Mixed Model For Analyzing Variability in Mortality Rates Over Time With Reference To Qatar Mortality Data

Mohamed Tawfik El- bolkiny
Assistant Proffesor
The Faculty of Administrative Science and Economics
Universtiy of Qatar

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Abstract

It was observed that variability in mortality rates over time in the state of Qatar is not of binomial nature. A method of analyzing these mortality rates in the heterogeneous population is presented in this paper. Three estimation approaches are also investigated. This variability was also examines for the total males and the total females. In each analysis it was observed that the variability is very much the same as what one would expect under a beta-binomial model. A significant finding, for the mortality data, was that estimates of the parameter Θ were remarkably homogeneous for each analysis. That is, the relative degree of extra binomiality was very much the same from month to month, regardless of the population size or the number of months in the analysis.

1-Introduction:

A recent effort in making detailed population projections for areas such as small states or municipalities suggests that there is an increasing interest in analyzing vital events in small populations. Unfortunately, analysis at the local level is fraught with much greater analytic problems than are analysis at the national, regional, or even large states level. One of

the major difficulties is that the stochastic variability of rate estimates for small populations reflects additional source of instability not as dramatically manifest in larger population groups. Thus mortality rates exhibit geographical variation on a worldwide scale, and within individual nation or state, even when adjusted for geographical differences in age, race, and sex. In other words geographic areas as large as nations or as small as municipalities are known to have varying patterns of mortality within their boundaries. This is true for all natural causes of death combined as well as for specific cause of death.

Similarly geographic areas are also known to have varying patterns of mortality from year to year or from month to month. This research will focus on the variation in the monthly mortality rates from 1984 to 1990 in the State of Qatar. The findings and the conclusions of this study will contribute significantly to the understanding of the differential mortality in the State of Qatar.

The analysis of the mortality rates has been discussed extensively from many different viewpoints ([1], [2], [3], [4], [5], [6], [7], [8], [9]). A common assumption is that the data follow a binomial distribution. However, a few writers have mentioned that in their experience, data which appear to be binomial proportions sometimes exhibit heterogeneity which results in greater variation than would be expected under the binomial distribution ([10], [11], [12], [13], [14], [15], [16], [17], [18], [19]).

The main topic of investigation in this research will be the variability encountered in mortality rates, over some period of time for a specified small population. One might consider the observed monthly mortality rates or crude death rates for the State of Qatar for 1984 - 1990. A statistician might expect such a set of data to behave in a binomial manner, apart from a possible time trend which might be caused by any of a number of factors. Since such mortality rates are typically small, he may expect the set of rates

to behave in a Poisson manner. The basic logic of the model is that at the individual level, events are governed by a binomial or Poisson process.

Conditions for the validity of assuming the binomial distribution for a group of lives observed during some time or age interval are that (a) each person of the group has the identical probability of death within the interval considered; (b) these lives represent statistically independent observations with respect to mortality. If the above conditions are fulfilled, then the deaths occurring in this group during the observation period may be considered a binomial distribution. In practice these conditions are never exactly satisfied. Condition (a) is violated to an appreciable extent for many groups of lives. The reason for this violation is that the mortality is not identical for each person in the group (i.e. monthly population). If we consider the average monthly mortality rate, it may be binomial within month but it may differ from month to month. Condition (b) may be also violated, since there is a close relation between friends, relatives and persons living near each other. Therefore use of the binomial distribution in demographic and actuarial work might appear to be a very questionable procedure. The same problem exists for the use of Poisson distribution, if we assume that the Poisson rate parameter is fixed.

In the actuarial work, the existence of these large variation in mortality probabilities has been known for many years [20]. This, at least indirectly, accounts for the fact that most actuaries do not usually view mortality estimation as a statistical problem. They regard variability in mortality data as being due to intrinsic roughness which is the target of the graduation techniques or smoothing devices. Thus most of actuarial research in this century, within the general area of mortality estimation, has been in the field of graduation. One cannot read the principal actuarial textbooks on mortality estimation without coming to the conclusion that within the mainstream of actuarial science, mortality estimation and estimation by using statistical distribution are almost unrelated.

Therefore, the objective of this research is to see if statistics can lead to a deeper understanding of these perplexing mortality variations, by finding or developing an alternative probability model which fits the data well. This model must be a general model for the analysis of mortality rates in multiple small populations. It must be designed to represent the estimates of the rates of vital events as a function of demographic variables, population heterogeneity, and selected, substantively relevant covariates.

At the individual level it is possible to model the binomial or Poisson rate parameter as a pre-selected function of covariates. However, in recognition of the facts that (a) individuals are heterogeneous in their risk to the process; and (b) the people are systematically packaged into small population "units" (i.e. monthly population), the total model of variation in small population dynamics will require that we "mix" individual level models. That is, our model of variation of rates over small populations requires that we view these rates as a realization of a mixture of individual level binomial distribution. In order to evaluate our model, we find it convenient to assume that the rate is constant over all individuals within a given small population (i.e. we assume all the individuals in each month have the same mortality rate), since we can not observe risk characteristics of individuals.

Consider x_i the observed number of deaths in the population cell i within some specified demographic group of size n_i. The n_i individuals are assumed to be demographically homogeneous so that individual differences in risk are not due to the mixing of demographic categories but instead are due to individual differences in genetic characteristics, environmental exposure, nutrition and lifestyle [21]. Because of dealing with population mortality data, individual differences in diseases susceptibility will not be

observed. Consequently, if the effects of such population heterogeneity are to be accounted for, then it is necessary to include parameters representing these effects in the probability density function employed to model the number of deaths in a cell. The choice of the model must be restricted to distributions (i.e.x_i takes only integer values) with two parameters (the first being related to the mean cell death rate, the second dealing with population heterogeneity). Given these restrictions, we find out that x_i can be modeled by the two parameter beta binomial distribution. The choice of beta binomial distribution is motivated by the fact that, it represents a compounding (i.e. weighted average) of binomial distributions with beta distributed levels of diseases susceptibility. The compounding increases the heterogeneity (variance in the model).

We could think about the general model for the mortality observations as a mixture of two binomial population, $B(n_1, q_1)$ and $B(n_2, q_2)$, in the unknown proportions of α and $(1 - \alpha)$. This three-parameter model could be a candidate for the underlying heterogeneous probability model for the mortality data. The model will require the estimation of the parameters p, q and α. Other more complicated alternatives, such as a mixture of three binomial or a mixture of two beta binomial distributions, may lead to a heterogeneous model.

We could also think about compounding the variance and instead of searching for a reasonable distribution to assign to q_i, the parameter of the binomial, we might leave the distribution unspecified, and merely assume that q_i has some distribution with mean (μ) and variance (V_q). We can find the expected value of the unconditional variance and the expected value of the variance of q_i and then we can estimate (V_q). (see appendix A)

Therefore

$$\hat{V}_q = \{1/(k-1)\}\{\sum_{i=1}^{K} n_i(q_i - q)^2 / n_i - \mu(1-\mu^2)/n_i .$$

where $\mu(1-\mu^2)/n_i$ .is a consistent estimate for

$$E\{q_i(1-q_i)/n_i\}$$

and where $q_i = x_i / n_i$ and $\mu = \sum_{i=1}^{K} x_i / \sum_{i=1}^{K} n_i = \sum_{i=1}^{K} n_i/q_i / \sum_{i=1}^{K} n_i$

These results will be investigated further when illustrative data are considered. We can use the actual data to estimate the unconditional variance of q_i then we can estimate the variance of q and compare it with the variance of the model. We can also fit the model parameters to the estimated variance, Var(q_i).

2 - The model

The binomial distribution could be used conditionally to model the number of deaths in a given month or a given period of time. This model is generalizable to a heterogeneous population model, if the q_is, the binomial parameters (hazard rates), have been obtained randomly from a beta distribution of the following from

$$p(q_i = q) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, q^{\alpha-1}(1-q)^{\beta-1}, & 0 < q < 1,\ a, b > 0 \\ 0 & \textit{elsewhere} \end{cases} \quad (1)$$

This distribution for the q_i is a reasonable one for the following reasons (1) The range is precisely that over which q_i must vary. (2) The distribution has a wide variety of shapes for differing values of the two

parameters, so that there is considerable flexibility in its form. (3) The mathematical convenience. The mean of this distribution is $\mu = \alpha/(\alpha + \beta)$ and its variance is $\sigma^2 = \alpha\beta/(\alpha + \beta)^2 (\alpha + \beta + 1)$ or $\sigma^2 = \Theta \mu (1 - \mu)$ where $\Theta = 1/(\alpha + \beta + 1)$.

Given q, the conditional distribution of the number (x_i) of deaths among n_i persons is assumed to be binomial

$$P\ (x_i = x/q) = \begin{cases} \binom{n}{x} q^x (1-q)^{n-x} & x = 0, 1, 2, \ldots, n \\ 0 & \textit{elsewhere} \end{cases} \qquad (2)$$

It follows that the marginal distribution of the number of deaths among n_i persons is as follows

$$P(x_i=x) = \int_0^1 \binom{n_i}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} q^{\alpha+x-1} (1-q)^{n+\beta-1}\, dq$$

therefore

$$p(x_i = x) = \begin{cases} \binom{n_i}{x} \frac{\Gamma(\alpha+\beta)\Gamma(\beta+n_i-x)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)\,\Gamma(\alpha)\;\Gamma(\beta)} & x = 0, 1, 2, .. \\ & \alpha, \beta > 0 \\ 0 & \textit{elsewhere} \end{cases} \qquad (3)$$

We now proceed to find the mean and the variance of the beta binomial distribution, via factorial moments ([22], [23], [24])

Let $x^{(r)}$ be defined by

$$x^{(r)} = x(x-1)(x-2)(x-3).....(x-r+1)$$

$$E(x^{(r)}) = \sum_{x=r}^{\infty} \binom{n_i}{x} \frac{x^{(r)}\ \Gamma(\alpha+x)\ \Gamma(\beta+n_i-x)\ \Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\ \Gamma(\alpha)\ \Gamma(\beta)}$$

$$= \frac{n^{(r)}\ \Gamma(\alpha+r)\ \Gamma(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+r)} = \frac{n^{(r)}\ (\alpha+r-1)^{(r)}}{(\alpha+\beta+r-1)^{(r)}}, \qquad (4)$$

Thus $E(x_i) = n_i\alpha/(\alpha+\beta) = n_i\mu$, **therefore**

$$E(x/n_i) = \alpha/(\alpha+\beta) = \mu \qquad (5)$$

and $$VAR(x_i) = \frac{n_i(n_i-1)\alpha(\alpha-1)}{(\alpha+\beta)(\alpha+\beta+1)} + \frac{n_i\alpha}{(\alpha+\beta)^2} - \frac{n_i^2\alpha^2}{(\alpha+\beta)^2}$$

$$= n_i^2\mu(1-\mu)\Theta + n_i\mu(1-\mu)(1-\Theta)$$

$$= n_i^2\mu(1-\mu)\Theta + n_i\mu(1-\mu)(1-\Theta)$$

$$= n_i\mu(1-\mu) + n_i^2\mu(1-\mu)\Theta(1-1/n_i)$$

Therefore (6)

$$VAR(x_i/n_i) = \mu(1-\mu)/n_i + \mu(1-\mu)\Theta(1-1/n_i)$$

Notice how the variance is partitioned into two components, the first component represents the usual binomial variability, Var(q), while the second component represents the additional variation. If α and β become

infinity so that $\alpha/(\alpha + \beta) = \mu$ is a constant, q, then var $(x_i/n_i) = q\ (1 - q)/n_i$, is a simple binomial variability. Therefore, one can think of Θ as a measure of the amount of variability in excess variation or a measure of extra-binomial variation. If Θ is very small or goes to zero, this excess variation (i.e. the second term of equation (6) can be ignored and binomial model can be used. On the other hand, if Θ is larger than zero the binomial variation should be disregarded, since it will provide a very small portion of the total variability, and other statistical models and procedures adopted.

The beta-binomial distribution which can be approximated to a negative binomial distribution, but the parameters involved in this distribution are not readily identified with the usual way of writing the negative binomial distribution. This different parameterization is natural in order to apply the model to the mortality data. It follows that the marginal distribution of the number of deaths when n, and β are large is approximately negative binomial distribution.

$$p(x_i = x) = \begin{cases} \{n_i/(n_i+\beta)\}^x \{\beta/(n_i+\beta)\}^\alpha \{\Pi\,[(\alpha+j-1)/\ j]\}, & \text{where } x = 0, 1, \ldots \\ 0 & \text{elsewhere} \end{cases} \tag{7}$$

This approximated model of beta binomial can be written as (see appedix B):

$$p(x_i = x) = \begin{cases} \binom{x+r+1}{r-1} P^r (1-P)^x, & x = 0, 1, .. \\ 0 & \text{elsewhere} \end{cases} \tag{8}$$

where $P = \left\{ \frac{\beta}{n+\beta} \right\}$

To find the mean and variance of the approximated form of beta binomial (negative binomial) we first find the general factorial moment.

Therefore

$$E(x^{(r)}) = \sum_{x=r}^{\infty} \frac{x^{(r)}\ (\alpha + x - 1)!}{(x-r)!(\alpha-1)!} \cdot \frac{\beta^{\alpha}\ n^{x}}{(n_i + \beta)^{(\alpha+B)}}$$

$$= (n_i / \beta)^r\ (\alpha + r - 1)^{(r)}$$

Therefore

$$E(x) = n_i\alpha/\beta \qquad \textbf{and} \qquad E(x/n_i) = \alpha/\beta \tag{9}$$

$$Var(x) = (n_i/\beta)^2\ \alpha(\alpha + 1) + n_i\alpha/\beta - (n_i\alpha/\beta)^2$$

$$= n^2_i\alpha/\beta^2 + n_i\alpha/\beta \qquad \text{and}$$

$$Var(x/n_i) = \alpha/n_i\beta + \alpha/\beta^2$$

The partitioning of the variance consists of two components. The first term represents binomial variability and the second term represents the additional variation.

3-Estimation procedures:

The maximum likelihood estimators of α and β cannot be derived in closed form in this case. Two simpler methods suggest themselves. The first

method is the method of moments and the second method is the method of the mean and zero to fit the beta-binomial. However these methods are inefficient (i. e. variances of estimators are too large). Therefore, an algorithm will be developed to obtain the maximum likelihood estimates numerically. Two other alternative estimation approaches are investigated. The first is the minimum chi-square method, and the second is the method of quasi-likelihood method introduced by Nelder and Wedderburn ([25], [26]) and used by ([27], [28] [11]).

3.1 - The method of maximum likelihood estimation (MLE):

From a theoretical point of view the most important general method of estimation so far known is the method of maximum likelihood. As a general method of estimation, it was first introduced by Fisher [29] and it has afterwards been further developed in a series of works by the same author. Important contributions have also been made by others. Allowing for differing numbers of persons at risk of dying, in each of k month by subscripting n, the likelihood function is

$$L = \prod_{i=1}^{K} \{n_i / (n_i + \beta)\}^{x} \{\beta/(n_i + \beta)\}^{\alpha} \{ \prod_{j=1}^{X_i} [(\alpha + j - 1)/ j]\} \qquad (11)$$

It is found out that the maximum likelihood estimator can not be derived analytically (see appendix B). Therefore the likelihood equations must be solved numerically by iterations. A large number of authors suggested the method of moment estimates as a convenient starting point for the iteration. Some authers [30], [31] draw attention to the inefficiency of the moment estimation in certain cases of the negative binomial law.

The standard errors of the MLE's of the parameters are the square roots of the diagonal elements of the inverse of the information matrix. The information matrix elements are asymptotically equivalent to the negative

expectation of the second partial derivatives of the likelihood function (see appendix C).

Using the state of Qatar mortality data the maximum likelihood estimators for α and β will be found and the information matrix, Φ, will be estimated. The purpose of developing this information matrix is to get estimates of the relevant variances. These variances will prove to be very useful when the data are considered. As an example we may use the "delta method" ([32], [12]) to find Var(μ) and Var (Θ).

Rcall $\mu = \alpha/(\alpha+\beta)$ **and** $\Theta = 1/(\alpha+\beta+1)$

$$\textbf{then } \mathbf{Var}(\mu) = H'\Phi^{-1}H$$

$$\textbf{and } \mathbf{Var}(\Theta) = G'\Phi^{-1}G$$

where $H' = \begin{pmatrix} \frac{\partial\mu}{\partial\alpha} & \frac{\partial\mu}{\partial\beta} \end{pmatrix}$, $G' = \begin{pmatrix} \frac{\partial\Theta}{\partial\alpha} & \frac{\partial\Theta}{\partial\beta} \end{pmatrix}$ **and** Φ^{-1} **is the inverse of the information matrix (i.e the variance-covariance matrix of** α **and** β**). The results are**

$$\mathbf{Var}(\mu) = \mathbf{Var}(\alpha)\{\beta/(\alpha+\beta)^2\}^2$$

$$- 2\,\mathbf{Cov}(\alpha,\beta)\{\alpha/(\alpha+\beta)^2\}\{\beta/(\alpha+\beta)^2\}$$

$$+ \mathbf{Var}(\beta)\{\alpha/(\alpha+\beta)^2\}^2 \qquad (12)$$

$$\textbf{and } \mathbf{Var}(\Theta) = \{1/(\alpha+\beta+1)^4\}\{\mathbf{Var}(\alpha) - 2\,\mathbf{Cov}(\alpha,\beta) + \mathbf{Var}(\beta)\} \qquad (13)$$

For large values of α and β, this matrix could be what is sometimes called "ill-behaved" (or unstable or almost singular) in that its determinant is very small relative to the elements of the matrix. In terms of MLE's the amounts of α and β, being highly correlated.

3.2 - The method of Minimum chi-square estimation (MCS):

Again consider that the mortality observation x_i is beta-binomial where i = 1, 2, ...,k. Since x_i the mortality observation and n_i the population sizes are very large, we could consider that the assumption of normal distribution is valid for large samples. In other words consider that x_i is approximately normal distribution with mean $n_i v$ and variance $n_i v\ (1 - \mu) / w_i = 1 + \Theta\ (n_i - 1)^{-1}$. As a result, if we take the vector of observation X

$$X \sim N\left(\begin{bmatrix} n_1\mu \\ n_2\mu \\ \cdot \\ \cdot \\ \cdot \\ n_k\mu \end{bmatrix}\begin{bmatrix} \mu(1-\mu)/\,w_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \cdot \\ & & n_K\mu(1-\mu)/\,w_k \end{bmatrix}\right)$$

Therefore

$$X^2 = \sum_{i=1}^{K} \frac{(x_i - n_i\,\mu)^2\ w_i}{n_i\,\mu(1-\mu)} \qquad (14)$$

is approximately chi-square with k-2 degrees of freedom, since α and β need to be estimated from the data. The procedure for finding MCS estimates is quite analogous to that for finding MLE. Moreover, the (asymptotic) properties of MCS estimates are similar to these of MLE's. It seems natural to attempt to determined the best values of the parameters μ and Θ so as to render X^2 as small as possible. We then have to solve the equations

$$\frac{\partial X^2}{\partial \mu} = \sum_{i=1}^{K} \frac{w_i (x_i - n_i \mu)^2}{\mu(1-\mu)} - \sum_{i=1}^{K} \frac{w_i (x_i - n_i \mu)^2 (1 - 2\mu)}{n_i \mu^2 (1-\mu)^2} = 0 \qquad (15)$$

$$\frac{\partial X^2}{\partial \Theta} = \sum_{i=1}^{K} \frac{w_i^2 (x_i - n_i \mu)^2 (n_i - 1)}{n_i \mu(1-\mu)} = 0 \qquad (16)$$

with respect to the unknown parameters μ and Θ and insert the values that we found into X^2.The limiting distribution for this method of estimation has been investigated by Pearson and Neyman [33], who used methods at multi-dimensional geometry of the type introduced by R. Fisher [29], [34]. It can be shown that for the large n_i the influence of the second term in equation [15] becomes negligible. Thus we have to solve the following two equations

$$\frac{\partial X^2}{\partial \mu} = \sum_{i=1}^{K} w_i (x_i - n_i \mu)^2 = 0 \qquad (17)$$

$$\frac{\partial X^2}{\partial \Theta} = \sum_{i=1}^{K} \frac{w_i^2 (x_i - n_i \mu)^2 (n_i - 1)}{n_i} = 0 \qquad (18)$$

and usually this will be much easier to deal with. The method of estimating the parameters μ and Θ from the system of equations (17) and (18) is called the modified minimum chi-square method. Under general conditions, both methods (MCS or MMCS) give the same limiting distribution for the

parameters estimates when n_i is large. In our analysis we found Θ must be equal to 1 for X^2 to be minimum and therefore the method of MCS is not a satisfactory method of estimation. Therefore the method of MCS will not be discussed further at this point.

3.3 - The method of quassi-likelihood estimation (MLE):

To define a likelihood function we have to specify the form of distribution of the observation, but defining a quasi-likelihood function need only specify a relation between the mean and variance of the observations. Wedderburn [26] defined the quasi-likelihood function, which can be used for estimation in the same way as a likelihood function. With constant variance this leads to least squares estimation. When other mean-variance relationships are specified, the quasi-likelihood sometimes turns out to be a recognizable likelihood function. For instance, for a constant coefficient of variation the quasi-likelihood function is the same as the likelihood function obtained by treating the observations of q_i, the mortaity rates as if they had a beta distribution.

Suppose we know the relationship between the mean $E(x_i)$ and the variance for the mortality observation x_i but we do not know the specific distribution for x_i. Usually $E(x_i)$ has a complicated structure in terms of other explanatory variables i.e.μ has the form $(1 + e^{-B})^{-1}$ where B is a linear model. Let us assume that q_is are independantly distributed on (0,1) with $E(q_i) = \mu$, var $(q_i) = \Theta\mu\ (1 - \mu)$ and assume that $E(x_i) = n_i\mu$ and Var $(x_i) = n_i\mu\ (1 - \mu)\ \{1 - \Theta\ (n_i - 1)\}$. Estimation of B is achieved by iterative use of weighted least squares equation,

$$Z' W V Z B = Z' W V Y \qquad (19)$$

In our study $\mu = 1/(1 + e^{-B})$ or B is scalar = - Log $\{ (1 - \mu) / \mu \}$ and a single equation results. The details are:

$$w_i = \frac{1}{(1 + \Theta(n_i - 1))} \quad , \quad \frac{\partial W_i}{\partial \Theta} = \frac{-(n_i - 1)}{\{1 + \Theta(n_i - 1)\}^2}$$

where

$$Z = \begin{pmatrix} 1 \\ 1 \\ . \\ 1 \end{pmatrix}_{K\times 1} \quad , \quad W = \begin{pmatrix} w_1 & 0 \ldots\ldots & 0 \\ 0 & w_2 \ 0 \ldots & 0 \\ . & . & . \\ 0 & & 0 \\ 0 & \ldots\ldots & w_k \end{pmatrix}_{K\times K}$$

$$V = \begin{pmatrix} n_1\mu(1-\mu) & & \\ & . & 0 \\ & . & \\ 0 & & . \\ & & n_k\mu(1-\mu) \end{pmatrix}_{K\times K} , WV = \begin{pmatrix} w_1 n_1\mu(1-\mu) & & \\ & . & 0 \\ & . & \\ 0 & & . \\ & & w_k n_k\mu(1-\mu) \end{pmatrix}_{K\times K}$$

$$Y = \begin{pmatrix} B + (x_1 - n_1\mu) / n_1\mu(1-\mu) \\ . \\ . \\ B + (x_k - n_k\mu) / n_k\mu(1-\mu) \end{pmatrix}_{K\times 1}$$

Therefore

$$Z' W V Z = \left(\sum_{i=1}^{K} w_i n_i \right) \mu(1-\mu)$$

$$y_i = B + (x_i - n_i \mu) / n_i \mu(1-\mu)$$

and since $Z' W V Z B = Z' W V Y$, then

$$\left(\sum_{i=1}^{K} w_i n_i \right) \mu(1-\mu)B = \sum_{i=1}^{K} w_i n_i \mu (1-\mu) \{ B + (x_i - n_i \mu) / n_i \mu(1-\mu) \}$$

$$= \sum_{i=1}^{K} w_i n_i \mu(1-\mu) B + \sum_{i=1}^{K} w_i (x_k - n_k \mu)$$

Thus

$$\sum_{i=1}^{K} w_i (x_k - n_k \mu) = 0 \qquad (20)$$

An ancillary equation based on the X^2 statistic is used to estimate Θ as,

$$\Theta - [X^2 - \sum_{i=1}^{K} \{w_i (1-w_i v_i q_i)\}] / [\sum_{i=1}^{K} \{w_i (n_i - 1)(1 - w_i v_i q_i)\}] = 0 \qquad (21)$$

It is readily apparent that one cannot obtain explicit expressions for the quasi-likelihood estimators of μ and Θ. Therefore, the estimation will be accomplished by numerical iterations. The estimation requires that initial estimates of the parameter values be provided. In our analysis we set $\mu = 10^{-3}$ and $\Theta = 10^{-4}$ with these "starting values," the iteration algorithm replaces at each iteration the current values of the parameters μ and Θ with better estimates until they converge.

4 - Application of the model to the State of Qatar mortality data:

A model for the analysis of rate differentials of vital events for a small populations over time has been presented earlier with the intent of providing demographers with a flexible framework for evaluating the effects of heterogeneity on aggregation bias. In our model heterogeneity is captured as extra-binomial variation. The methodology presented earlier was applied to the analysis of monthly mortality data for the State of Qatar for the years 1984 - 1990. **Table (D.1)** in appendix **D,** n_i represents the population size

for each month, x_i represents the number of mortalities for each month, and $q_i = \{(x_i / n_i)\ (365/D_m)\}$, where D_m represents the number of days in the month. Now the results of assuming that q_i the binomial parameter, has an unspecified distributional form with variance V_{qi} will be examined using the State of Qatar mortality data. As it pointed earlier, the unconditional variance, $Var(q_i)$, will be estimated as

$$\hat{var}(\hat{q}_i = \{1/(k-1)\}\{\sum_{i=1}^{K} n_i(q_i - q)^2\} / n_i$$

To begin the analysis it is necessary to estimate what has been called V_q the difference between the unconditional variance and the binomial contribution to that conditional variance. The estimates are presented in **Table (D. 2)** in appendix **D,** subscripted because of the differing n_i' s. Note that, there is additional variability in the eighty four points. This additional variability, represented by V_q for all points accounts for more than 93% of the total variability.

We turn to the beta binomial distribution as a candidate for the underlying probability for the data in Table (**D.1).** Recall that, for this distribution, $E(x_i/n_i) = \alpha/(\alpha + \beta)$ and the $Var(x_i / n_i) = \mu(1 - \mu)/n_i + \mu(1-\mu) \Theta (1 - 1/n_i)$. As mentioned the first component in the variance represent the usual binomial variation, $Var(q_i)$, and the second component represents the additional variation. The problem now is to estimate the parameters α and β. We will first consider the maximum likelihood estimators for α and β, as derived earlier. The natural logarithm of the likelihood function for the beta binomial model is

$$\text{Log } L = \sum_{i=1}^{K} \{\alpha \log \beta - \text{Log } x_i! + x_i \text{ Log } n_i!$$

$$- (\alpha + x_i)\text{Log}(\beta + n_i) + [\sum_{j=1}^{X_i} \text{Log}(\alpha + j - 1)]\}.$$

When first partial derivatives were taken it was noted that analytical solutions for the maximum likelihood estimators α and β were not possible. As a result iterative procedures were developed to obtain the estimates. The estimation procedure requires that initial estimates of the parameters values be provided. In our analysis we set $\hat{\alpha} = 10^{-04}$ and $\hat{\beta} = 10^{-04}$ as initial values for the parameters. Based on these iterative procedures, we obtain estimates: $\hat{\alpha}$ = .12971, $\hat{\beta}$ = 700.7865, $\hat{\mu}$ = 2.251539E-03, and $\hat{\Theta}$ = 1.73335E-02. Also we may estimate S.E.($\hat{\alpha}$), S.E.($\hat{\beta}$), S.E.($\hat{\mu}$) and S.E.($\hat{\Theta}$) using the actual data, and the form of the information matrix. The inverse of the estimated information matrix is

$$\Phi^{-1} = \begin{pmatrix} 2.019126861E-01 & -1.090846500E-01 \\ -1.090846500E+01 & 4.574625816EE+04 \end{pmatrix}$$

Then S.E.($\hat{\alpha}$) = 1.420959816E-02, S.E.($\hat{\beta}$) = 6.763598021E + 01, S.E.($\hat{\mu}$) = 1.207804925E-03 S.E.($\hat{\Theta}$) = 1.376752728E-04 Since we considered Θ as a measure of the amount of extra-binomial variation. Let us test for binomiality by assuming

$$H_0: \Theta = 0 \quad \text{vs.} \quad H_A: \Theta > 0.$$

Assuming $\hat{\Theta}$ is approximately normal with mean $\hat{\Theta}$ and variance V, then the 95% confidence interval for $\hat{\Theta}$ + 1.64 {S.E.($\hat{\Theta}$)} or (0.0171077, 0.0175593). Since $\Theta = 0$ does not fall in the C.I., H_0 will be rejected and H_A (i.e. there exist extra variation) will be accepted. Therefore the beta binomial distribution is favored to fit the mortality data instead of the binomial distribution. The problem is now how to test whether the model fits the data adequately. The test outlined earlier is the classic textbook asymptotic (large sample) statistical test for goodness of fit. In other words we proceed to compare the agreement between expectation under the

probability model and the actual observation. The resulting χ^2 - test gives $x^2 = 2.34472$, which is very small in comparison with χ^2 from the table with 81 degrees of freedom. As the maximum likelihood fit is entirely satisfactory, there exists good reason to use the beta binomial law (negative binomial approximation) as a mathematical model of the mortality differential

The estimators of the beta-binomial parameters were validated with the Qatar monthly mortality data for 1984 - 1990. The parameters of beta binomial were estimated using a randomly selected half of the data and were tested for goodness of fit using the other half. The results are $\hat{\alpha} = .129956$, $\hat{\beta} = 699.98764$, $\hat{\mu} = 2.25878E-03$ and $\hat{\Theta} = 1.42629E-03$. Testing for goodness of fit using the other half of the data we obtain $x^2 = 1.25023$, which is very small comparing with χ^2 with 39 degrees of freedom. Therefore the prameters $\hat{\mu}$ and $\hat{\Theta}$ that depend on $\hat{\alpha}$ and $\hat{\beta}$ are very satisfactory. As mentioned earlier, the first component of the Var (x_i / n_i) represents the usual binomial variation, Var(q_i), and the second component represents the additional variation or the extra-binomial variation. **Table (D.3)** shows the values of binomial variation and the extra variation for each month in the State of Qatar mortality data. It is clear from this table that the additional variation accounts for more than 99%, for almost all cases, of the total variability.

We have seen that, for the State of Qatar data, the binomial portion of the total variability in monthly mortality rates over time, (1984 - 1990), was very small, and that a much better fit was obtained under a beta binomial (or negative binomial) model, in which the additional variation is characterized by the parameter Θ. In order to gain a better understanding of the phenomenon of extra-binomial variation, an empirical investigation was undertaken to estimate the beta binomial parameters for the mortality data over time for total male and total female as well.

In **Table (1)**, $\hat{\alpha}$, $\hat{\beta}$ represent the maximum likelihood estimates of the beta binomial parameters, for each group. Similarly $\hat{\mu}$ represents the estimated beta binomial parameter and $\hat{\Theta}$ the measure of the extra-binomial variation for each group. **Table (2)** shows S.E ($\hat{\alpha}$), S.E ($\hat{\beta}$), S.E ($\hat{\mu}$) and S.E ($\hat{\Theta}$). Then, the goodness of fit for the beta binomial and for the binomial model were run in each of the data sets. The results are compared in **Table (3)**. It is quite clear that the binomial model is unsatisfactory. By contrast, none of the χ^2 values for fitted beta binomial is significant. It is obvious that beta-binomial fits the data quite well.

Table (1)

Maximum Likelihood For Beta Binomial Parameters (α, β) The Expected Mean (μ) and The Measure of the Extra Binomial Variation (Θ), For Qatar Mortality Data (1984 - 1990)

Group	α	β	μ	Θ
Total Males	1.2937 E- 01	700.8911	2.24534 E-03	1.73309E- 02
Total Females	1.2987 E- 01	700.8784	2.24534 E- 02	1.73309E- 02
Total Population	1.2971 E- 01	700.7864	2.25154 E- 03	1.73335E- 02

Table (2)

The Standard Error For The Parameters α, β, μ,Θ

For Qatar Mortality Data (1984 - 1990)

Group	SE (α)	SE (β)	SE (μ)	SE (Θ)
Total Males	1.47566 E- 01	2.18208 E + 02	2.299485 E- 04	4.44089 E- 04
Total Females	1.47878 E- 01	2.14179 E + 02	2.245058 E- 04	4.35878 E- 04
Total Population	1.42096 E- 01	6.76359 E + 02	2.207805 E- 04	4.37675 E- 04

Table (3)

Comparaison of Chi - Square Values For

Beta-Binomial and Binomial Distributions

Group	X^2 for Beta - Binomoial	X^2 for Binomial
Total Males	2.8511 E + 00	1.25375 E + 03
Total Females	5.5327 E + 00	2.51535 E + 03
Total Population	2.3447 E + 00	1.53509 E + 03
Tabulated x^{2} **	1.03152 E + 00	1.04426 E + 03

(** Note that degress of freedom for χ^2 when testing beta- binomial parameters = 84-1-2= 81, since the estimated parameters are two, where the of for χ^2 when testing beta- binomial parameters = 84-1-1= 82, since the estimated parameters are one).

We will next consider the method of quasi likelihood estimates, defined and introduced by [26]. In this method we do not have to specify the form of distribution of the observations, but need only specify the relation between the mean and variance of the observations and we could define quasi likelihood and use it for estimation. To examine the extra-binomial variation let us assume as [27] does, that the q_is are independently distributed on (0, 1) with $E(q_i) = \mu$, $Var(q_i) = \Theta\mu(1 - \mu)$ and assume that, x_i conditionally binomial $B_i(n_i, q_i)$ so that unconditionally $E(x_i) = n_i \mu$ and $Var(x_i) = n_i \mu (1 - \mu) \{1 + \Theta (n_i - 1)\}$. This model is fitted by the Gauss-Newton method and applied to the State of Qatar mortality data 1984 - 1990. It was found that, using this iterative procedure that $\hat{\mu}$ = 2.117588E-03 and $\hat{\Theta}$ = 2.7878E-03 for Qatar data. Williams's method estimates the unknown parameters in the linear predictor for a given value of Θ. The method uses a common estimate of Θ obtained through making the generalized chi-square statistic for the "maximal" model equal to its degrees of freedom

Testing for goodness of fit using the State of Qatar mortality data, x^2 = 82.9967 which is very close to k - 1 = 83 the degrees of freedom, and this implies that the estimates μ and Θ are very satisfactory. Note that Θ the measure of extra-binomial variation, is smaller than Θ under MLE method of estimation. Solving the equations (20, 21) using Qatar data for eighty eight month, we again estimate the parameters μ and Θ for total males and total females.

Results are presented in column 2 and 3 of **Table (4).** Column 4 in the table presents the value of (x^2) for the test for goodness of fit, where column 5 represents the difference between 83, number of degrees of freedom, and (x^2). Since the values of (x^2)is very close to the number of the degrees of freedom, we can conclude that the estimation of the parameters is very acceptable.

Table (4)

The Quasi- likelihood Parameters μ, Θ and The Test For Goodness of Fit X^2

For Qatar Mortality Data (1984 - 1990)

Year	μ	Θ	X^2	$83 - X^2$
Total Males	2.010863E-03	2.77568 E-03	83.0017	- 0.0017
Total Females	2.175893E-03	2.82782 E-03	829986	+ 0.0014
Total Population	2.117588E-03	2.78785 E-03	82.9967	+ 0.0033

5 - Summary and conclusions:

It was observed that variability in mortality rates over times for large populations is not of binomial nature. The approach has been to assume that this excessive variability is caused by heterogeneity arising from the parameters q_i of the product of the probability density Functions of beta and binomial. This distribution has many names, possibly the most descriptive name is the beta-binomial distribution.

Furthermore, the variance can be separated into a binomial component and an extra-binomial component. The aim of this paper has been then to characterize this extra-binomiality by means of the estimation of the parameter Θ (i.e. the measure of the extra-binomial variability), that depends on α and β the parameters of the beta-binomial distribution. The procedure for estimating this parameter was that of maximum likelihood estimation. An iterative method was used to find the maximum likelihood estimates using mortality data. Previous discussions in the literature of

estimation procedures for the parameters of the beta-binomial distribution have not dealt with unequal n_i or such large samples, as was done here.

Due to the typically small mortality rate and the large population size which were considered, the beta-binomial model was approximated by a form of the negative binomial distribution. For this form of approximation of beta-binomial, contrary to the binomial for which the mean exceeds the variance, the variance is greater than the mean.

Variability in monthly mortality rates for the total males and total females for the same period of time (1984 - 1992) were examined with the model too. The most interesting finding in this study lies in characterizing the excess variability, or the extra-binomiality, via the compound distribution. Another notable finding, for the total population mortality data, the total males and the total females mortality data, was that estimates of the parameter β were remarkably homogeneous for each analysis. That is, the relative degree of extra binomiality was very much the same from month to month regardless of the population size or the number of months these analyses deal with. The finding supports our initial assumption that the unobservable binomial parameters are random variables that follow beta distribution.

The motivation for choosing beta distribution was, as discussed earlier, in section 2, that beta distribution has a variety of shapes. These shapes are very similar for each analysis. It is significant that our investigation supports this finding though it has adopted a different modeling approach, hence a different analysis by estimating the parameters of the beta binomial distribution using the actual mortality data and calculating the monthly mortality rates as $E(x_i/n_i)$. By arriving at the same results arrived at by the most established methods in the previous studies (i.e. the adjusted mortality rates or the standardized mortality rates) we add credibility to our approach. Thus our study adds a new model for and method of estimating yearly mortality rates.

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Appendix (A)

Computational expressions for compounding the variance:

Let x_i be a binomial random variable, i.e. $B_i (n_i, q_i)$ and let $q_i = x_i/n_i$, i =1, 2, 3, ..., k, where k is the number of the independent random variables q_i. Thus the mean and the variance of q_i, given q_is are $E(q_i/q) = q$ and var $(q_i/q_i) = q_i (1- q_i)/n_i$. For the unobservable q_i we specify no distribution but only label the moments as $E(q_i) = \mu$ and $VAR(q_i) = V_q$. Suppose we consider the unconditional variance, Var (q_i), is $\sigma^2\mu/n_i$ where $\sigma^2\mu = \Sigma n_i (q_i - \mu)^2 / (k-1)$ is an unbiased estimator. We can find the expected value of the unconditional variance and the expected value of the variance of q_i are:

$$var(q_i) = E\{E[q_i - EE(q_i/q_i)]^2\}$$

$$= E\{E[q_i - E(q_i)]^2\}$$

$$= E\{E[(q_i - q_i) + (q_i - E(q_i)]^2\}$$

$$= E\{E[q_i - q_i]^2\} + 2 E\{E(q_i - q_i) (q_i - E(q_i) \} + E\{ q_i - E(q_i)\}^2$$

$= E\{ q_i(1-q_i)/n_i.\}$ (given)

$+ 0$ (since $E\{ q_i - E(q_i)\} = 0$)

$+ V_q$ (by definition)

Therefore

$$V_q = var(q_i) - E\{ q_i(1-q_i)/n_i.\}$$

and since

$$E\{ q_i(1-q_i)/n_i.\} = (1/n_i)\{E(q_i) - E(q_i^2)\}$$

$$=(1/n_i) \{ \mu - (\mu^2 - V_q)\}$$

Thus

$$V_q = var(q_i) - \{ [\mu - (\mu^2 - V_{q,i})] / n_i \}$$

$$V_q = var(q_i) - \mu (1 - \mu^2) /n_i - V_{q,i} / n_i$$

If n_i is very large, then

$$V_q = \text{var}(q_i) - \mu(1 - \mu^2)/n_i$$

Now estimating Vq, and since $\{1/(k-1)\}\{\Sigma n_i (q_i - \mu)^2/n_i\}$ is an unbiased estimate for var (q_i).

Therefore

$$V_q = \{1/(k-1)\}\{\Sigma n_i(q_i - q)^2/n_i - \mu(1 - \mu^2)/n_i .$$

where $\mu(1 - \mu^2)/n_i$ is a consistent estimate for

$$E\{q_i(1 - q_i)/n_i\}$$

and where $q_i = x_i / n_i$ and $\mu = \Sigma x_i / \Sigma n_i = \Sigma n_i q_i / \Sigma n_i$

Appendix (B)

Computational expressions for approximation of Beta-Binomial

It is important to note the conditions under which the beta-binomial distribution leads to the negative binomial distribution. This relationship can be expressed, examined and verified as follows. Beginning with the beta binomial distribution (3), consider that n is large, μ is very small, and we want the variance of x_i to be moderate. This implies that $\mu = \alpha / (\alpha + \beta)$ goes to zero, and $(n_i / (\alpha + \beta + 1)$ and n_i/β are moderate. Then the limit of this distribution is

$$\lim_{n_i \to \infty} P(x_i = x) = \lim_{n_i \to \infty} \binom{n_i}{x} \frac{\Gamma(\alpha + x)\ \Gamma(\beta + n_i - x)\ \Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + n_i)\ \Gamma(\alpha)\ \Gamma(\beta)}$$

$$= \frac{\Gamma(\alpha + x)}{x!\ \Gamma(\alpha)} \lim_{n_i \to \infty} \frac{\Gamma(n_i + 1)}{\Gamma(n_i - x + 1)} \cdot \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)} \cdot \frac{\Gamma(\beta + n_i - x)}{\Gamma(\alpha + \beta + n_i)}$$

and since $\lim_{n_i \to \infty} \frac{\Gamma(N + a)}{\Gamma(N + b)} = N^{b-a} = 1$ by definition

then $\lim_{n_i \to \infty} \frac{\Gamma(N + a)}{\Gamma(N + b)} = N^{a-b}$

Therefore the limit becomes

$$= \frac{\Gamma(\alpha + x)}{x!\,\Gamma(\alpha)}\; n_i^{x}\; \beta^{\alpha}\; (n_i + \beta)^{-(\alpha+x)}$$

$$= \{n_i/(n_i+\beta)\}^{x}\; \{\beta/(n_i+\beta)\}^{\alpha}\; \{\prod_{j=1}^{x} [(\alpha + j - 1)/j]\}$$

This approximation results in a negative binomial distribtion defined by equation (7).

Appendix (C)

Computational expressions for maximum likelihood and the information matrix

$$L = \prod_{i=1}^{K} \{n_i/(n_i+\beta)\}^{x}\; \{\beta/(n_i+\beta)\}^{\alpha}\; \{\prod_{j=1}^{X_i} [(\alpha + j - 1)/j]\} \qquad \text{(C.1)}$$

Therefore

$$\text{Log } L = \sum_{i=1}^{K} \{\alpha \log \beta - \text{Log } x_i! + x_i \text{ Log } n_i!$$

$$- (\alpha + x)\text{Log}(\beta + n_i) + [\sum_{j=1}^{K} \text{Log}(\alpha + j - 1)]\}. \qquad \text{(C.2)}$$

$$\frac{\partial LogL}{\partial \alpha} = \sum_{i=1}^{k} \{\log \beta - \log (\beta + n_i) + [\sum_{j=1}^{k} 1/(\alpha + j \qquad \text{(C.3)}$$

$$\frac{\partial LogL}{\partial \beta} = \sum_{i=1}^{K} \{\alpha/\beta - (\alpha + x)/(\beta + n_i)\} = 0. \qquad \text{(C.4)}$$

$$\frac{\partial^2 LogL}{\partial \alpha^2} = \sum_{i=1}^{K} \left[\sum_{j=1}^{X_i} 1/(\alpha + j - 1)^2 \right], \qquad \text{(C.5)}$$

$$\frac{\partial^2 LogL}{\partial \beta^2} = - K\alpha /\beta^2 + \sum_{j=1}^{K} (\alpha + x)/(\beta + n_i)^2 \qquad \text{(C.6)}$$

$$\frac{\partial^2 LogL}{\partial \alpha \beta} = K / \beta - \sum_{i=1}^{K} 1/(\beta + n_i) \qquad \text{(C.7)}$$

The information matrix is defind by

$$\Phi = \begin{pmatrix} -E\left(\frac{\partial^2 LogL}{\partial \alpha^2}\right) & -E\left(\frac{\partial^2 LogL}{\partial \alpha \beta}\right) \\ -E\left(\frac{\partial^2 Logl}{\partial \alpha \beta}\right) & -E\left(\frac{\partial^2 LogL}{\partial \beta^2}\right) \end{pmatrix} \qquad \text{(C.8)}$$

and therefore the variances and covariances matrix are given by the inverse of the information matrix Φ^{-1}.

APPENDIX (D)

Table (D.1)

Population Size, Number of Deaths and Death Rates for The State of Qatar

(1984 - 1990)

Year	Month	n_i *	X_i **	$Q_i \times 10^4$ ***
1984	JAN	351191	65	2.17922
	FEB	351797	47	1.74157
	MAR	352383	55	1.83772
	APR	353017	55	1.89556
	MAY	353730	48	1.59772
	JUN	354359	51	1.75105
	JUL	354996	57	1.89053
	AUG	355656	57	1.88702
	SEP	356338	47	1.60475
	OCT	357060	50	1.64877
	NOV	357753	51	1.73444
	DEC	358496	59	1.93776
1985	JAN	359167	84	2.75368
	FEB	359804	67	2.42741
	MAR	360479	67	2.18840
	APR	361158	51	1.71808
	MAY	361924	63	2.04953
	JUN	362599	71	2.38234
	JUL	363311	63	2.04171
	AUG	364050	57	1.84351
	SEP	364775	54	1.80111
	OCT	365501	60	1.93283
	NOV	366189	76	2.52505
	DEC	366927	81	2.59918
1986	JAN	367683	74	2.36968
	FEB	368297	55	1.94670
	MAR	369079	56	1.78640
	APR	369745	74	2.43501
	MAY	370494	68	2.16102
	JUN	371246	71	2.32685
	JUL	372059	63	1.99370
	AUG	372856	63	1.98944
	SEP	373634	60	1.95378
	OCT	374468	58	1.82366
	NOV	375300	73	2.36655
	DEC	376085	69	2.16020

Table (D.1) (Continue)

Year	Month	n_i *	X_i **	$Q_i \times 10^4$ ***
1987	JAN	376799	68	2.06236
	FEB	377399	59	2.03792
	MAR	378218	62	1.93010
	APR	378967	74	2.37576
	MAY	379676	61	1.89168
	JUN	379741	63	2.01848
	JUL	380483	76	2.35185
	AUG	381240	54	1.66773
	SEP	382091	59	1.87870
	OCT	382907	64	1.96797
	NOV	383741	74	2.34620
	DEC	384586	76	2.32676
1988	JAN	385356	91	2.78042
	FEB	386088	63	2.12711
	MAR	386902	61	1.85635
	APR	387687	63	1.97711
	MAY	388470	77	2.33380
	JUN	389345	72	2.24993
	JUL	390278	59	1.77996
	AUG	390996	79	2.37895
	SEP	391819	65	2.01836
	OCT	392805	59	1.76850
	NOV	393693	75	2.31780
	DEC	394569	94	2.80502
1989	JAN	395329	93	2.76984
	FEB	396059	61	2.00773
	MAR	396900	71	2.10624
	APR	397707	65	1.98848
	MAY	398579	67	1.97921
	JUN	399391	82	2.49797
	JUL	400164	66	1.94195
	AUG	401025	59	1.73225
	SEP	401904	66	1.99799
	OCT	402883	67	1.95806
	NOV	403788	67	2.01880
	DEC	404630	83	2.41519

Table (D.1) (Continue)

Year	Month	n_i *	X_i **	Q_i x 10^4 ***
1990	JAN	405449	88	2.55551
	FEB	406190	79	2.53532
	MAR	407163	70	2.02423
	APR	408026	63	1.87856
	MAY	408923	63	1.81397
	JUN	409721	65	1.93018
	JUL	410605	63	1.80654
	AUG	411443	76	2.17488
	SEP	412342	68	2.00643
	OCT	413214	70	1.99459
	NOV	414069	77	2.26251
	DEC	414881	89	2.52579

* The population estimation depends only on the 1986 cencus and the natural growth since the monthly or the annual net migration is not available .

** Source: Central Statistical Organization, Vital Statistics, Annual Bulleten (Births and Deaths) 1984 - 1990 .

*** q_i is the annualized monthly mortality rates .

$$\text{or } q_i = \left(\frac{X_i}{n_i}\right)\left(\frac{360}{D_m}\right)$$

where D_m represents the number of days in each month .

Table (D.2)

Binominal and Extra - Binominal Variatibility

for the State of Qatar Mortality Data

Year	Month	Var (q_i)	B.V *	$Vq_{;i}$ **	Per ***
1984	JAN	9.27470 E- 08	6.19170 E- 09	8.68014 E- 08	9.33241 E- 01
	FEB	9.25873 E- 08	4.94188 E- 09	8.66519 E- 08	9.46625 E- 01
	MAR	9.24333 E- 08	5.20554 E- 09	8.65078 E- 08	9.43683 E- 01
	APR	9.22673 E- 08	5.35942 E- 09	8.63524 E- 08	9.41914 E- 01
	MAY	9.20813 E- 08	4.50956 E- 09	8.61783 E- 08	9.51026 E- 01
	JUN	9.19179 E- 08	4.93280 E- 09	8.60254 E- 08	9.46335 E- 01
	JUL	9.17529 E- 08	5.31543 E- 09	8.58710 E- 08	9.42068 E- 01
	AUG	9.15827 E- 08	5.29573 E- 09	8.57117 E- 08	9.42175 E- 01
	SEP	9.14074 E- 08	4.49622 E- 09	8.55476 E- 08	9.50811 E- 01
	OCT	9.12226 E- 08	4.61001 E- 09	8.53746 E- 08	9.49464 E- 01
	NOV	9.10459 E- 08	4.83974 E- 09	8.52093 E- 08	9.46843 E- 01
	DEC	9.08572 E- 08	5.39477 E- 09	8.50327 E- 08	9.40624 E- 01
1985	JAN	9.06874 E- 08	7.64574 E- 09	8.48738 E- 08	9.15691 E- 01
	FEB	9.05269 E- 08	6.73010 E- 09	8.47235 E- 08	9.25656 E- 01
	MAR	9.03574 E- 08	6.05753 E- 09	8.45649 E- 08	9.32960 E- 01
	APR	9.01875 E- 08	4.74897 E- 09	8.44059 E- 08	9.47343 E- 01
	MAY	8.99966 E- 08	5.65127 E- 09	8.42273 E- 08	9.37206 E- 01
	JUN	9.98291 E- 08	6.55453 E- 09	8.40705 E- 08	9.27033 E- 01
	JUL	8.96530 E- 08	5.60826 E- 09	8.39057 E- 08	9.37445 E- 01
	AUG	8.94710 E- 08	5.05456 E- 09	8.37354 E- 08	9.43506 E- 01
	SEP	8.92932 E- 08	4.92870 E- 09	8.35690 E- 08	9.44803 E- 01
	OCT	8.91158 E- 08	5.27794 E- 09	8.34030 E- 08	9.40774 E- 01
	NOV	8.89462 E- 08	6.87790 E- 09	8.32442 E- 08	9.22673 E- 01
	DEC	8.87695 E- 08	7.06523 E- 09	8.30788 E- 08	9.20409 E- 01
1986	JAN	8.85870 E- 08	6.42963 E- 09	8.29080 E- 08	9.27420 E- 01
	FEB	8.84393 E- 08	5.27539 E- 09	8.27698 E- 08	9.40350 E- 01
	MAR	8.82519 E- 08	4.83175 E- 09	8.25944 E- 08	9.45250 E- 01
	APR	8.80929 E- 08	6.56961 E- 09	8.24456 E- 08	9.25424 E- 01
	MAY	8.79149 E- 08	5.82020 E- 09	8.22790 E- 08	9.33797 E- 01
	JUN	8.77368 E- 08	6.25309 E- 09	8.21123 E- 08	9.28729 E- 01
	JUL	8.75451 E- 08	5.34788 E- 09	8.19329 E- 08	9.38913 E- 01
	AUG	8.73579 E- 08	5.32506 E- 09	8.17577 E- 08	9.39043 E- 01
	SEP	8.71760 E- 08	5.21891 E- 09	8.15875 E- 08	9.40134 E- 01
	OCT	8.69819 E- 08	4.86112 E- 09	8.14058 E- 08	9.44113 E- 01
	NOV	8.67890 E- 08	6.29083 E- 09	8.12253 E- 08	9.27516 E- 01
	DEC	8.66079 E- 08	5.73151 E- 09	8.10558 E- 08	9.33822 E- 01

Table (D.2) (Continu

Year	Month	Var (q_i)	B.V *	$Vq_{,i}$ **	Per ***
1987	JAN	8.64438 E- 08	5.46208 E- 09	8.09022 E- 08	9.36813 E- 01
	FEB	8.63063 E- 08	5.38890 E- 09	8.07736 E- 08	9.37561 E- 01
	MAR	8.61195 E- 08	5.09329 E- 09	8.05987 E- 08	9.40858 E- 01
	APR	8.59492 E- 08	6.25415 E- 09	8.04394 E- 08	9.27234 E- 01
	MAY	8.57887 E- 08	4.97293 E- 09	8.02892 E- 08	9.42033 E- 01
	JUN	8.57741 E- 08	5.30468 E- 09	8.02754 E- 08	9.38155 E- 01
	JUL	8.56068 E- 08	6.16668 E- 09	8.01189 E- 08	9.27965 E- 01
	AUG	8.54368 E- 08	4.36719 E- 09	7.99598 E- 08	9.48884 E- 01
	SEP	8.52465 E- 08	4.90765 E- 09	7.97817 E- 08	9.42430 E- 01
	OCT	8.50649 E- 08	5.12944 E- 09	7.96117 E- 08	9.39700 E- 01
	NOV	8.48800 E- 08	6.09967 E- 09	7.94386 E- 08	9.28138 E- 01
	DEC	8.46935 E- 08	6.03596 E- 09	7.92641 E- 08	9.28732 E- 01
1988	JAN	8.45243 E- 08	7.19514 E- 09	7.91057 E- 08	9.14875 E- 01
	FEB	8.43640 E- 08	5.49767 E- 09	7.89557 E- 08	9.34834 E- 01
	MAR	8.41865 E- 08	4.78908 E- 09	7.87896 E- 08	9.43113 E- 01
	APR	8.40160 E- 08	5.08968 E- 09	7.86301 E- 08	9.39420 E- 01
	MAY	8.38467 E- 08	5.99365 E- 09	7.84716 E- 08	9.28517 E- 01
	JUN	8.36583 E- 08	5.76575 E- 09	7.82953 E- 08	9.31080 E- 01
	JUL	8.34583 E- 08	4.55263 E- 09	7.81081 E- 08	9.45450 E- 01
	AUG	8.33050 E- 08	6.06986 E- 09	7.79646 E- 08	9.27137 E- 01
	SEP	8.31300 E- 08	5.14086 E- 09	7.78009 E- 08	9.38159 E- 01
	OCT	8.29214 E- 08	4.49427 E- 09	7.76056 E- 08	9.45801 E- 01
	NOV	8.27343 E- 08	5.87368 E- 09	7.74306 E- 08	9.29005 E- 01
	DEC	8.25507 E- 08	7.08913 E- 09	7.72586 E- 08	9.14124 E- 01
1989	JAN	8.23920 E- 08	6.98701 E- 09	7.71101 E- 08	9.15198 E- 01
	FEB	8.22401 E- 08	5.05909 E- 09	7.69680 E- 08	9.38484 E- 01
	MAR	8.20658 E- 08	5.29555 E- 09	7.68049 E- 08	9.35472 E- 01
	APR	8.18993 E- 08	4.98992 E- 09	7.66491 E- 08	9.39073 E- 01
	MAY	8.17201 E- 08	4.95584 E- 09	7.64814 E- 08	9.39356 E- 01
	JUN	8.15540 E- 08	6.23882 E- 09	7.63259 E- 08	9.23501 E- 01
	JUL	8.13964 E- 08	4.84346 E- 09	7.61784 E- 08	9.40495 E- 01
	AUG	8.12217 E- 08	4.31207 E- 09	7.60149 E- 08	9.46910 E- 01
	SEP	8.10440 E- 08	4.96138 E- 09	7.58486 E- 08	9.38782 E- 01
	OCT	8.08471 E- 08	4.85060 E- 09	7.56643 E- 08	9.40003 E- 01
	NOV	8.06659 E- 08	4.98956 E- 09	7.54947 E- 08	9.38145 E- 01
	DEC	8.04981 E- 08	5,95447 E- 09	7.53376 E- 08	9.26030 E- 01

Table (D.2) (Continue)

Year	Month	Var (q_i)	B.V *	$Vq_{,i}$ **	Per ***
1990	JAN	8.03354 E- 08	6.28681 E- 09	7.51855 E- 08	9.21743 E- 01
	FEB	8.01889 E- 08	6.22588 E- 09	7.50483 E- 08	9.22360 E- 01
	MAR	7.99973 E- 08	4.96148 E- 09	7.48689 E- 08	9.37979 E- 01
	APR	7.98281 E- 08	4.59537 E- 09	7.47106 E- 08	9.42434 E- 01
	MAY	7.96530 E- 08	4.42792 E- 09	7.45467 E- 08	9.44410 E- 01
	JUN	7.94978 E- 08	4.70187 E- 09	7.44015 E- 08	9.40855 E- 01
	JUL	7.93267 E- 08	4.39175 E- 09	7.42413 E- 08	9.44637 E- 01
	AUG	7.91651 E- 08	5.27448 E- 09	7.40901 E- 08	9.33374 E- 01
	SEP	7.89925 E- 08	4.85617 E- 09	7.39286 E- 08	9.38524 E- 01
	OCT	7.88258 E- 08	4.81739 E- 09	737726 E- 08	9.38886 E- 01
	NOV	7.86630 E- 08	5.45173 E- 09	7.36203 E- 08	9.30695 E- 01
	DEC	7.85091 E- 08	6.07261 E- 09	7.34762 E- 08	9.22651 E- 01

B.V = [q(1-q) / ni] represents the binominal variation .

Vq,i = [var (qi) - q (1- q) / ni] represents the extra variation .

Per represents percentage of the extra binominal variation to the total variatibility .

Table (D.3)

Binominal and Extra - Binominal Variatibility using beta - binominal parameters for the state of Qatar mortality data

Year	Month	B. V	Extra V*	Per
1984	JAN	5.26191 E- 10	2.63597 E- 07	9.98008 E- 01
	FEB	5.25285 E- 10	2.63597 E- 07	9.98011 E- 01
	MAR	5.24412 E- 10	2.63597 E- 07	9.98015 E- 01
	APR	5.23470 E- 10	2.63597 E- 07	9.98018 E- 01
	MAY	5.22415 E- 10	2.63597 E- 07	9.98022 E- 01
	JUN	5.21487 E- 10	2.63597 E- 07	9.98026 E- 01
	JUL	5.20552 E- 10	2.63597 E- 07	9.98029 E- 01
	AUG	5.19586 E- 10	2.63597 E- 07	9.98033 E- 01
	SEP	5.18591 E- 10	2.63597 E- 07	9.98036 E- 01
	OCT	5.17542 E- 10	2.63597 E- 07	9.98040 E- 01
	NOV	5.16340 E- 10	2.63597 E- 07	9.98044 E- 01
	DEC	5.15469 E- 10	2.63597 E- 07	9.98048 E- 01
1985	JAN	5.14506 E- 10	2.63597 E- 07	9.98052 E- 01
	FEB	5.13595 E- 10	2.63597 E- 07	9.98055 E- 01
	MAR	5.12634 E- 10	2.63597 E- 07	9.98059 E- 01
	APR	5.11670 E- 10	2.63597 E- 07	9.98063 E- 01
	MAY	5.10587 E- 10	2.63597 E- 07	9.98067 E- 01
	JUN	5.09637 E- 10	2.63597 E- 07	9.98070 E- 01
	JUL	5.08638 E- 10	2.63597 E- 07	9.98074 E- 01
	AUG	5.07605 E- 10	2.63597 E- 07	9.98078 E- 01
	SEP	5.06596 E- 10	2.63597 E- 07	9.98082 E- 01
	OCT	5.05590 E- 10	2.63597 E- 07	9.98086 E- 01
	NOV	5.04628 E- 10	2.63597 E- 07	9.98089 E- 01
	DEC	5.03625 E- 10	2.63597 E- 07	9.98093 E- 01
1986	JAN	5.02590 E- 10	2.63597 E- 07	9.98097 E- 01
	FEB	5.01752 E- 10	2.63597 E- 07	9.98100 E- 01
	MAR	5.00689 E- 10	2.63597 E- 07	9.98104 E- 01
	APR	4.99787 E- 10	2.63597 E- 07	9.98108 E- 01
	MAY	4.98777 E- 10	2.63597 E- 07	9.98111 E- 01
	JUN	4.97766 E- 10	2.63597 E- 07	9.98115 E- 01
	JUL	4.96679 E- 10	2.63597 E- 07	9.98119 E- 01
	AUG	4.95617 E- 10	2.63597 E- 07	9.98123 E- 01
	SEP	4.94585 E- 10	2.63597 E- 07	9.98127 E- 01
	OCT	4.93483 E- 10	2.63597 E- 07	9.98131 E- 01
	NOV	4.92389 E- 10	2.63597 E- 07	9.98136 E- 01
	DEC	4.91362 E- 10	2.63597 E- 07	9.98139 E- 01

Table (D.3) (Continue)

Year	Month	B. V	Extra V*	Per
1987	JAN	4.90430 E- 10	2.63597 E- 07	9.98143 E- 01
	FEB	4.89651 E- 10	2.63597 E- 07	9.98146 E- 01
	MAR	4.88590 E- 10	2.63597 E- 07	9.98150 E- 01
	APR	4.87625 E- 10	2.63597 E- 07	9.98154 E- 01
	MAY	4.86714 E- 10	2.63597 E- 07	9.98157 E- 01
	JUN	4.86631 E- 10	2.63597 E- 07	9.98157 E- 01
	JUL	4.85682 E- 10	2.63597 E- 07	9.98161 E- 01
	AUG	4.84718 E- 10	2.63597 E- 07	9.98165 E- 01
	SEP	4.83638 E- 10	2.63597 E- 07	9.98169 E- 01
	OCT	4.82607 E- 10	2.63597 E- 07	9.98172 E- 01
	NOV	4.81558 E- 10	2.63597 E- 07	9.98176 E- 01
	DEC	4.80500 E- 10	2.63597 E- 07	9.98180 E- 01
1988	JAN	4.79540 E- 10	2.63597 E- 07	9.98184 E- 01
	FEB	4.78631 E- 10	2.63597 E- 07	9.98188 E- 01
	MAR	4.77624 E- 10	2.63597 E- 07	9.98191 E- 01
	APR	4.76657 E- 10	2.63597 E- 07	9.98195 E- 01
	MAY	4.75696 E- 10	2.63597 E- 07	9.98199 E- 01
	JUN	4.74627 E- 10	2.63597 E- 07	9.98203 E- 01
	JUL	4.73493 E- 10	2.63597 E- 07	9.98207 E- 01
	AUG	4.72623 E- 10	2.63597 E- 07	9.98210 E- 01
	SEP	4.71630 E- 10	2.63597 E- 07	9.98214 E- 01
	OCT	4.70446 E- 10	2.63597 E- 07	9.98218 E- 01
	NOV	4.69385 E- 10	2.63597 E- 07	9.98222 E- 01
	DEC	4.68343 E- 10	2.63597 E- 07	9.98226 E- 01
1989	JAN	4.67443 E- 10	2.63597 E- 07	9.98230 E- 01
	FEB	4.66581 E- 10	2.63597 E- 07	9.98233 E- 01
	MAR	4.65593 E- 10	2.63597 E- 07	9.98237 E- 01
	APR	4.64648 E- 10	2.63597 E- 07	9.98240 E- 01
	MAY	4.63631 E- 10	2.63597 E- 07	9.98244 E- 01
	JUN	4.62689 E- 10	2.63597 E- 07	9.98248 E- 01
	JUL	4.61795 E- 10	2.63597 E- 07	9.98251 E- 01
	AUG	4.60803 E- 10	2.63597 E- 07	9.98255 E- 01
	SEP	4.59796 E- 10	2.63597 E- 07	9.98259 E- 01
	OCT	4.58678 E- 10	2.63597 E- 07	9.98263 E- 01
	NOV	4.57650 E- 10	2.63597 E- 07	9.98267 E- 01
	DEC	4.56698 E- 10	2.63597 E- 07	9.98270 E- 01

Table (D.3) (Continue)

Year	Month	B. V	Extra V*	Per
1990	JAN	4.55775 E- 10	2.63597 E- 07	9.98274 E- 01
	FEB	4.54944 E- 10	2.63597 E- 07	9.98277 E- 01
	MAR	4.53857 E- 10	2.63597 E- 07	9.98281 E- 01
	APR	4.52897 E- 10	2.63597 E- 07	9.98285 E- 01
	MAY	4.51903 E- 10	2.63597 E- 07	9.98289 E- 01
	JUN	4.51023 E- 10	2.63597 E- 07	9.98292 E- 01
	JUL	4.50052 E- 10	2.63597 E- 07	9.98296 E- 01
	AUG	4.49136 E- 10	2.63597 E- 07	9.98299 E- 01
	SEP	4.48156 E- 10	2.63597 E- 07	9.98303 E- 01
	OCT	4.47211 E- 10	2.63597 E- 07	9.98306 E- 01
	NOV	4.46287 E- 10	2.63597 E- 07	9.98310 E- 01
	DEC	4.45154 E- 10	2.63597 E- 07	9.98313 E- 01

* Extra $V_{\cdot} = \mu(1-\mu)\Theta(1-1/n_i)$ represents the extra variation .