The Multicommodity Multiperiod Assignment Problem 11: Theoretical Results

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Abstract

The multiperiod assignment problem is an important specialization of the three dimensional assignment problem, which is a generalization of the classical (two dimensional) assignment problem. This model describes the optimization problem of assigning people to activities (jobs) over several time periods. In the most general case, there is a cost of assigning a person to an activity in each time period, and a cost of transferring a person from one activity in each period to another activity in the following period. The number of time periods is not restricted to equal the number of persons and activities.

We focus on the special characteristics and properties of noninteger solutions to the linear programming relaxation of the multicommodity, multiperiod assignment problem. The propositions highlight special conditions that must be present for any solution to the problem to be fractional, basic and fesible.

KEY WORDS:

programming : Integer

Networks/Graphs

Programming: Assignment

Manpower Planning

Introduction:

In [6], we presented the multicommodity, multiperiod assignment problem (MCMAP). The multiperiod assignment problem was formulated as an integer multicommodity network flow problem. A specialized branch and bound algorithm was developed and implemented as MAP. Tested problems illustrated superiority of the algorithm over the commercial mixed-integer programming package MPSX/MIP/370 [11]. A great deal of model understanding can be realized by considering the linear programming relaxation of MCMAP. Here, we focus on the special characteristics and properties of noninteger solutions to the linear programming relaxation of the multicommodity, multiperiod assignment problem. Special properties of the basis matrix for the linear programming relaxation LMCMAP are discussed. The propositions highlight special conditions that must be present for any solution to the problem to be fractional, basic and feasible. The terms fractional solution and noninteger solution refer to the noninteger assignment of flow through arcs of the networks of the n commodities of LMCMAP.

Of special interest are basic cycles. A cycle in the network of commodity i is said to be a basic cycle if it is formed entirely by basic arcs corresponding to a basic feasible solution to LMCMAP. Unless otherwise specified, a cycle of commodity j, is formed by arcs with positive flow assignments. This however, does not apply to basic cycles. The graphical representations of trees and cycles are given with respect to the node labeling scheme of the networks of the model presented in [6]. Note that no two assignment arcs in the networks of LMCMAP are incident to any single node, except for the source and sink arcs. Therefore, no cycle may be formed entirely by assignment arcs. This will be discussed in Proposition 3. The network of commodity i of a 3 by 4 multicommodity, multiperiod assignment problem is illustrated in Figure 1. The basic arcs and cycles of the three commodities corresponding to a basic feasible solution to the linear programming relaxation of the 3 by 4 problem are presented in Figures 2, 3 and 4.

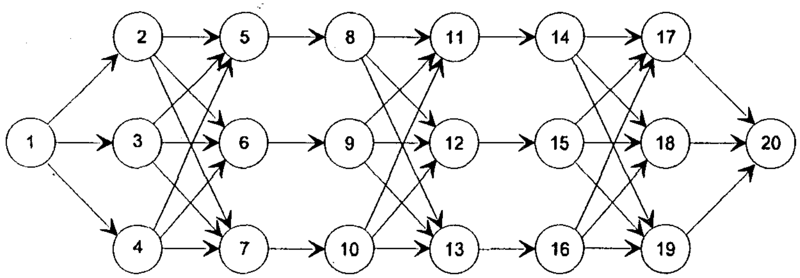

Figure 1: The Network of Commodity i of a 3 by 4 Multicommodity, Multiperiod Assignment Problem, for i = 1,...,n.

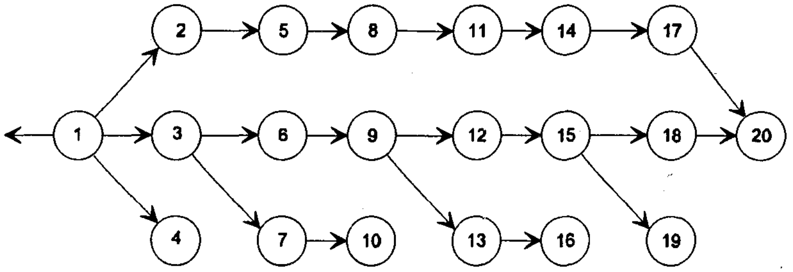

Figure 2: The Basic Arcs of Commodity 1 Corresponding to a Basic Feasible Solution to the Linear Programming Relaxation of the 3 by 4 Multicommodity, Multiperiod Assignment Problem.

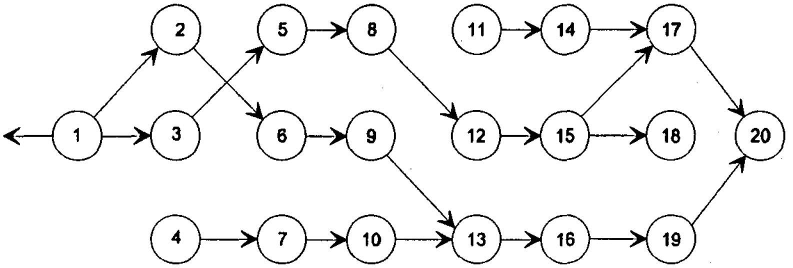

Figure 3: The Basic Arcs of Commodity 2 Corresponding to a Basic Feasible Solution to the Linear Programming Relaxation of the 3 by 4 Multicommodity, Multiperiod Assignment Problem.

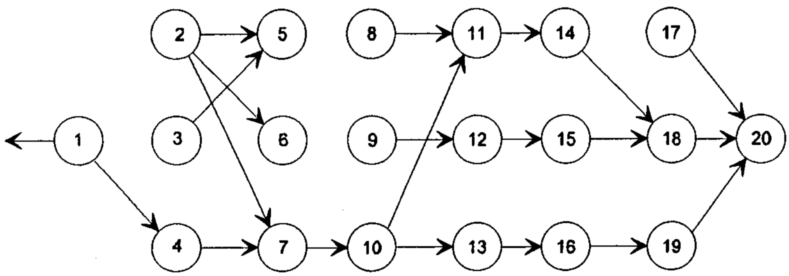

Figure 4: The Basic Arcs of Commodity 3 Corresponding to a Basic Feasible Solution to the Linear Programming Relaxation of the 3 by 4 Multicommodity, Multiperiod Assignment Problem.

For motivational details on the model and its application, see [1, 4, 5, 6, 7]. For related work, see [8, 9, 10, 14]. For standard network terminology and definitions, see [2, 3, 12, 13].

2. The Mathematical Model

For a full derivation and definition of the model see [6]. Let be the flow through arc . The Multicommodity, multiperiod, assignment problem, MCMAP, may be stated mathematical as follows:

$$\text{(MCMAP)} \quad Min \sum_{i=1}^{n} \sum_{\in (s,v)E^i} c^i_{sv}\, x^i_{sv} \tag{1}$$

subject to :

$$\sum_{\in SN^i_E} x^i_{1s} = 1 \; ; \; i = 1, \ldots, n, \tag{2}$$

$$x^i_{1j} - \sum_{\Sigma \in N^2_B} x^i_{js} = 0 \; ; \; i = 1, \ldots, n \; ; \; j = 1, \ldots, n, \tag{3}$$

$$\sum_{S \in N_E^{t-1}} x_{sj}^i - x_{j,j+n}^i \qquad = 0 \; ; \; i = 1, \ldots, n \qquad (4)$$

$$; \; j \in N_B^t$$
$$; \; t = 2, \ldots, T-1,$$

$$x_{j-n,j}^i - \sum_{S \in N_B^{t+1}} x_{js}^i \qquad = 0 \; ; \; i = 1, \ldots, n \qquad (5)$$

$$; \; j \in N_E^t$$
$$; \; t = 2, \ldots, T-1,$$

$$\sum_{S \in N_E^{T-1}} x_{sj}^i - \qquad = 0; \; i = 1, \ldots, n \qquad (6$$
$$; \; j \in N_B^T$$
$$; \; p \in SK,$$

$$- \sum_{S \in N_B^T} x_{sp}^i \qquad = -1 \; ; \; i = 1, \ldots, n \qquad (7)$$

$$; \; p \in Sk,$$

$$\sum_{i=1}^{n} x_{sv}^i \qquad \leq 1 \; ; \; (s,v) \in E_A. \qquad (8)$$

$$x_{sv}^i \geq 0, \text{ integer} \qquad ; \; i = 1, \ldots, n \qquad (9)$$

$$; \; (s,v) \in E^i$$

The objective function (1) is to be minimized. Constraint sets (2) through (7) are the conservation of flow relations. When T=2, constraint set (4) is dropped from the problem. Constraint set (8) is the mutual capacity relation son the assignment arcs. Constraint set (9) imposes the nonnegativity and integrality conditions associated with flow assignments. There are n[2+2n(T-1)] conservation of flow constraints and nT mutual arc capacity constraints. By definition, the n networks of n commodities are identical, directed acyclic networks. Every path in the network of commodity is directed away from the source node and toward the sing node, for i = 1, ... , n. We define LMCMAP to be the linear programming relaxation of MCMAP. The special structure of the constraint matrix of LMCMAP can be viewed by rearranging its columns so that they take the form illustrated in Figure 5.

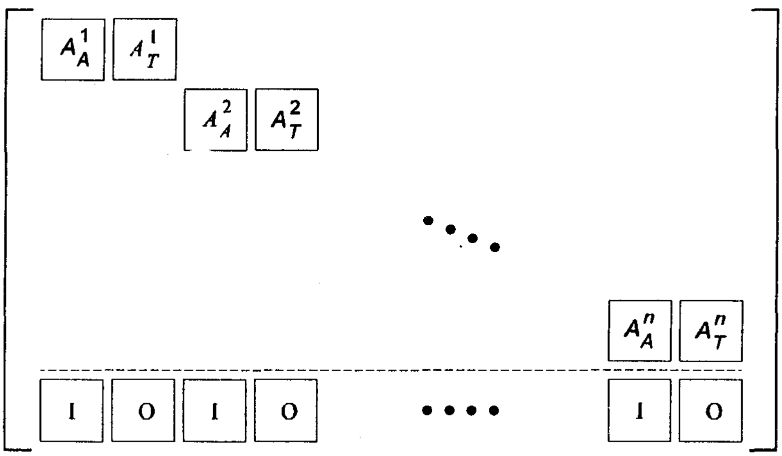

Figure 5: The Constraint Matrix of the Multicommodity Multiperiod Assignment Problem. The columns AA correspond to assignment arcs of commodity i; the columns AT correspond to transfer arcs of commodity i. The identity matrix I is of order nT, and matrix 0 consists of zeros.

3. Theoretical Results

The following propositions define necessary conditions for the existence of noninteger basic feasible solutions to LMCMAP.

Proposition 1:

In every fractional feasible solution to LMCMAP, arcs of commodity i that are assigned fractional flows form at least two unique paths from the source node to the sink node.

Proof:

Assume that arc $(s,v)^i$ of period t, is assigned fractional flow. Arc $(s,v)^i$ is either an assignment arc or a transfer arc. Let X^i_{sv} be the flow assigned to arc $(s,v)^i$. One of the following two cases must hold:

Case 1: Arc $(s,v)^i$ is an assignment arc. To satisfy the flow conservation constraints, the total flow assigned to all assignment arcs of period t in commodity i must equal 1. That

is, (10) must be true:

$$\sum_{(s,v)^i \in E^t_A} x^i_{sv} = 1 \tag{10}$$

where $s \in N^t_B,\ v \in N^\tau_E$.
By (10) and the assumption that,

$$0 < x^i_{sv} < 1 \tag{11}$$

the total flow assigned to the assignment arcs of period in commodity i, excluding the flow assigned to $(s,v)^i$, is also a positive fraction that must be assigned to at least one other assignment arc in period, t, say arc $(u,w)^i$ This generates at least one more unique path from the source node to the sink node. Thus, there must exist a minimum of two unique paths from the source node to the sink node.

Case 2: Arc $(s,v)^i$ is a transfer arc. A similar aargument as in Case 1 applies, subject to (12) below.

(12)

<u>**Q E D**</u>

An important result that follows from Proposition 1 is the existence of sycles in fractional feasible solutions to LMCMAP.

Corollary 1:

In every fractional feasible solution to LMCMAP, there exists at least one cycle formed by arcs with positive fractional flows.

Proof:

The proof follows by noting the definition of the networks of the n commodities and the existence of at least two unique paths from the source node to the sink node, formed by arcs with positive fractional flows, as described in Proposition 1.

QED

Corollary 2:

In every fractional basic feasible solution to LMCMAP, there exists a basic cycle.

Proof:

Because the solution is fractional, by Corollary 1 there exists a cycle that is formed by arcs with fractional flow assignments. By the definition of LMCMAP, an arc with a fractional flow assignment in a basic feasible solution to LMCMAP must be basic. Therefore, there exists at least one basic cycle in every fractional basic feasible solution to LMCMAP.

QED

Corollary 3:

If a fractional flow is assigned to a subset of assignment arcs of commodity i in period t, then that subset consists of at least two assignment arcs.

Proof:

By the discussion presented in the proof of Proposition 1, and by the flow conservation constraints, if one assignment arc in period t is assigned a fractional flow f, $0 < f < 1$, then a total flow of $1-f$ must be assigned to at least one other assignment arc of the same time period.

QED

Another property of basic feasible solutions is that no two basic cycles of two commodities may be identical.

Proposition 2:

There exists no basic feasible solution to LMCMAP such that two basic cycles in two commodities are identical.

Proof:

Assume contradicting the desired result, that there exists a basic feasible solution to LMCMAP such that two basic cycles c^i and c^i in commodities i and j are identical. It will be shown that the corresponding basis is singular.

Recall that A^k is the submatrix of the constraint matrix consisting of columns corresponding to arcs of the network of commodity k. Let z^i and z^i be the submatrices of A^i and A^i that consist of columns corresponding to arcs in cycles c^i and c^i, respectively. Also, let S^j and S^j be the submatrices of the constraint matrix of the problem, consisting of entries from the linking constraint rows corresponding to the assignment arcs in cycles and , respectively. Because the cycles c^i and c^i are identical, $S^j = S^j$. Let S = $S^j = S^j$ Let Z be the matrix consisting of the submatricesZ^j,S^j andS^j. The matrix Z is represented as

$$Z = \left[\begin{array}{c|c} 0 & 0 \\ Z^i & : \\ 0 & 0 \\ 0 & 0 \\ : & Z^j \\ 0 & 0 \\ \hline S & S \end{array}\right] \qquad (13$$

Let z^i_h be a column in z^j , z^i_h be a column in Z^i , and $s^i_h = s^i_h$ be the corresponding column in S. Also, let e^i_h be the arc to which the column z^i_h corresponds. Then, Z may represented as follows:

$$Z = \left[\begin{array}{c|c} 0 & 0 \\ z_1^i \; z_2^i \; \ldots \; z_v^i & \vdots \\ 0 & 0 \\ 0 & 0 \\ \vdots & z_1^j \; z_2^j \; \ldots \; z_v^j \\ 0 & 0 \\ \hline s_1 \; s_2 \; \ldots \; s_v & s_1 \; s_2 \; \ldots \; s_v \end{array}\right] \tag{14}$$

The columns $z^i_{h,}$ where $h = 1, \ldots, v$, are linearly dependent because the correspond to arcs in a cycle. Therefore, it is possible to transform a column z^i_h, for a given h, into a vector of zeros by elementary column operations. This is achieved by multiplying every column of commodity in Z, by the orientation of the corresponding arc, and replacing the column corresponding to arc e^i_h by the sum of these columns.

The same procedure may be applied to any column z^i_h for $h = 1, \ldots, v$. The elementary column operations performed on the columns of the two commodities in Z are shown below:

$$\left[\frac{z^i_v}{s^i_v}\right] = \sum_{k=1}^{v} O\left(e^j_k\right)\left[\frac{z^j_k}{s^i_k}\right] = \left[\frac{o}{w^i}\right] \tag{15}$$

$$\left[\frac{z^j_v}{s^j_s}\right] = \sum_{k=1}^{v} O\left(e^j_k\right)\left[\frac{z^j_k}{s^j_k}\right] = \left[\frac{o}{w^j}\right] \tag{16}$$

Because the cycles c^i are c^i idendtical, $O(e^j_k) = \pm O(e^j_k)$. This implies that $w^i = \pm\, w^j$. Therefore

$$\left[\frac{z_v^i}{s_v^i}\right] = \pm\left[\frac{z_v^j}{s_v^j}\right] = \pm\left[\frac{o}{w}\right] \tag{17}$$

That is, the resulting new columns in Z are linearly dependent.

Because the columns in Z are in the basis matrix, the basis matrix is singular. Thus there exists no basic feasible solution to LMCMAP such that two basic cycles in two commodities are identical

QED

Basic cycles are necessary for the existence of fractional solutions to LMCMAP (Corollary 1). To maintain the linear independence of the columns of a basis with basic cycles, every cycle must consist of assignment arcs and transfer arcs as shown in Propositions 3.

Proposition 3:

In a basic feasible solution to LMCMAP, every basic cycle of commodity i consists of arcs from both sets E^i_T (assignment arcs) and E^i_T (transfer arcs).

Proof:

Assume contradicting the desired result, that there exists a basic cycle c^i of commodity *i*, in some basic feasible solution that is formed entirely by arcs from exactly one of the two sets E^i_A or E^i_T

Case 1: The basic cycle c^i is formed entirely by assignment arcs. This is only possible when the cycle is formed entirely by source and sink arcs, due to the fact that no two assignment arcs are incident to any single node except for the source and sink arcs. This is only possible for a one period problem, which reduces to a standard assignment problem, which is totally unimodular. Therefore, no cycle may be formed entirely by assignment arcs. Additionally, by the model assumptions $T \geq 2$.

Case 2: The basic cycle c^i is formed entirely transfer arcs. It will be shown that the corresponding basis matrix is singular. By definition of LMCMAP, the transfer arcs of the n commodities are not linked by mutual capacity constraints. Therefore, the arcs in c^i are not linked together. Because columns corresponding to arcs in a cycle in a pure network flow problem are linearly dependent, the columns in the basis matrix corresponding to arcs in c^i must be linearly dependent. This implies that the basis matrix must be singular, a contradiction.

Therefore, in a basic feasible solution to LMCMAP, every basic cycle of commodity i must be formed by arcs from both sets E^i_A and E^i_T

QED

Another important property of fractional basic feasible solutions to LMCMAP is:

Proposition 4:

In every fractional basic feasible solution to LMCMAP, assignment arcs with fractional flows must be in basic cycles of at least two commodities.

Proof:

Assume that in a given fractional basic feasible solution, a subset of the assignment arcs of commodity i, of period t are assigned fractional flows. Let arc $(s, v)^i$ be in that subset. Let $x^i_{sv} = f, O < f < 1$. By the Proposition of [6] that states that the assignment mutual arc capacity constraints of LMCMAP are implicit equality constraints [15], the linking constraints on the assignment arcs are always tight.

Therefore

(18)

$h \neq i$

Thus, there exists at least one arc $(s, v)^h$, where $h = 1...., n$ and $h \pm i$, such that $O < x^h_{sv} < 1$. By Corollary 2, fractional flow is allowed only in cycles. The every assignment arc of commodity i that is assigned fractional flow must also be in a cycle of at least one other commodity.

QED

The next result follows immediately:

Corollary 5:

In every fractional feasible solution to LMCMAP, arcs of at least two commodities are assigned fractional flows.

Proof:

The proof follows directly from Proposition 4.

Q E D

Although any of the nT slack variables associated with the nT assignment arcs may basic some restrictions apply when the solution is fractional and feasible, as is shown in the next proposition and corollary.

Proposition 5:

There exists no fractional basic feasible solution to LMCMAP such that all slack variables associated with the mutual arc capacity constraints (8) are basic.

Proof:

Assume contradicting the desired result, that there exists a fractional basic feasible solution in which all slack variables for the linking constraints are basic. Let B be the basis matrix corresponding to that solution. It will be shown that B is singular.

Because all slack variables associated with the linking constraints are basic, there exists all identity submatrix I of order nT in B corresponding to the nT slack variables of the nT assignment arcs. Let N be the submatrix in B consisting of columns corresponding to the basic arcs of the n commodities. Let S be the submatrix in B consisting of the entries in the linking constraint rows under the columns of N. The matrix B may be represented in the form given in (19):

$$B = \left[\begin{array}{cc} N & O \\ \hline S & I \end{array}\right] \qquad (19)$$

The matrix B' may be obtained by performing elementary column operations on B to zero out the entries in S. This is achieved by adding a negative multiple of each column of the slack variable columns to the corresponding columns in

$$\left[\frac{N}{S}\right].$$

The resulting matrix B is given by

$$B' = \left[\frac{N \quad O}{O \quad I}\right] \tag{20}$$

By Propositions 4, the submatrix N contains columns that correspond to arcs in cycles in at least two commodities. Thus there is linear dependence in some of the columns of N. Therefore matrix B' is singular and its determinant is zero. Because B" and B are row equivalent matrices, B' and B have the same determinant. Thus B is singular, which contradicts the assumption that the solution is basic. This implies that there exists no fractional basic feasible solution to LMCMAP such that all slack variables to the mutual capacity constraints (8) are basic.

QED

The above restriction is associated with the basic assignment arcs. Corollary 6 specifies a restriction on slack variables associaed with mutual capacity constraints of assignment arcs in a basic cycle.

Corollary 6:

There exists no fractional basic feasible solution to LMCMAP for which all slack variables, associated with the mutual capacity constraints of the assignment arcs in any basic cycle are basic.

Proof:

By an argument similar to that presented in the proof Proposition 5, it is possible to zero out the entries in the rows of the linking constraints in the columns corresponding to arcs in any basic cycle. Therefore the columns corresponding to the arcs in a basic cycle are linearly dependent. Thus the solution cannot be basic.

QED

The next proposition shows there exists an integer basic feasible solution to every n by T problem. This proposition proves that the branch and bound algorithm presented in [6] to solve MCMAP does converge to an optimal basic feasible solution in a finite number of steps.

Proposition 6:

There exists a basic feasible solution corresponding to every feasible multiperiod assignment.

Proof:

Consider the n by T problem. It will be shown that for every integer feasible solution, a basis matrix may be constructed.

Let person i be assigned to job in period t. By (8), no person is assigned the same job as another in each time period. By (3) and the flow conservation constraints, the arcs of commodity i that are assigned a flow of 1 form a unique path from the source node to the sink node. Let be that path of commodity i, where

$$p^i = \left\{ \begin{array}{c} (1, j_1^i + 1), (j_1^i + 1, J_2^i + q_2), (j_2^i + q_2, j_2^i + q_2 + n), \\ (j_2^i + q_2 + n, j_3^i + q_3), (j_3^i + q_3, j_3^i + q_3 + n), \ldots, \\ (j_{T-1}^i + q_{T-1} + n, j_T^i), (j_T^i + qT, p,)| \\ qt = n(2t-3) + 1; p = 2n(T-1) + 2 \end{array} \right\} \quad (21)$$

See Figure. If $X^{ii}_{sv} = 1$, then $X^{ii}_{sv} = 0$, for h = 1, ..., n, and $h \neq i$, in order to satisfy the linking constraints i.e . $J^1_t \neq J^2_t \neq J^{11}_t \neq$ for $t = 1, \ldots T$. Therefore, in every integer feasible solution to an n by T problem, there exist a total of n paths from the source node to the sink node in the n commodities, where the intersection of these paths is empty.

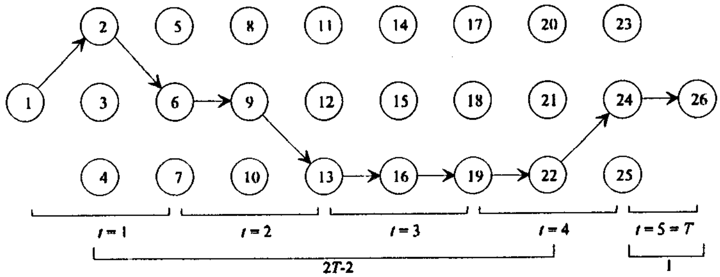

Figure 6: Path in Commodity i of a 3 by 5 Multicommodity, Multiperiod Assignment Problem Formed by Basic Arcs with Positive Flow of 1. There are *(2T-1)* Arcs in P^i

By definition of the networks of the n commodities, consists of (2T-1) arcs. Let T^i be a tree consisting of arcs in and the node set N, where $i = 1, ..., n$. Let the path Y^i be defined as follows:

$$Y^i = P^i \setminus \left\{ (j^i_T + n(2t - 3) +, 2n(T-1) + 2) \right\} \qquad (22)$$

That is, is a path that consists of all but the last arc in . Include the , for i = 1, ... , n , and h –i, to generate n acyclic connected spanning trees. Each tree consists of 2 = 2n(T-1) nodes and 1+2n(T-1) arcs. Root arcs are added to each tree.

Without loss of generality, let the root arc of every tree be an arc directed away from the source node. See Figures 2, 3, and 4 for examples of such trees.

Let B be a submatrix of the constraint matrix of MCMAP, consisting of columns corresponding to arcs in the n trees and the nT columns associated with the nT slack variables of the nT mutual capacity constraints in B. Because there are no cycles in any of the n constructed trees, the columns of B are linearly independent. The matrix B is square with linearly independent columns that correspond to arcs with feasible flow assignments. Therefore B is a basis matrix. Thus there exists a basic feasible solution corresponding to every integer feasible solution to LMCMAP.

QED

The properties outlined in the proof may be applied to any given integer feasible solution to LMCMAP to generate a corresponding basis matrix.

Corollary 7:

There exists an integer basic feasible solution to the linear programming relaxation of MCMAP, LMCMAP.

Proof:

The proof follows immediately form Proposition 6.

QED

3. Concluding Remarks

We have discussed special characteristics and properties of noninteger solutions to the linear programming relaxation of the multicommodity, multiperiod assignment problem. The results indicate that a specialization of a multicommodity network flow algorithm should solve such problems efficiently.

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