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# STABILITY ANALYSIS OF DIGITAL PD CONTROL APPLIED TO FLEXIBLE SYSTEMS

**Fanni, M.** and **Al-Salem, N.**
Mechanical Engineering Department,
University of Qatar,
P. O. Box 2713,
Doha, Qatar

E-mail: mfanni@qu.edu.qa & nabeel@qu.edu.qa

## ABSTRACT

The work shows that the sampling in a digital PD controller of a flexible system may cause instability, while the continuous PD control system is stable. A procedure is introduced to define the stability regions analytically in the gain space. A closed form stability criterion is derived for a system consists of a single rigid mode and one flexible mode. The procedure is applied successfully also for the case of single rigid mode considering the computation time in addition to sampling period. A comparison with the literature shows that this procedure is straightforward, gives more accurate result and even corrects some stability regions published in the literature. A computer program is constructed using MATLAB to determine the stability region for a system that consists of a single rigid mode and arbitrary number of flexible modes. This simulates many practical systems such as flexible robot arms, disk-drive servos and antenna systems. The results show that the stability regions can be approximated by a right triangle in the gain space. Analytical expressions are derived to define such triangles.

**Keywords:** Stability analysis, PD control, sampled-data systems, digital control, flexible systems.

## 1. INTRODUCTION

Stability analysis of feedback controlled flexible systems is increasingly attracting many researchers. A stability condition is derived for the attitude control of flexible spacecraft [1]. It is shown that if the PID controller is unconditionally stable, assuming the satellite to be rigid, then structural flexibility cannot destabilize it. PD control is proved to guarantee stability for flexible robot arms in [2]. A collocated PD control and non-collocated PD control for flexible arm are investigated in [3]. It is shown that the first control makes the system stable while the other does not, when the system damping is excluded. A new strategy for stability analysis of a feedback controlled flexible arm without truncation is presented in [4]. The stability study of collocated PD control with tip acceleration feedback is conducted and verified experimentally. An independent joint PD control and modal feedback is used in [5] for vibration control of multi-link flexible robots. The asymptotic stability of the closed-loop system is proved via Lyapunov arguments. A control strategy that ensures the exponential stability of the tracking error in the workspace of a class of flexible robots is presented in [6]. Stability analysis for a fuzzy-logic control of flexible robots is presented in [7-9].

The studies mentioned above have commonly focused on continuous control systems. When control systems are implemented digitally, the sample-and-hold effect has to be taken into account. In practice it has been acknowledged that a system may become unstable when the feedback gains are high. It is shown in [10, 11] that a stable high-gain continuous PD control systems, second- and first-order systems, may become unstable when implemented digitally. Stability criteria have been derived for second-order system in the following cases: zero computation time, one-sampling-period computation time and general computation time. The stability regions are determined numerically in the last two cases.

In this work, it is shown that a collocated PD control of flexible system, which guarantees stability, may become unstable when implemented digitally even with small gains. The aim of this work is to define the stability region for such systems in a form suitable for design purposes.

The paper is organized as follows. Section 2 describes the dynamic systems to be considered, and shows how a continuous, stable PD control applied to flexible system becomes unstable when the controller is implemented digitally. A method for establishing the stability regions for systems described by low order characteristic equations is introduced in section 3. The results of second order system are compared with the literature. In section 4, the equations describing the boundary of stability regions are derived for a system consists of single rigid mode and one flexible mode. A simplified method to define the stability region is also introduced. General procedures to establish the stability regions for a system, which consists of one rigid mode and arbitrary number of flexible modes, are introduced in section 5. An approximated method to define the stability regions for such systems is proposed.

## 2. COLLOCATED PD CONTROL OF FLEXIBLE SYSTEMS

The flexible system under consideration consists of one rigid mode and arbitrary number of flexible modes. This simulates many practical systems like flexible robot arm, disk-drive servo and antenna servo system. The state equations for such systems are:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (1)$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & -\omega_1^2 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_2^2 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & -\omega_n^2 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ b_0 \\ 0 \\ b_1 \\ 0 \\ b_2 \\ \dots \\ \dots \\ 0 \\ b_n \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} b_0 & 0 & b_1 & 0 & b_2 & 0 & \dots & \dots & b_n & 0 \\ 0 & b_0 & 0 & b_1 & 0 & b_2 & \dots & \dots & 0 & b_n \end{bmatrix}$$

**x** is the modal coordinate vector, u is the control input, **y** is the vector of the displacement and velocity at a coordinate where the control input is applied, n is the number of flexible modes considered, $\omega_i$ (i=1, ..n) are the frequencies of the flexible modes, $b_0$ is the component of weighted rigid mode shape at a coordinate where the control input is applied and $b_i$ (i=1,...n) are similar to $b_0$ but for flexible modes. For a collocated PD control the control law is:

$$u = -k_p(y_1 - y_{1d}) - k_d y_2 \qquad (2)$$

Where $k_p$ and $k_d$ are the proportional and derivative gains, $y_1$ and $y_2$ are the component of vector **y** and $y_{1d}$ is the desired position. It is shown [3, 4] that the continuous system described above is always stable for $k_p>0$ and $k_d>0$. To illustrate our point of the effect of digitizing the controller action, we will consider a system of four flexible modes has the following parameters: $\omega_1$=20, $\omega_2$=60, $\omega_3$=110, $\omega_4$=220, $b_0$=1, $b_1$=4, $b_2$=3, $b_3$=2, $b_4$=0.5, $k_p$=20, $k_d$=0.1 and $y_{1d}$=0. The simulation of the system is carried out through construction of SIMULINK-models within the environment of the MATLAB-software [12]. The time response of the system is shown in **Fig. 1**. The response shows that the system is stable. When the PD control is applied digitally with zero-order hold and sampling period of 0.01, the time response indicates instability as shown in **Fig. 2**.

These results indicate that, even with small gains, the system becomes unstable.

## 3. REFINEMENT OF STABILITY CRITERIA FOR SECOND ORDER SYSTEMS

Considering a second order system described by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u \qquad (3)$$

Stability criteria for digital PD control is derived in [10, 11]. For zero computation time and sampling period, T, the characteristic equation is obtained as:

$$\lambda^2 - (2 - \alpha - \frac{\beta}{2})\lambda + 1 - \alpha + \frac{\beta}{2} = 0 \qquad (4)$$

where,

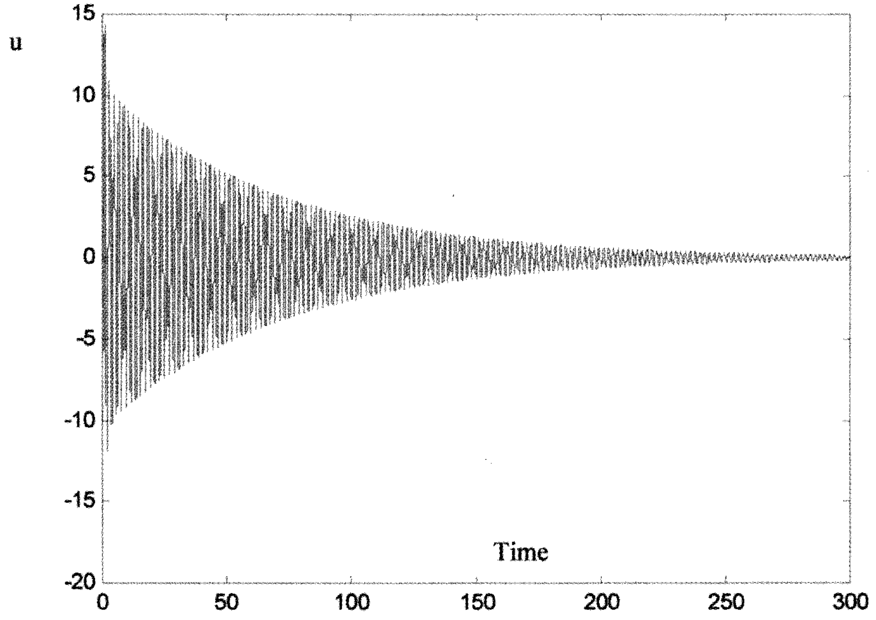
$$\alpha = bk_d T, \qquad \beta = bk_p T^2$$

Using root locus and some geometrical relations, the following stability criterion is given in [10].

$$\alpha < 2, \qquad \beta < 2\alpha \qquad (5)$$

In the case of one-sampling-period computation time, the characteristic equation is obtained as:

$$\lambda^3 - 2\lambda^2 + (\alpha + \frac{\beta}{2} + 1)\lambda - \alpha + \frac{\beta}{2} = 0 \qquad (6)$$



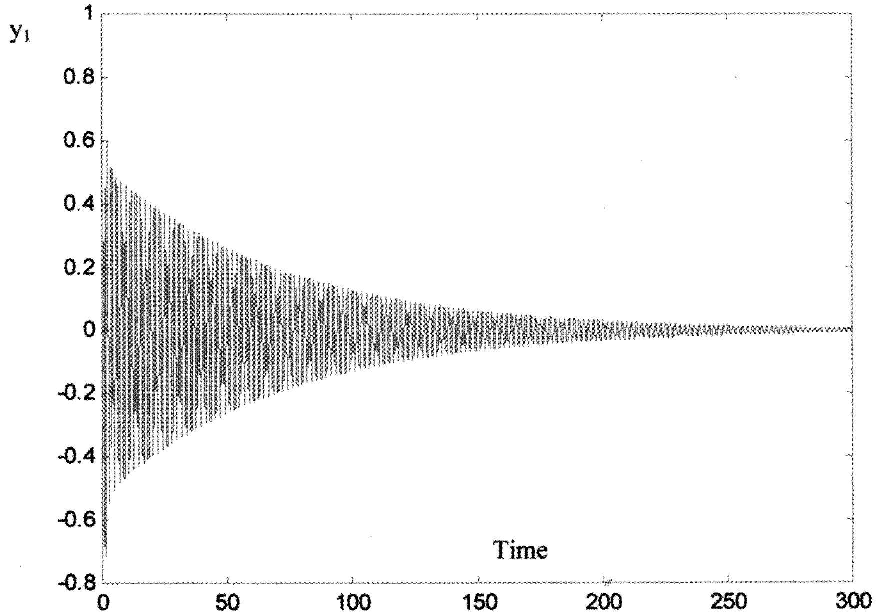




**Fig. 1. Time response of a continuous PD control**

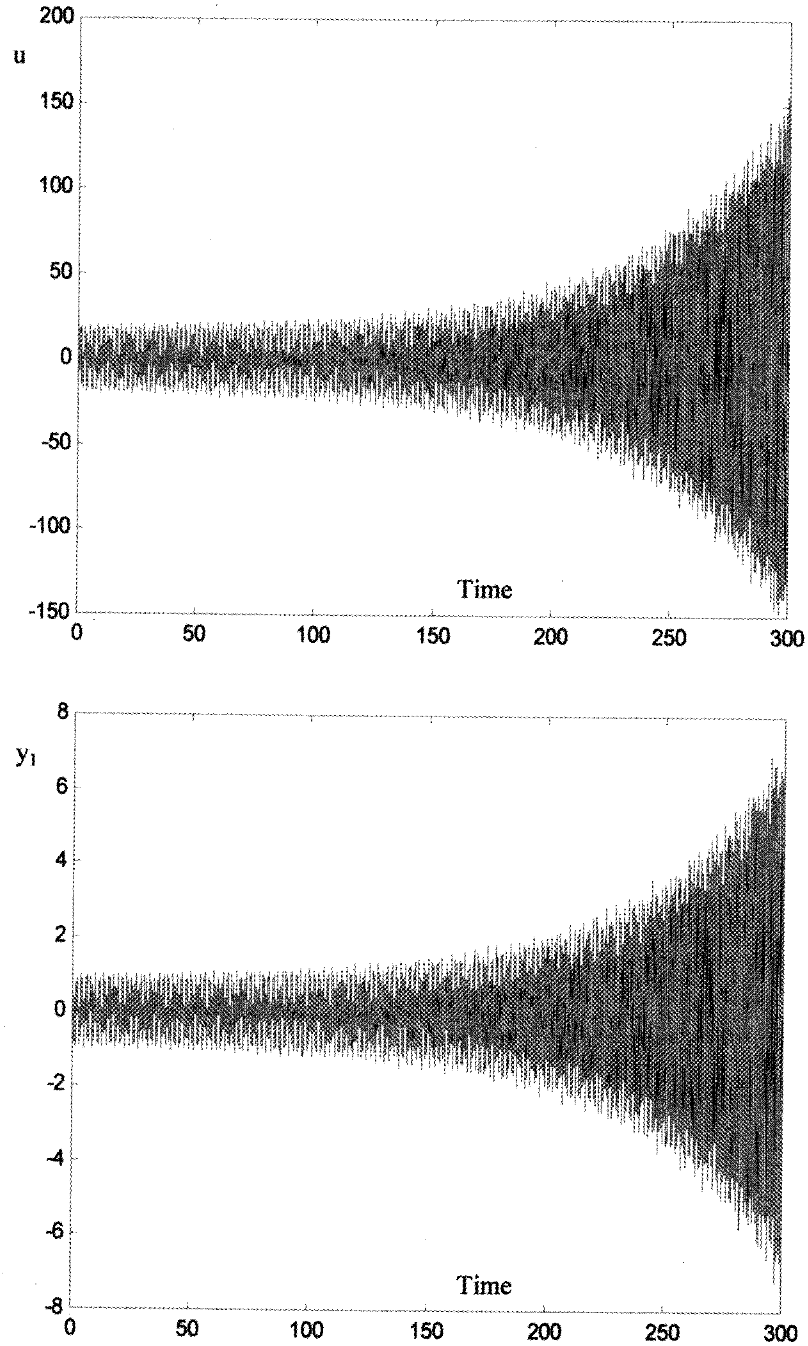



**Fig. 2. Time response of a digital PD control**

Using root locus and numerical analysis, the boundary of the stability region is found by numerical solution of the characteristic equation (6) for β under the following condition:

$$|\lambda| = 1, \qquad 0 < \alpha < 1 \qquad (7)$$

For a general computation time 'τ', the characteristic equation is obtained as:

$$\lambda^3 + (-2 - \alpha\gamma - \beta\gamma + \frac{\beta}{2} + \frac{\beta\gamma^2}{2} + \alpha)\lambda^2 + (1 + 2\alpha\gamma + \beta\gamma + \frac{\beta}{2} - \beta\gamma^2 - \alpha)\lambda + \frac{\beta\gamma^2}{2} - \alpha\gamma = 0 \qquad (8)$$

where, $\gamma = \frac{\tau}{T}$

A nomograph describing the stability region for various γ's is obtained by numerical solution of the characteristic equation (8) under the conditions |λ|=1. Through setting β=0 in equation (8), the critical value of α as a function of γ for $\beta \to 0^+$ is given by the following equations:

$$\alpha = \begin{cases} \dfrac{2}{1-2\gamma} & \text{for} \quad 0 \le \gamma \le 0.25 \\ \dfrac{1}{\gamma} & \text{for} \quad 0.25 < \gamma \le 1 \end{cases} \qquad (9)$$

Thus, the boundaries of the stability regions are defined numerically for the last two cases in [10]. For design purposes, it may be better to define these boundaries by analytical expressions.

In [13], stability conditions of low order polynomials, up to the fifth order, are presented. These conditions are simplification of the Jury test [14]. The polynomials can be expressed as:

$$f(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots\dots\dots + a_{n-1}\lambda + a_n \qquad (10)$$

Ref. [13] states that, the critical stability constraints that determine the boundary of the stability region in the coefficient space are given by the first condition of the Jury test:

$$f(1) > 0 \qquad and \qquad (-1)^n f(-1) > 0 \qquad (11)$$

and the last condition in the Jury test which is given below after simplifications for n=2,3,4 respectively.

$$a_0 - a_2 > 0 \tag{12}$$

$$a_3^2 - a_0^2 < a_3 a_1 - a_0 a_2 \tag{13}$$

$$a_4^3 + 2a_4 a_2 a_0 + a_3 a_1 a_0 - a_4 a_0^2 - a_2 a_0^2 - a_4 a_1^2 - a_4^2 a_0 - a_4^2 a_2 - a_3^2 a_0 + a_0^3 + a_4 a_3 a_1 > 0 \tag{14}$$

In this work, it is found that the critical stability constraints could also be obtained from the first, the last and the second last conditions of Routh-Hurwitz method using bilinear transformation.

The general procedure to define the stability region in the gain space and hence the stability criteria, in a closed form, are introduced here as follows:

a) Arrange the system characteristic equation using independent dimensionless terms to reduce the parameters of the system. The gains should be included in two dimensionless terms at most. These terms are called gain parameters. The other terms are called system parameters.
b) Apply the critical stability constraints to the characteristic equation after replacing the '<' and '>' symbols by '=' symbol.
c) Arrange the terms in the resulted equations to form polynomials of one gain-parameter like β for example.
d) Solve the resulted polynomials analytically if possible.
e) Make plots of the solutions for various system parameters, and make simple checks for points inside the intersected areas to discard the uncritical solutions.
f) Find the intersections between the critical solutions to define their applicable ranges.
g) Use the equations for the critical solutions and the intersections to express the stability criteria in closed form.

To carry out the above procedure, it is recommended to use the symbolic toolbox and plotting capability of the MATLAB [12]. The following remarks are stated for the above procedures:

1) It seems a little bit tricky to notice that, although the stability criteria are derived from the critical stability constraints, these constraints are not sufficient to check for system stability. The critical constraints may define several regions. Some of them are stable while the others are not. The benefit of the uncritical constraints (the other stability conditions) is to discard the unstable regions rather than to define stable ones.

2) It seems that similar results could be obtained by applying the above procedures to the whole stability conditions instead of the critical ones. In general, this is not true. The solutions of some conditions could almost be so coincided that the plotting cannot define which one bounds the stability region. In some cases, even the comparison between coincided solutions using numerical values can result in wrong conclusion due to round off error as it is noticed in the case of single-rigid/single-flexible mode system discussed in the next section. In addition, The large number of the solutions with many system parameters make the process so complicated that, a clear conclusion may not be reached.
3) It is noticed that, some critical solutions define stability regions only for certain ranges of system parameters. For the remained ranges of system parameters, other critical solutions become active (see the third case below). This indicates the importance of making many plots that scan the ranges of parameters, similar to root-locus method where different loci are obtained for different parameters to study the system.

Applying the procedure for the case of zero computation time (that is applying constraints (11,12) for equation (4)), results in the same stability criteria (5) with the additional conditions ($\alpha > 0$, $\beta > 0$). Thus, the stability region is a right triangle in the gain space as given in [10]. The described procedure is easier and straightforward approach than that of [10,11].

Applying the above procedure on the second case [10] of the one-sampling-period computation time yields the solutions plotted in **Fig. 3**. By simple check, the stability region could be defined as the area bounded by solution 3, $S_3$, and solution 1, $S_1$, (the horizontal axis). It is exactly the same as given in [10]. The intersections between solution 1 and solution 3 occurs at $\alpha = 0$, $\alpha = 1$. Thus the stability criteria could be expressed here analytically in a closed form that is more suitable for design purposes as:

$$\begin{aligned} &0 < \alpha < 1 \\ &0 < \beta < 2\alpha + \sqrt{9 - 8\alpha} - 3 \end{aligned} \tag{15}$$

Applying the above procedure for the case of a general computation time, the resulted stability criteria is obtained as:

$$for \quad 0 \le \gamma \le 0.25 \rightarrow \begin{cases} 0 < \alpha \le \dfrac{2}{1-2\gamma} & then \quad 0 < \beta < g(\alpha,\gamma) \\ \dfrac{2}{1-2\gamma} < \alpha < \dfrac{2+4\gamma-8\gamma^2}{2\gamma^2-2\gamma+1} & then \quad \dfrac{2+2\alpha\gamma-\alpha}{\gamma(\gamma-1)} < \beta < g(\alpha,\gamma) \end{cases}$$

$$for \quad 0.25 < \gamma \le 1 \quad \rightarrow \quad 0 < \alpha < \frac{1}{\gamma} \quad , \quad 0 < \beta < g(\alpha,\gamma) \tag{16}$$

where for $\gamma \neq 0.5$,

$$g(\alpha,\gamma) = \frac{3\alpha\gamma^2 - \alpha\gamma - 2\gamma - 1 + \sqrt{\alpha^2\gamma^2 - 2\alpha^2\gamma^3 + 2\alpha\gamma - 6\alpha\gamma^2 + \alpha^2\gamma^4 - 4\alpha\gamma^3 + 1 + 4\gamma + 4\gamma^2}}{\gamma^2(2\gamma - 1)}$$

and for $\gamma = 0.5$,

$$g(\alpha,\gamma) = \frac{4\alpha(\alpha - 2)}{-8 + \alpha}$$

The stability regions are plotted for various γ's in **Fig. 4.** A comparison between this plot and the nomograph published in [10] shows one difference. Instead of the inclined lines in **Fig. 4**, vertical lines are presented in the nomograph of [10]. A check using the roots of the characteristic equation (8) indicates the correctness of the plot in **Fig. 4** against the nomograph in [10]. So, the boundaries of the stability regions are not expressed in analytical forms only but they are also defined in more accurate way.

## 4. SINGLE-RIGID/SINGLE-FLEXIBLE MODE SYSTEM

The state equations for single-rigid/single-flexible mode system can be obtained from (1) by setting n=1. The discretized model for this system with zero-order hold (ZOH) can be derived as:

$$\begin{aligned} \mathbf{x}^*(k+1) &= \mathbf{A_d}\mathbf{x}^*(k) + \mathbf{B_d}u^*(k) \\ \mathbf{y}^*(k) &= \mathbf{C}\mathbf{x}^*(k) \end{aligned} \tag{17}$$

where, **C** is the same as in continuous system, equations (1), for n=1 and,

$$\mathbf{A_d} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\omega_1 T) & \sin(\omega_1 T)/\omega_1 \\ 0 & 0 & -\omega_1 \sin(\omega_1 T) & \cos(\omega_1 T) \end{bmatrix}, \quad \mathbf{B_d} = \begin{bmatrix} T^2 b_0 / 2 \\ T b_0 \\ -b_1(\cos(\omega_1 T) - 1)/\omega_1^2 \\ b_1 \sin(\omega_1 T)/\omega_1 \end{bmatrix}$$

For PD control with zero computation time,

$$u^*(k) = -k_p(y_1^*(k) - y_{1d}) - k_d y_2^*(k) \tag{18}$$

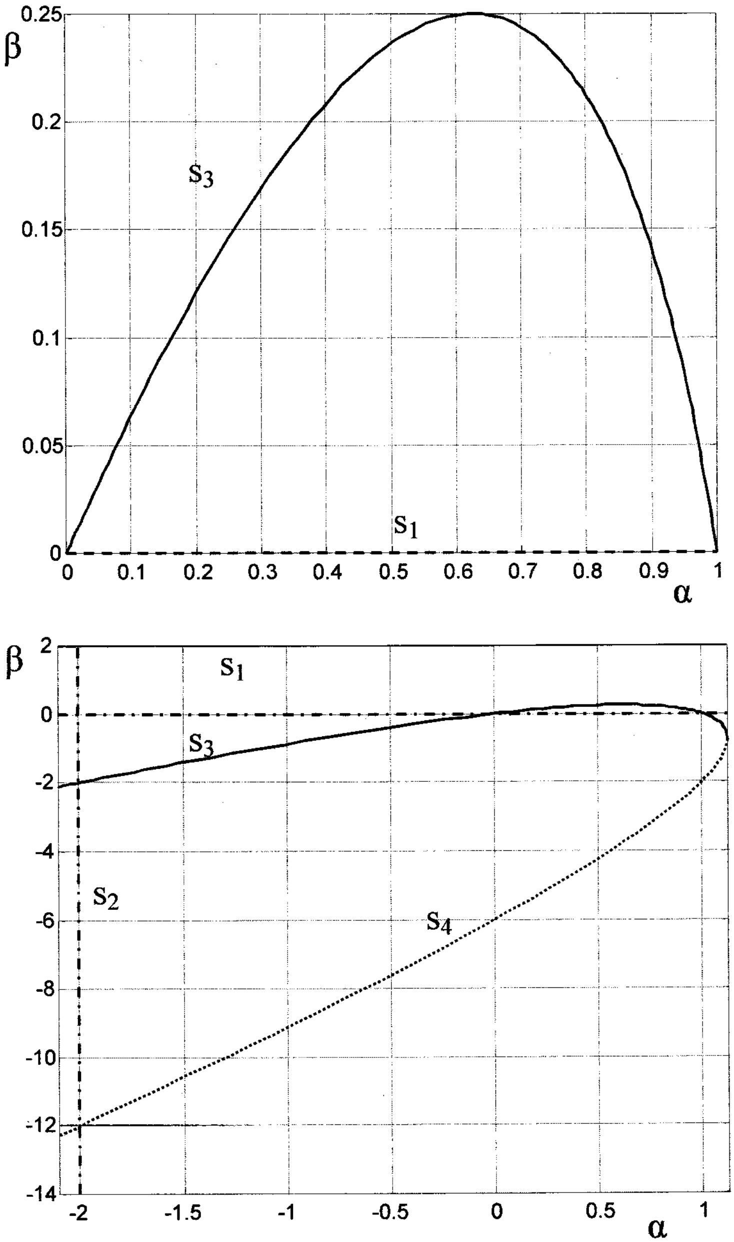



**Fig. 3. Stability region for one-sampling-period computation time**

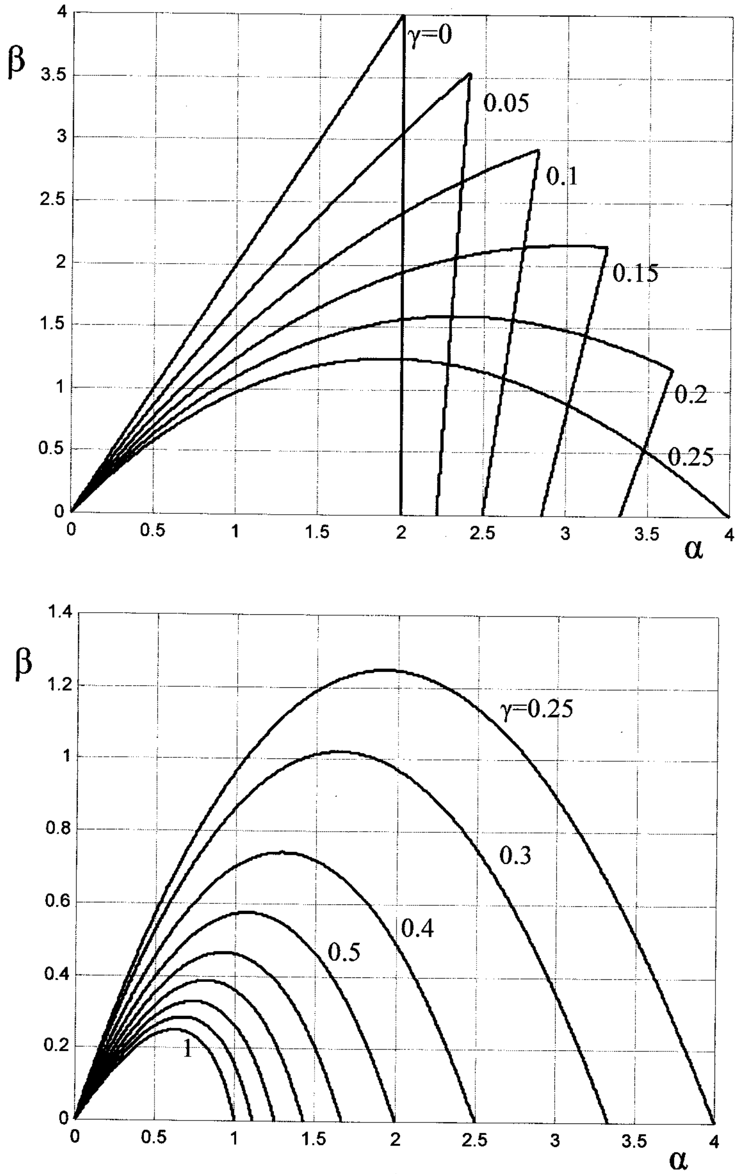



**Fig. 4. Stability regions for various computation times**

Substitution of equation (18) into equation (17) leads to

$$\mathbf{x}^{*}(k+1)=\begin{bmatrix} 1-\frac{T^2b_0^2k_p}{2} & T-\frac{T^2b_0^2k_d}{2} & -\frac{T^2b_0b_1k_p}{2} & -\frac{T^2b_0b_1k_d}{2} \\ -Tb_0^2k_p & 1-Tb_0^2k_d & -Tb_0b_1k_p & -Tb_0b_1k_d \\ \frac{b_0b_1(\cos(\omega_1T)-1)k_p}{\omega_1^2} & \frac{b_0b_1(\cos(\omega_1T)-1)k_d}{\omega_1^2} & \cos(\omega_1T)+\frac{b_1^2k_p(\cos(\omega_1T)-1)}{\omega_1^2} & \frac{\sin(\omega_1T)}{\omega_1}+\frac{b_1^2k_d(\cos(\omega_1T)-1)}{\omega_1^2} \\ -\frac{b_0b_1\sin(\omega_1T)k_p}{\omega_1} & -\frac{b_0b_1\sin(\omega_1T)k_d}{\omega_1} & -\sin(\omega_1T)\left(\omega_1+\frac{b_1^2k_p}{\omega_1}\right) & \cos(\omega_1T)-\frac{b_1^2\sin(\omega_1T)k_d}{\omega_1} \end{bmatrix}\mathbf{x}^{*}(k) \tag{19}$$

The characteristic equation of the system can be reduced to:

$$\begin{aligned}
&\lambda^4+\left(-2(1+\cos(e_1))+\frac{r_1\sin(e_1)\alpha}{e_1}+\frac{r_1\beta}{e_1^2}(1-\cos(e_1))+\frac{\beta}{2}+\alpha\right)\lambda^3+\\
&\left(4\cos(e_1)-2\alpha\cos(e_1)-\frac{r_1\beta}{e_1^2}(1-\cos(e_1))-\alpha+2-\frac{3r_1\sin(e_1)\alpha}{e_1}-\beta\cos(e_1)+\frac{\beta}{2}\right)\lambda^2+\\
&\left(-2+\alpha-2\cos(e_1)+\frac{3r_1\sin(e_1)\alpha}{e_1}+2\alpha\cos(e_1)-\frac{r_1\beta}{e_1^2}(1-\cos(e_1))-\beta\cos(e_1)+\frac{\beta}{2}\right)\lambda+\\
&\left(-\alpha+\frac{\beta}{2}+1-\frac{r_1\sin(e_1)\alpha}{e_1}+\frac{r_1\beta}{e_1^2}(1-\cos(e_1))\right)=0
\end{aligned} \tag{20}$$

where,

$$r_1=\frac{b_1^2}{b_0^2},\qquad e_1=\omega_1T,\qquad \alpha=Tb_0^2k_d,\qquad \beta=T^2b_0^2k_p$$

By setting $b=b_0^2$, the definition of α and β is the same as those for the second order system [10]. Note that, this formulation of the characteristic equation satisfies requirement (i) of the procedure presented in the previous section. From practical point of view, the sampling rate should be selected more than twice as fast as the resonance [15]. This means that $2\pi/T>2\omega_1$ and consequently $e_1<\pi$.

Applying the procedures of the previous section to this fourth-order polynomial of equation (20), the stability criterion can be derived as:

$$\begin{aligned}
&e_1\neq 2\pi i,\qquad i=0,1,2,\ldots..\\
&0<\alpha<\alpha_{\max}(r_1e_1)\\
&0<\beta<h(r_1,e_1,\alpha)
\end{aligned} \tag{21}$$

where,

$$\alpha_{\max}(r_1,e_1)=\frac{2+2\cos(e_1)}{1+\cos(e_1)+\dfrac{2r_1\sin(e_1)}{e_1}},$$

$$h(r_1,e_1,\alpha)=\frac{h_n(r_1,e_1,\alpha)}{h_d(r_1,e_1,\alpha)},$$

$$\begin{aligned}h_n &= e_1\alpha^2 r_1^2(-e_1 s^2+2s+2sc^2+ce_1s^2-4sc)+e_1\alpha^2 r_1(2e_1-e_1^2 s+ce_1^2 s+2e_1c^2-4ce_1)\\ &+e_1\alpha r_1(-2e_1+2ce_1^2 s+2c^3 e_1-6e_1c^2-e_1^2 s+6ce_1-c^2e_1^2 s)+e_1\alpha r_1(c-1)(2c-2+se_1)\\ &\sqrt{-2ce_1^2+\alpha^2 e_1^2-2\alpha e_1^2+2\alpha ce_1^2+e_1^2c^2+e_1^2-2r_1\alpha sce_1+2r_1\alpha^2 se_1+2r_1\alpha se_1+r_1^2\alpha^2 s^2},\end{aligned}$$

$$h_d = r_1(c-1)\left\{\alpha r_1(-4c^2+2se_1+8c-4-2sce_1)+\alpha(e_1^3 s+2ce_1^2-2e_1^2)+2e_1^2c^2+2e_1^2-4ce_1^2\right\},$$

$$c=\cos(e_1),\qquad s=\sin(e_1)$$

Using the above expressions, the stability regions are plotted for various system parameters, $r_1$ and $e_1$, as shown in **Figs. 5, 6**.

The plots show that the stability regions are almost right triangles in the gain space similar to the case of a second order system with zero computation time. Although the hypotenuses in these plots are not exact straight lines as indicted by the equation $\beta$=h($r_1$,$e_1$, $\alpha$) of (21), where h is a nonlinear function of $\alpha$ as shown previously, they can be very well approximated by straight lines. **Fig. 5** shows that, the range of $\alpha$ increases as $r_1$ decreases. As a limit, when $r_1 \rightarrow 0$, $\alpha_{max} \rightarrow 2$ which corresponds to the range of second order system (single rigid mode) as shown previously (equation (5)). The inclination of hypotenuse is slightly affected by the changes of $r_1$. But **Fig. 6** indicates great influence of hypotenuse inclination by the change of $e_1$. As $e_1$ decreases from $\pi$ to 0, the inclination increases from 0 to 2. The limit value of the inclination is 2, which equals the corresponding value for single rigid mode (see equation (5)).

It is desired to approximate the function, $\beta$=h($r_1$,$e_1$, $\alpha$), by a straight line, $\beta$= $\theta(r_1,e_1)$ $\alpha$. In doing so, the stability region is defined by only two parameters, $\alpha_{max}$ and the line inclination, $\theta$. This helps in the design process especially in some method such that given in [16] which needs these two parameters $\alpha_{max}$ and $\theta$. The proposed straight line is defined as a line passing through the origin and through the mid point of the curve, h, (the horizontal coordinate of the mid point is $\alpha_{max}/2$). The line inclination, $\theta$, can now be found as:

$$\theta(e_1,r_1)=\frac{\theta_n(e_1,r_1)}{\theta_d(e_1,r_1)} \qquad (22)$$

where,

$$\theta_n = -e_1 r_1(-3e_1 - c^3 e_1 - 2s + 3c^2 e_1 + ce_1 - 6sc^2 + 8sc) - e_1(\sqrt{M}(2 - 2c - e_1 s) - 2e_1{}^2 s - e_1{}^2 cs - 2c^3 e_1 + e_1{}^2 c^2 s + 2ce_1),$$

$$\theta_d = r_1(4c^2 - 4c^3 + 6se_1 + 2sc^2 e_1 + 4c - 4 - 8sce_1) + 2c^3 e_1{}^2 - 2e_1{}^2 c + e_1{}^3 s + sce_1{}^3,$$

$$M = (1+c)(c^3 e_1^2 - r_1{}^2 c^3 + 2r_1 sc^2 e_1 + e_1{}^2 c^2 + 7r_1^2 c^2 - 15r_1{}^2 c - 2r_1 sce_1 + 4r_1 se_1 + 9r_1{}^2)$$

Where c and s have the same definitions as in relation (21). Note that both $\alpha_{max}$ and $\theta$ are functions of the system parameters, $r_1$ and $e_1$, only. In order to discover how good the curve, h, is approximated by this straight line in the range ($0 \rightarrow \alpha_{max}$), A percentage deviation, D, is introduced. D is defined as the root-mean-square of the error between the curve and the line, divided by the mean value of the curve, and multiplied by one hundred. D is calculated for different system parameters that scan their practical ranges. The results are shown in **Fig. 7**. A horizontal logarithmic scale is selected to represent the results in a clear way. The results show that, the deviation becomes high in small ranges of the parameters (high $e_1$ and small $r_1$). Practically speaking, the approximation becomes good for $r_1>1$ and any value of $e_1$ or for $e_1< \pi/2$ and any value of $r_1$.

## 5. SINGLE-RIGID/MULTIPLE-FLEXIBLE MODE SYSTEM

Beginning with the state equations (1) and using symbolic tools of MATLAB-Software, one can obtain the discretized model of single-rigid/multiple-flexible mode system with zero-order hold (ZOH), as well as its characteristic equation. Comparing the characteristic equations for systems having 1,2,3 and 4 flexible modes show that the characteristic equations for single-rigid/multiple-flexible mode systems can be expressed in terms of the following dimensionless parameters:

$$\alpha = Tb_0^2 k_d, \qquad \beta = T^2 b_0^2 k_p, \qquad e_i = \omega_i T, \qquad r_i = \frac{b_i^2}{b_0^2}, \qquad i = 1,.....n$$

Applying the first condition of Jury test, equation (11), on the characteristic equations for n=1,2,3,4 and comparing the results, yields after lengthy and complicated reduction the following general conditions for arbitrary number of flexible modes:

$$\beta \prod_{i=1}^{n} \left(1 - \cos(e_i)\right) > 0 \qquad \left(or \quad \beta > 0, \quad e_i \neq 2\pi k, \quad i = 1,...n, \quad k = 0,1,2,....\right) \tag{23}$$

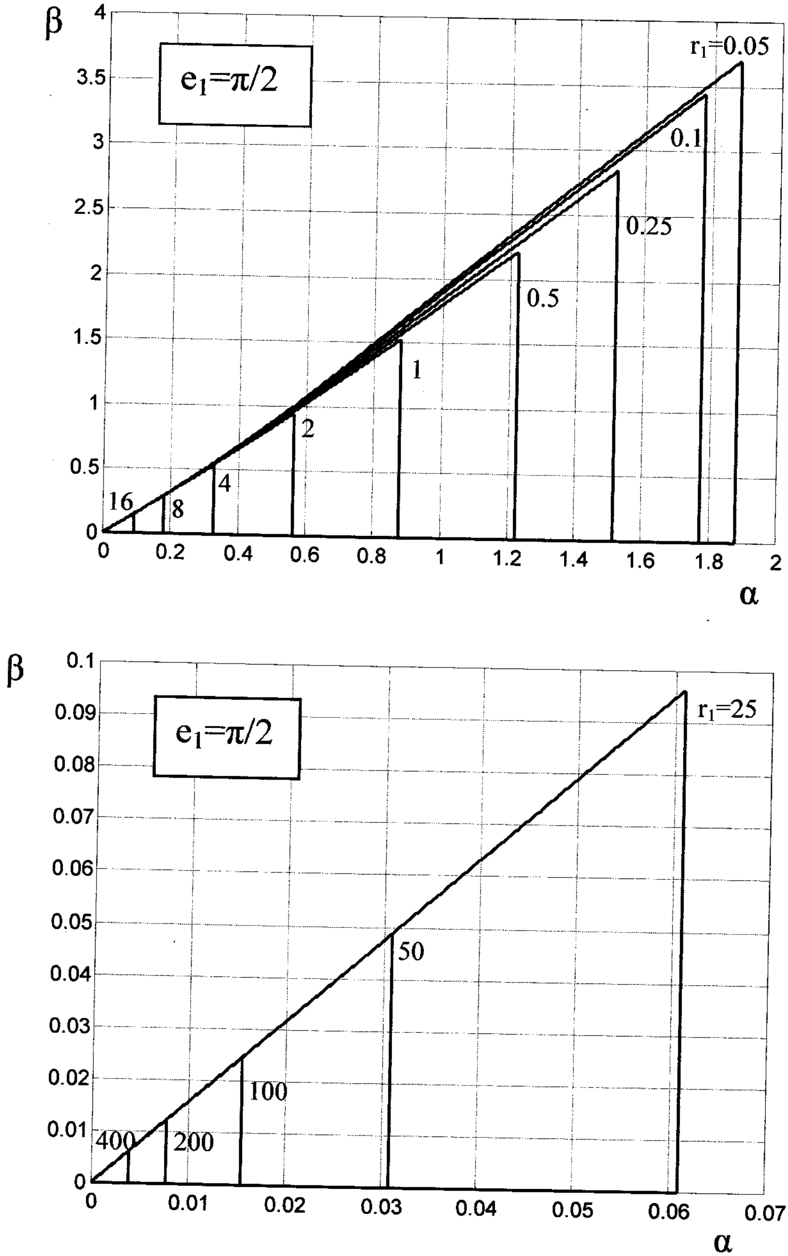



**Fig. 5. Stability regions for different values of $r_1$-parameter**

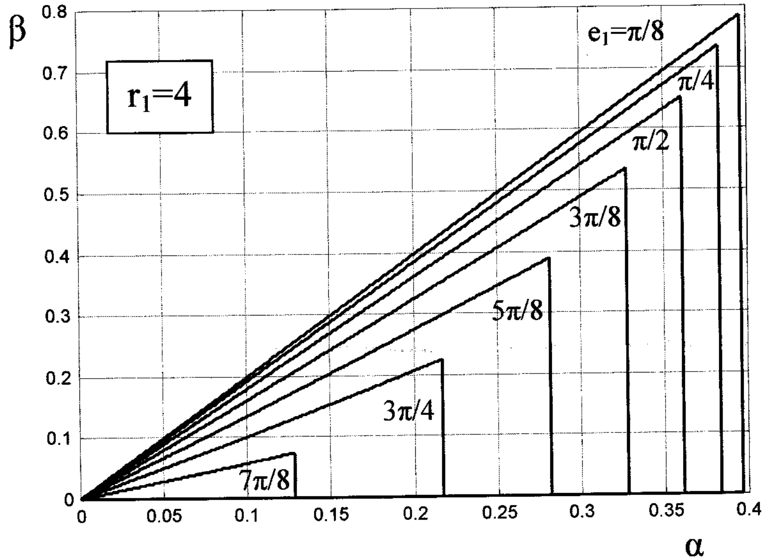



**Fig. 6. Stability regions for different values of $e_1$-parameter**

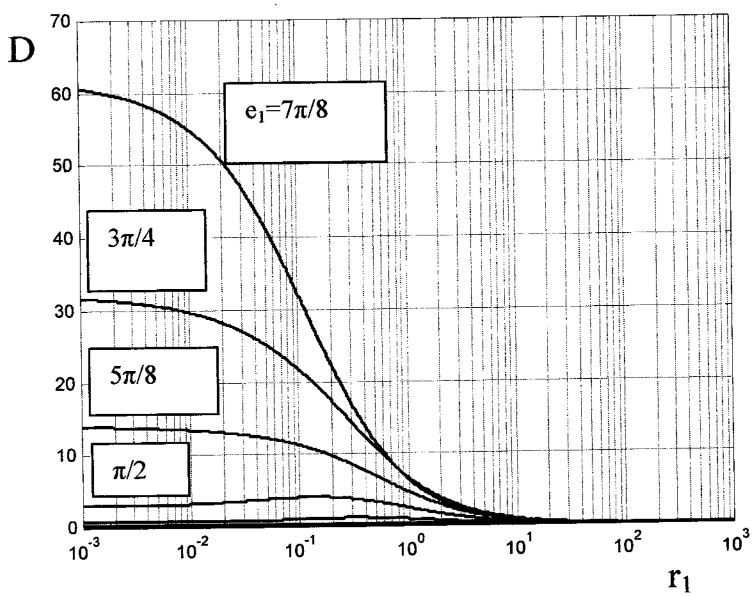



**Fig. 7. Percentage deviation from linearity for different system parameters**

$$\alpha < \frac{2\prod_{i=1}^{n}\left(1+\cos(e_i)\right)}{\prod_{i=1}^{n}\left(1+\cos(e_i)\right)+\sum_{i=1}^{n}\frac{2r_i}{e_i}\sin(e_i)\prod_{j=1,j\neq i}^{n}\left(1+\cos(e_j)\right)} \qquad (24)$$

Since the flexible system of n>1 has characteristic equations of order higher than 5, the stability criteria method used in previous section can not be applied. The application of full Jury test or Routh-Hurwitz method to these flexible systems exceeds the limitations of symbolic tools of MATLAB-Software.

The stability regions for these systems are obtained here using numerical technique described in the flowchart shown in **Fig. (8)**.

Where N is the number of points on one boundary of the stability region. The calculation of $\bar{A}$ begins with replacement of $\omega_i$, $b_0$, and $b_i$ by $e_i$, 1, and $r_i$ respectively in equation (1) where i=1,2,...n, and replacement of $k_p$, and $k_d$ with $\beta$ and $\alpha$ respectively in equation (2). Then MATLAB-functions are used to convert the continuous system to discretized system with T=1. The conversion is similar to the previous case of single-rigid/single-flexible mode system but applied here numerically rather than symbolically. The discretized system can be expressed as:

$$\mathbf{x}^*(k+1) = \overline{\mathbf{A}}\mathbf{x}^*(k) \qquad (25)$$

Relations (23) and (24) are used in this program to limit the search space. However, it is expected that these relations define the boundaries of the stability region after replacement "<" and ">" with "=" as will be shown later on. This expectation arises during the study of stability region of single-rigid/double-flexible mode system using full Routh-Hurwitz method and symbolic tools of MATLAB-Software. It is found that two boundaries of the stability region for various system parameters are expressed using relations (23) and (24) as in single-rigid/single-flexible mode system.

**Figures (9-12)** show stability regions for different number of flexible modes as well as for various system parameters.

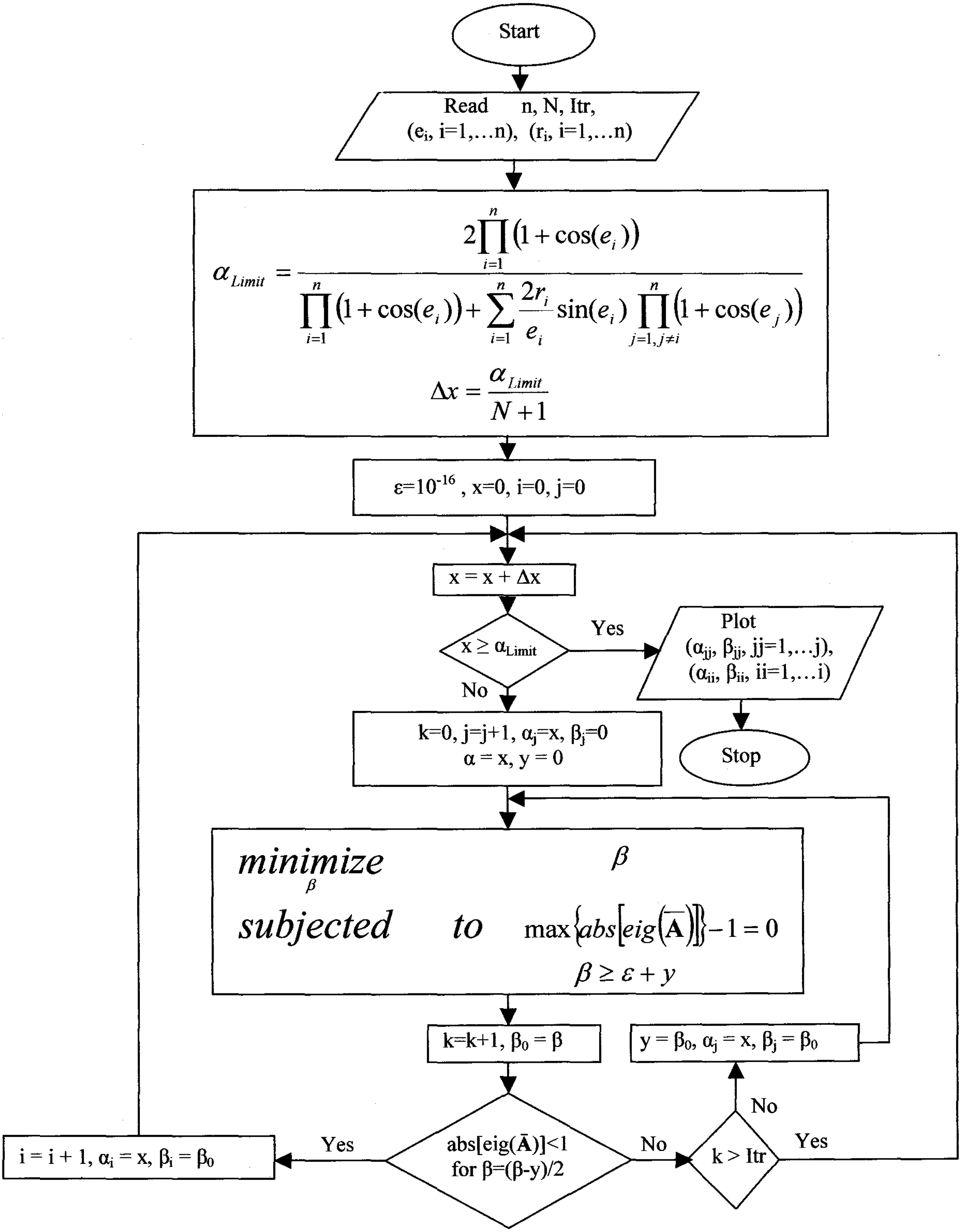



**Fig. 8. Flowchart for determination of stability regions for arbitrary number of flexible modes**

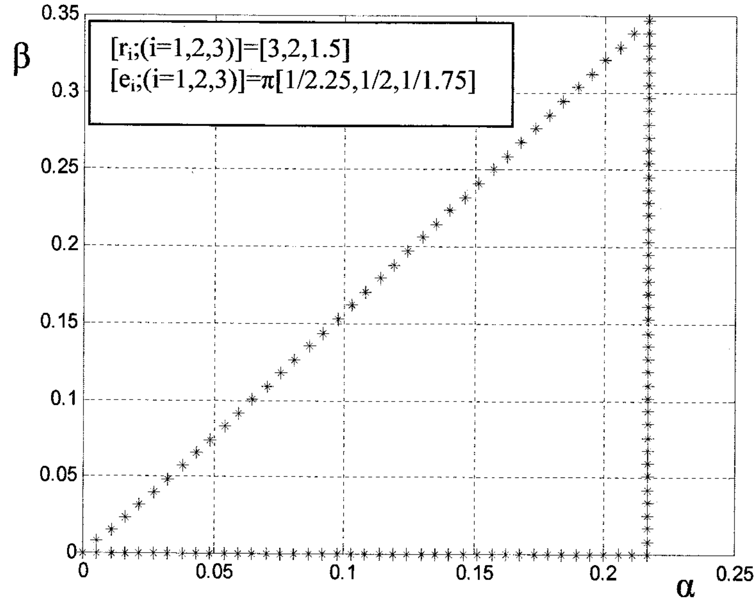



**Fig. 9. Stability region for n = 3**

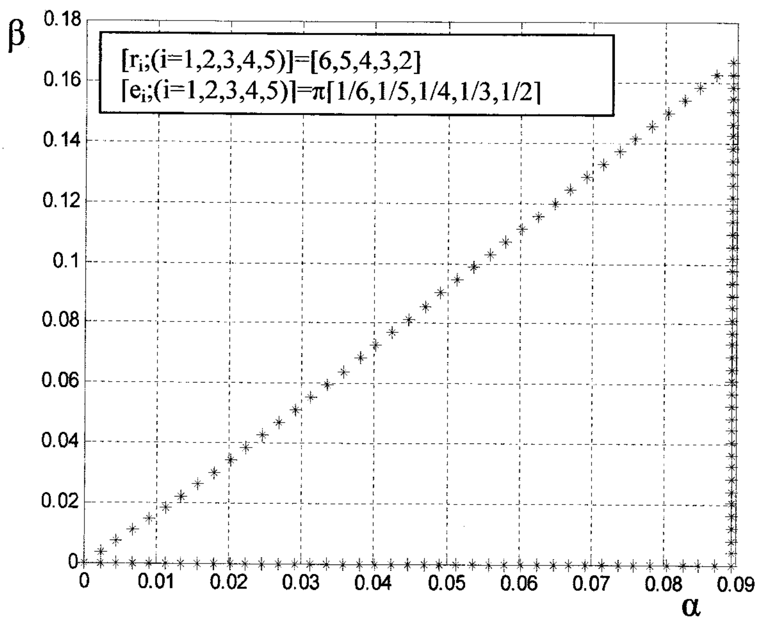



**Fig. 10. Stability region for n = 5**

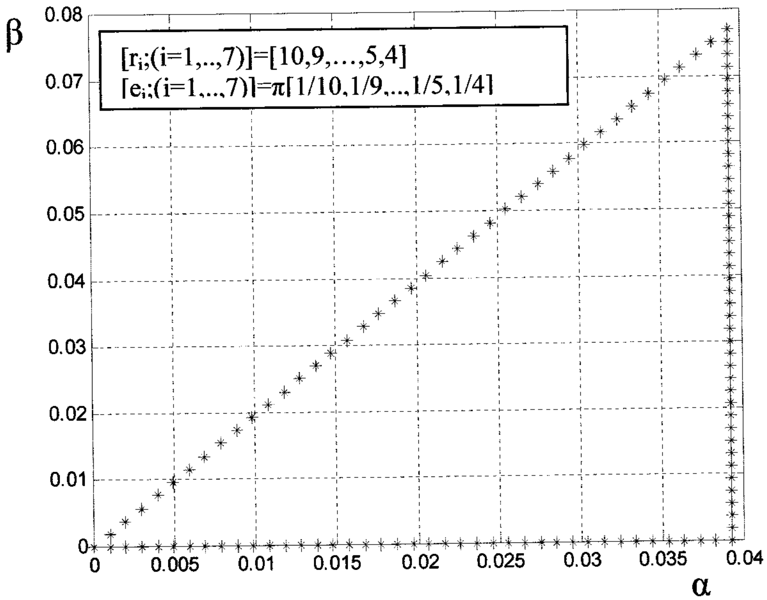



**Fig. 11. Stability region for n = 7**

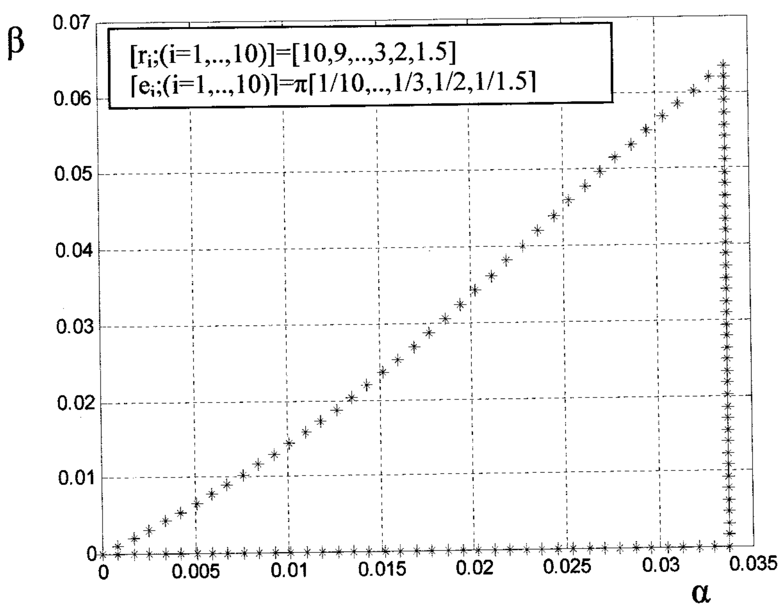



**Fig. 12. Stability region for n = 10**

The initial analysis of these figures, as well as others not shown here, indicate that, the stability region of such systems consists of three boundaries that approximate right triangle as in single-rigid/single-flexible mode system. The two perpendicular sides could be described by relations (23) and (24). The third one could be approximated by a line whose inclination could be approximated as a weighted average of the inclinations of the corresponding lines for single-rigid/single-flexible mode systems as follow:

$$\theta(e_1, r_1, \ldots\ldots, e_n, r_n) = \frac{\sum_{i=1}^{n} \frac{\theta_n(e_i, r_i)}{\theta_d(e_i, r_i)} e_i}{\sum_{i=1}^{n} e_i} \qquad (26)$$

Where $\theta_n$ and $\theta_d$ are defined as in equation (22) after replacing "1" by "i".

In order to examine such proposal, the program described above is executed for representative large-number of cases, in which the system parameters are set randomly.

In all case, it is found that two boundaries of the stability region are described using relations (23) and (24) without any exceptions. For examination of relation (24), similar program to the above described one is used, in which $\alpha$ is determined for equally spaced set of $\beta$. Some of the distributions of the percentage deviation of the third boundary-inclination (after being linearly interpolated) from that described by equation (26) are plotted in **Figs. (13-16)** for systems of different number of flexible modes. These figures indicate that the approximation of the third boundary is quite well. The average percentage deviation is found to be about 2.5%, while the maximum percentage deviation is about 10%. These values are important to assign suitable safety factor when applying the proposed expression. It is to be mentioned that other expressions than equation (26) are suggested and tested. They give less accurate result than the proposed equation (26). **Figures (17-20)** show stability regions for the proposed expressions against the stability regions obtained numerical by the above-described program. The parameters in these figures are the same as those of **Figs. (9-12)**. The figures indicate clearly the validity of the proposed expressions.

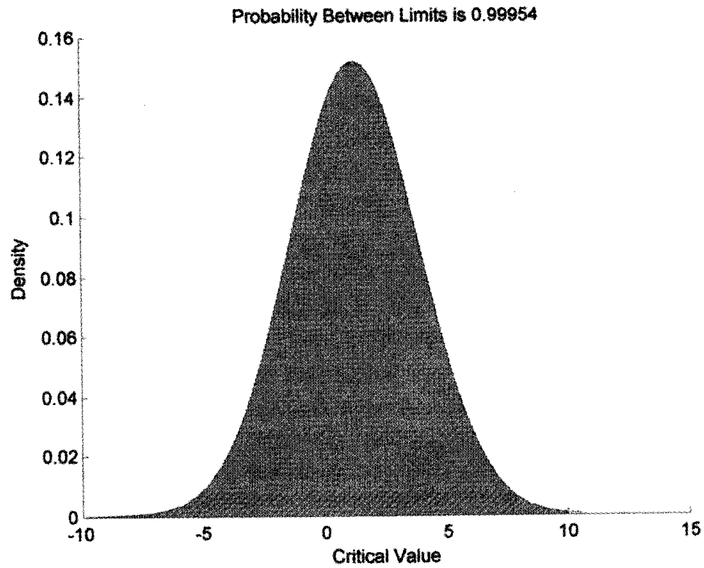



**Fig. 13. Deviation distribution for n=2**

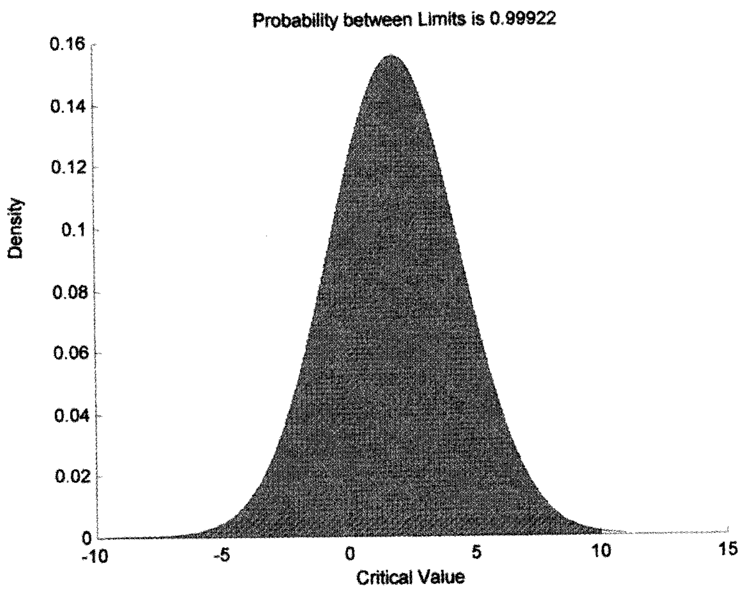



**Fig. 14. Deviation distribution for n=6**

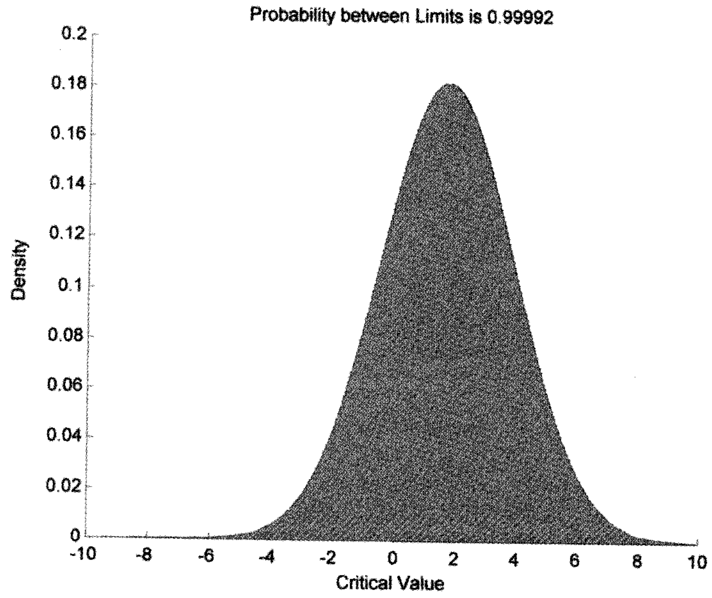



**Fig. 15. Deviation distribution for n=8**

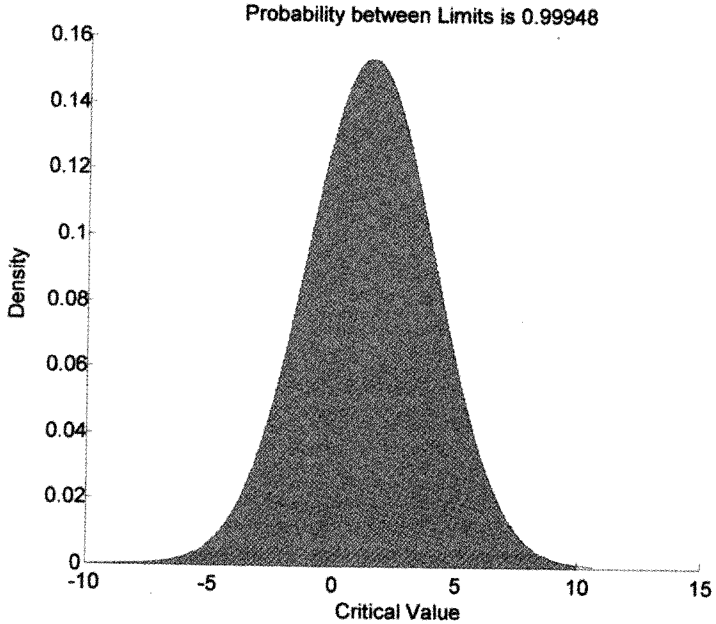



**Fig. 16. Deviation distribution for n=10**

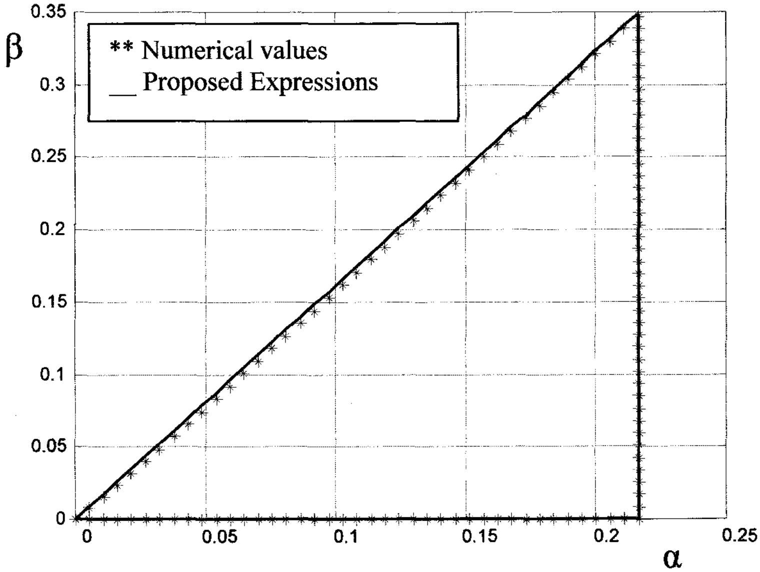



**Fig. 17. Proposed expressions against numerical values for n = 3**

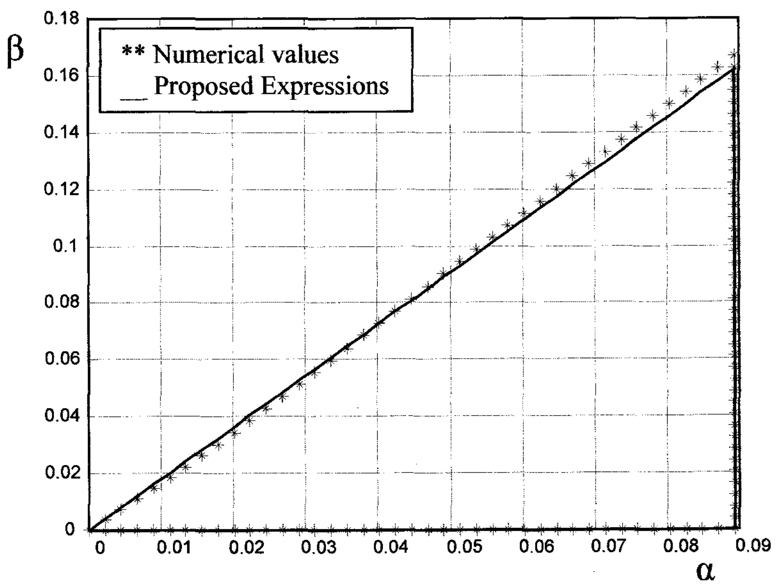



**Fig. 18. Proposed expressions against numerical values for n = 5**

Although the exact boundaries of the stability regions for single-rigid/multiple-flexible mode systems are obtained numerically in this section, the procedure described in section 3 to define the stability region analytically has great influence on the outcomes of this section. On one hand, relations (23, 24), that obtained through partial application of the procedure, limit the search space of the numerical method used to define the stability region (see **Fig. 8**). On the other hand, the derivation of equation (26), that used to define the stability region approximately in this section, is based on equation (22). The derivation of equation (22) is based on relation (21). Finally, relation (21) is obtained through direct application of the procedure.

It is suggested to use the approximated expressions (equations (22, 26)) in the iterative design process. Then the final design can be checked using exact expression (21) in the case of single-rigid/single-flexible mode system or the computer program **(Fig. 8)** in the case of single-rigid/multiple-flexible mode system.

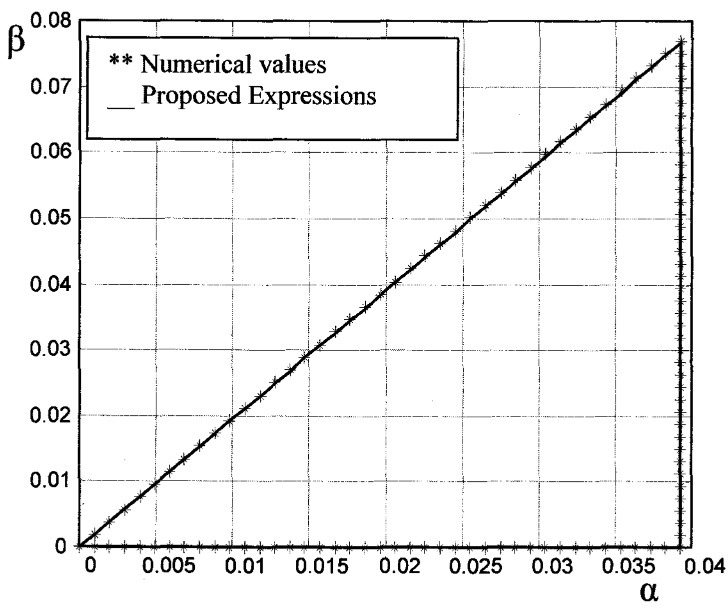



**Fig. 19. Proposed expressions against numerical values for n = 7**

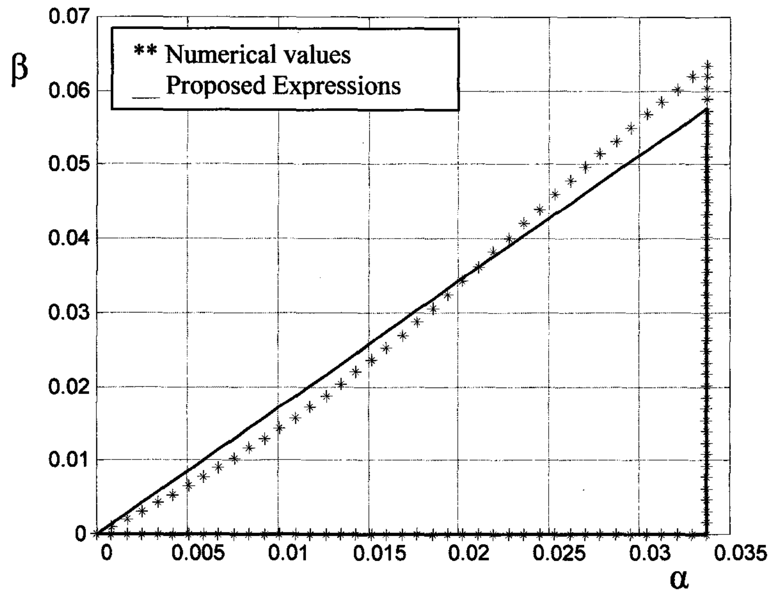



**Fig. 20. Proposed expressions against numerical values for n = 10**

## 6. CONCLUSION

The work introduces a procedure to define the stability regions in the gain space to determine the stability criteria for digitally controlled systems using analytical expressions. This is accomplished by means of the critical stability constraints of Jury test and the capability of MATLAB-Software. The procedures are applied successfully for second order system controlled digitally by PD algorithm. The results obtained for the case of general computing time (**Fig. 4**) show some deviations from those obtained numerically in the literature [10]. A direct check shows that the results represented here are the correct ones. The procedure is then applied to digital PD control system that has one rigid mode and one flexible mode. The obtained closed form stability criteria, relation (21), show that the stability region is almost a right triangle in the gain space. The two perpendicular sides of the proposed triangle coincide exactly with two boundaries of the stability region. These two sides can be determined using analytical expressions ($\beta=0$, $\alpha=\alpha_{max}(r_1,e_1)$, see relation (21)). The third side approximates very well the third boundary and can be determined using equation (22). In the case of single-rigid/multiple-flexible mode system, the proposed procedures can not be applied since the critical stability constraints of Jury test are defined for systems having low order characteristic equations (less than 6). A computer program is constructed that can determine accurately the stability region for a digital PD control system that has single rigid mode and arbitrary number of flexible modes using a numerical method. Such system has practical importance. It simulates flexible robot arm as an example. Similar to the previous case, the obtained stability region is almost right triangle in the gain space. Closed form expressions are derived, that limit the gains for arbitrary number of flexible modes and various system parameters through the application of the first condition of Jury test, relations (23-24). It is found that the two perpendicular sides of the proposed triangle coincide exactly with two boundaries of the stability region and can be determined using relations (23-24) ($\beta=0$, $\alpha=\alpha_{limit}$, see **Fig. 8**). The third side approximates the third boundary very well. This is proved statistically. The third side can be determined using equation (26). Thus the stability regions for digitally controlled flexible systems are found by two methods. The first one gives the exact boundaries of the stability regions (relations (21) and **Fig. 8**). The second one gives approximate boundaries. The first one is suitable for checking of the final design, while the second one is suitable for iterative design process.

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