\, dt. \tag{3.22}$$

By (3.7) and (3.22) we get

$$\|I_1(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \le C\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\,\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}\,. \tag{3.23}$$

Now, since

$$\left\|T^0_{\Omega,P}(f)\right\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \le \|I_1(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} + \|I_2(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}\,,$$

by (3.21) and (3.23) we get

$$E(d) \le E\left(\frac{d}{2}\right) + C\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\,\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}\,.$$

Since $d = 2^k$ with $k \ge 1$ we get

$$E(2^k) \le E(2^{k-1}) + C\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\,\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}$$

and hence

$$E(2^k) \le E(2) + Ck\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\,\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}\,. \tag{3.24}$$

We need now to estimate $E(2)$. To this end, we write $P(x) = \sum_{|\alpha|=2} a_\alpha x^\alpha$. Without loss of generality, we may assume that $\sum_{|\alpha|=2} |a_\alpha| = 1$. Write

$$\begin{aligned} T^0_{\Omega,P} f(x) &= \int_{t\le t_0-1} (\sigma_{t,P} * f(x) - \sigma_t * f(x))\, dt + \int_{t\le t_0-1} \sigma_t * f(x)\, dt \\ &:= J_1 f(x) + J_2 f(x). \end{aligned} \tag{3.25}$$

By following the same argument as in the proof of Theorem 1.1 we get

$$\|J_2(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \le C\,\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \tag{3.26}$$

for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $q \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$. Now, we turn our attention to $\|J_1(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}$. Since

$$\left|e^{iP(y)} - 1\right| \le \sum_{|\alpha|=2} |a_\alpha|\,|y|^2 \le 2^{t+1} \text{ whenever } |y| \le 2^{t+1} \le 1 \text{ for } t \le t_0-1,$$

we get

$$|J_1 f(x)| \le \int_{t\le t_0-1} 2^{t+1} \left(|\sigma_t| * |f|(x)\right) dt.$$

By the last inequality and (3.7) we get

$$\|J_1(f)\|_{\dot F_p^{0,q}(\mathbf{R}^n)} \le C\,\|f\|_{\dot F_p^{0,q}(\mathbf{R}^n)}\,. \tag{3.27}$$

By combining (3.25), (3.27) and (3.24)

$$E(2^k) \le C(k+1)\,\|f\|_{\dot F_p^{0,q}(\mathbf{R}^n)}\,. \tag{3.28}$$

The case of general d is now trivial. If $2^{k-1} < d \le 2^k$, then

$$E(d) \le E(2^k) \le C(k{+}1)\,\|f\|_{\dot F_p^{0,q}(\mathbf{R}^n)} \le C(\log d{+}1)\,\|f\|_{\dot F_p^{0,q}(\mathbf{R}^n)}\,. \tag{3.29}$$

We shall now treat the term $\left\|T^{\infty}_{\Omega,P}(f)\right\|_{\dot F_p^{0,q}(\mathbf{R}^n)}$. Write $T^{\infty}_{\Omega,P}f(x)$ as

$$T^{\infty}_{\Omega,P}f(x) = \int_{t>t_0-1} \sigma_{t,P} * f(x)\,dt := \int_{t>t_0-1} F_t f(x)\,dt.$$

Since

$$\left\|T^{\infty}_{\Omega,P}(f)\right\|_{\dot F_p^{0,q}(\mathbf{R}^n)} \le \int_{t>t_0-1} \|F_t(f)\|_{\dot F_p^{0,q}(\mathbf{R}^n)}\,dt, \tag{3.30}$$

we just need to estimate $\|F_t(f)\|_{\dot F_p^{0,q}(\mathbf{R}^n)}$. By following an argument that is similar to the one in the proof of (3.3), we have

$$\|F_t(f)\|_{\dot F_p^{0,q}(\mathbf{R}^n)} \le C\left\|\left(\int_{\mathbf{R}} \left|\sigma_{t,P} * S_{2^{(t+s)}}f\right|^q\,ds\right)^{1/q}\right\|_p. \tag{3.31}$$

As in the proof of Theorem 1.1, we need to consider three cases: (1) $p = q = 2$, (2) $p = q$ and (3) $p > q$. Now if $p = q = 2$, by (3.31) and Plancherel's theorem we obtain

$$\begin{aligned} \|F_t(f)\|^2_{\dot F_2^{0,2}(\mathbf{R}^n)} &\le C\int_{\mathbf{R}}\int_{\mathbf{R}^n} \left|\sigma_{t,P} * S_{2^{(t+s)}}f(x)\right|^2\,dx\,ds \\ &\le C\int_{\mathbf{R}}\int_{\mathbf{R}^n} \left|\hat\phi(2^{t+s}\xi)\hat\sigma_{t,P}(\xi)\hat f(\xi)\right|^2\,d\xi\,ds \\ &\le C\int_{\mathbf{R}}\int_{\mathbf{\Delta}_{t+s}} \left(\hat\phi(2^{t+s}\xi)\right)^2 \left|\hat\sigma_{t,P}(\xi)\right|^2 \left|\hat f(\xi)\right|^2\,d\xi\,ds. \end{aligned} \tag{3.32}$$

Now we need to estimate $|\hat{\sigma}_{t,P}(\xi)|$. By definition and a change of variable, we have

$$\hat{\sigma}_{t,P}(\xi) = \int_{\mathbf{S}^{n-1}} \Omega(y) \left(\int_{2^{-1}}^{1} e^{i(P(2^{(t+1)}uy) - 2^{(t+1)}uy\cdot\xi)} \frac{du}{u} \right) d\sigma(y).$$

By Lemma 2.1 we get

$$\left| \int_{2^{-1}}^{1} e^{i(P(2^{(t+1)}uy) - 2^{(t+1)}uy\cdot\xi)} \frac{du}{u} \right| \le C \left| 2^{j_0(t+1)} P_{j_0}(y) \right|^{-\frac{1}{d}} .$$

By combining the last estimate with the trivial estimate

$$\left| \int_{2^{-1}}^{1} e^{i(P(2^{(t+1)}uy) - 2^{(t+1)}uy\cdot\xi)} \frac{du}{u} \right| \le 1,$$

we obtain

$$\left| \int_{2^{-1}}^{1} e^{i(P(2^{(t+1)}uy) - 2^{(t+1)}uy\cdot\xi)} \frac{du}{u} \right| \le C \left(\log 2^{j_0(t+1)} \right)^{-(\alpha+1)} \left(d + \alpha + \log \frac{1}{|P_{j_0}(y)|} \right)^{\alpha+1} .$$

By the last inequality and since $(a+b)^\theta \le 2^{\theta-1}\left(a^\theta + b^\theta\right)$ (for $\theta \ge 1$ and $a, b \ge 0$) we get

$$\begin{aligned} \left| \int_{2^{-1}}^{1} e^{i(P(2^{(t+1)}uy) - 2^{(t+1)}uy\cdot\xi)} \frac{du}{u} \right| &\le C\left(j_0(t+1)\right)^{-(\alpha+1)} \left[(d+\alpha)^{\alpha+1} + \left(\log \frac{1}{|P_{j_0}(y)|} \right)^{\alpha+1} \right] \\ &\le C(t+1)^{-(\alpha+1)} \left[1 + \left(\log \frac{1}{|P_{j_0}(y)|} \right)^{\alpha+1} \right]. \end{aligned}$$

Since $P_{j_0} \in \mathcal{H}(n; m)$ and $\|P_{j_0}\| = 1$, we get

$$|\hat{\sigma}_{t,P}(\xi)| \le C(t+1)^{-(\alpha+1)}. \tag{3.33}$$

Therefore, by (3.32)–(3.33) and by Plancherel's theorem we get

$$\|F_t(f)\|_{\dot{F}_2^{0,2}(\mathbf{R}^n)} \leq C(t+1)^{-(\alpha+1)} \|f\|_{\dot{F}_2^{0,2}(\mathbf{R}^n)} . \tag{3.34}$$

As for the cases $p = q$ and $p > q$, we follow the same argument as in the proof of Theorem 1.1 (in dealing with these cases the factor $e^{iP(y)}$ being harmless) to get

$$\|F_t(f)\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \leq C \|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \tag{3.35}$$

for $p \geq q$. Now, the rest of the proof will follow by (3.30), (3.34)–(3.35) and the same argument as in the proof of (3.17). This completes the proof of Theorem 1.2. □

Proof of Theorem 1.4: Define the family of measures $\{\lambda_t : t \in \mathbf{R}\}$ by

$$\lambda_t * f(x) = \frac{1}{2^t} \int_{|y| \leq 2^t} \frac{\Omega(y')}{|y|^{n-1}} f(x-y)\, dy.$$

It is easy to see that

$$\mathcal{M}_{\Omega,q} f(x) \sim \left(\int_{\mathbf{R}} |\lambda_t * f(x)|^q \; dt \right)^{1/q} .$$

Write

$$\lambda_t * f = \int_{\mathbf{R}} (\lambda_t * S_{2^{t+s}} f) \; ds,$$

where $S_{2^t} f(x) = \phi_{2^t} * f(x)$ with ϕ given as above and satisfies the condition $\int_{\mathbf{R}} \hat{\phi}_{2^t}(\xi)\, dt = 1$ instead of $\int_{\mathbf{R}} \left|\hat{\phi}_{2^t}(\xi)\right|^2 dt = 1$. Now, by Minkowski's inequality we have

$$\|\lambda_t * f\|_{L^q(\mathbf{R})} \leq \int_{\mathbf{R}} A_s(f)\, ds,$$

where $A_s f(x) = \left(\int_{\mathbf{R}} |\lambda_t * S_{2^{t+s}} f(x)|^q \; dt \right)^{1/q}$. By invoking the estimates

$$\left|\hat{\lambda}_t(\xi)\right| \leq \min\left\{ \left|2^t \xi\right|, \left(\log \left|2^t \xi\right|\right)^{-1-\alpha} \right\}$$

from [**13**] and following the same arguments as in the proof of (3.16) we obtain

$$\|A_s(f)\|_{L^p(\mathbf{R})} \leq C\,(1+|s|)^{-\eta} \|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)} \tag{3.36}$$

for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, $q \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and for some $\eta > 1$. Thus we have

$$\begin{aligned}\|\mathcal{M}_{\Omega,q}(f)\|_{L^p(\mathbf{R})} &\le C\,\|\lambda_t * f\|_{L^p(L^q(\mathbf{R}),\mathbf{R}^n)}\\ &\le \int_{\mathbf{R}} \|A_s(f)\|_{L^p(\mathbf{R})}\, ds \le C\,\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}\end{aligned}$$

for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, $q \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$. Theorem 1.4 is proved. □

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H. M. Al-Qassem:
Department of Mathematics and Physics
Qatar University
Doha-Qatar
E-mail address: husseink@qu.edu.qa

L. C. Cheng:
Department of Mathematics
Bryn Mawr College
Bryn Mawr, PA 19010
U.S.A.
E-mail address: lcheng@brynmawr.edu

Y. Pan:
Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260
U.S.A.
E-mail address: yibiao@pitt.edu

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