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# Common fixed Points for Multimaps in Menger Spaces

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# النقاط الثابتة لدالة متعددة المتغيرات في فضاءات منجر

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الهدف من هذا البحث هو الحصول على نظرية نقطة ثابتة لدالة متعددة المتغيرات في فضاءات منجر.

**Key words and phrases**: *Menger spaces, common fixed point, multimaps*

## ABSTRACT

The aim of this paper is to obtain a common fixed point theorem for multivalued mappings in Menger spaces. Of course this is a new result in Menger spaces. 2000 Mathematics Subject Classification: 47H10, 54H25.

## Introduction

Menger [6] introduced the notion of probabilistic metric spaces, which is generalization of metric space, and the study of these spaces was expanded rapidly with the pioneering work of Schweizer and Sklar [12], [13]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis.

Recently, fixed point theorems in Menger spaces have been proved by many authors including Bylka [1], Pathak, Kang and Baek [8], Stojakovic [16], [17], [18], Hadzic [4], [5], Dedeic and Sarapa [3], Rashwan and Hedar [11], Mishra [7], Radu [9], [10], Sehgal and Bharucha-Reid [14], Cho, Murthy and Stojakovic [2].

## Preliminaries

Let R denote the set of reals and $R^+$ the non-negative reals. A mapping $\mathbf{F} : R \rightarrow R^+$ is called a distribution function if it is non- decreasing and left continuous with inf **F** =0 and sup **F** = 1. We will denote by L the set of all distribution functions.

A probabilistic metric space is a pair (X, **F** ), where X is non empty set and **F** is a mapping from XxX to L.

For (u, v) $\in$ XxX, the distribution function **F** (u, v) is denoted by Fu, v.
The function Fu, v are assumed to satisfy the following conditions:
($P_1$) Fu, v (x) = 1 for every x > 0 if and only if u = v,
($P_2$) Fv, u (0) = 0 for every u, v $\in$ X,
($P_3$) Fu, v (x) = Fv, u(x) for every u, v $\in$ X,
($P_4$) if Fu, v (x) = 1 and Fv, w(y) = 1 then Fu, w (x + y) = 1 for every u, v, w $\in$ X and x, y > 0.
In metric space (X, d) the metric d induces a mapping **F** : X x X $\to$ L such that
**F** (u, v) (x) = Fu, v (x) = H( x – d(u, v))
For every u, v $\in$ X and x $\in$ R, where H is a distributive function defined by

$$H(x) = \begin{cases} 0 & x \le o \\ 1 & x > 0. \end{cases}$$

**Definition 1.** A function t: [0, 1] x [0, 1] $\to$ [0, 1] is called a T- norm if it satisfies the following conditions:
($t_1$) t (a, 1) = a for every a $\in$ [0,1] and t(0, 0) = 0,
($t_2$) t(a, b) = t(b, a) for every a, b $\in$ [0,1],
($t_3$) If c $\ge$ a and d $\ge$ b then t(c, d) $\ge$ t(a, b),
($t_4$) t(t(a, b), c) = t(a, t(b, c)) for every a, b, c $\in$ [0, 1].
**Definition 2**. A Menger space is a triple (X, **F**, t), where (X, **F** ) is a PM-space and t is a T-norm with the following condition:
($P_5$) Fu, w (x+y) $\ge$ t ( Fu, v (x), F v, w (y)) for every u, v, w $\in$ X and x, y $\in$ $R^+$.

The concept of neighbourhood in PM-spaces was introduced by Schwizer-Sklar [12]. If u $\in$ X, $\varepsilon > 0$ and $\lambda \in (0, 1)$, then an $(\varepsilon, \lambda)$-neighbourhood of u, denoted by $U_u(\varepsilon, \lambda)$ is defined by

$$U_u(\varepsilon, \lambda) = \{v \in X : Fu, v (\varepsilon) > 1- \lambda\}$$

If (X, F, t) is a Menger space with the continuous T- norm t, then the family

$$\{U_u(\varepsilon, \lambda) : u \in X, \varepsilon > 0 \text{ and } \lambda \in (0, 1)\}$$

of neighbourhoods induces a Hausdorff topology on X and if $\sup_{a<1} t(a, a) = 1$, it is metrizable.

An important T-norm is the T-norm t(a, b) = min{a, b} for all a, b $\in$ [0,1] and this is the unique T-norm such that t(a, a) $\ge$ a for every a $\in$ [0,1]. Indeed if it satisfies this condition, we have

$$\min\{a, b\} \le t(\min\{a, b\}, \min\{a, b\}) \le t(a, b)$$
$$\le t(\min\{a, b\},1) = \min\{a, b\}$$

Therefore, t = min.
In the sequel, we need the following definitions due to Radu [9].
**Definition 3.** Let (X, **F**, t) be a Menger space with continuous T- norm t. A sequence $\{x_n\}$ of points in X is said to be convergent to a point x $\in$ X if for every $\varepsilon > 0$

$$\lim_{n \to \infty} Fx_n, x (\varepsilon) = 1.$$

**Definition 4.** . Let (X, F, t) be a Menger space with continuous T-norm t. A sequence $\{x_n\}$ of points in X is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda) > 0$ such that $Fx_n, x_m(\varepsilon) > 1 - \lambda$ for all $m, n \geq N$.
**Definition 5.** A Menger space (X, **F**, t) with the continuous T-norm t is said to be complete if every Cauchy sequence in X converges to a point in X.
**Lemma 1** [13, 15]. Let $\{x_n\}$ be a sequence in a Menger space (X, **F**, t), where t is a continuous T- norm and $t(x, x) \geq x$ for all $x \in [0, 1]$. If there exists a constant $k \in (0, 1)$ such that $Fx_n, x_{n+1}(kx) \geq Fx_{n-1}, x_n(x)$ for all $x > 0$ and $n \in N$, then $\{x_n\}$ is a Cauchy sequence in X.
We introduce the following concept for multivalued mappings in Menger space (X, **F**, t):
We denote by CB(X) the set of all nonempty, bounded and closed subsets of X. We have
$F^{\nabla}y, B(x) = \max\{Fy, b(x) : b \in B\}$
$F_{\nabla}A, B(x) = \min\{\min_{a\in A}\{F^{\nabla}a, B(x)\}, \min_{b\in B}\{F^{\nabla}A, b(x)\}\}$
for all A, B in X and $x > 0$.

## Main Results

**Theorem 1.** Let (X, **F**, t) be a complete Menger space with $t(x, y) = \min\{x, y\}$ for all x, $y \in [0, 1]$. Let S, T: $X \to CB(X)$ satisfying:

$$F_{\nabla}Su, Tv(kx) \geq \min\{Fu, v(x), F^{\nabla}u, Su(x), F^{\nabla}v, Tv(x), F^{\nabla}u, Tv((2-\alpha)x)), F^{\nabla}v, Su(x)\} \quad (1.1)$$

For all $u, v \in X$, $x \geq 0$, where $k \in (0, 1)$ and all $\alpha \in (0, 2)$. Then S and T have a common fixed point.
**Proof.** Let $x_0$ be an arbitrary point in X and $x_1 \in X$ is such that $x_1 \in Sx_0$ and
$Fx_0, x_1(kx) \geq F^{\nabla}x_0, Sx_0(kx) - \varepsilon$,
$x_2 \in X$ is such that $x_2 \in Tx_1$ and
$Fx_1, x_2(kx) \geq F^{\nabla}x_1, Tx_1(kx) - \varepsilon/2$.
Inductively $x_{2n+1} \in X$ is such that $x_{2n+1} \in Sx_{2n}$ and
$Fx_{2n}, x_{2n+1}(kx) \geq F^{\nabla}x_{2n}, Sx_{2n}(kx) - \varepsilon/2^{2n}$,
$x_{2n+2} \in X$ is such that $x_{2n+2} \in Tx_{2n+1}$ and
$Fx_{2n+1}, x_{2n+2}(kx) \geq F^{\nabla}x_{2n+1}, Tx_{2n+1}(kx) - \varepsilon/2^{2n+1}$.
Now we show that $\{x_n\}$ is a Cauchy sequence.
By (1.1) for all $x \geq 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$, we write

$$\begin{aligned} Fx_{2n+1}, x_{2n+2}(kx) &\geq F^{\nabla}x_{2n+1}, Tx_{2n+1}(kx) - \varepsilon/2^{2n+1} \\ &\geq F_{\nabla}S_{2n}, Tx_{2n+1}(kx) - \varepsilon/2^{2n+1} \\ &\geq \min\{Fx_{2n}, x_{2n+1}(x), F^{\nabla}x_{2n}, Sx_{2n}(x), F^{\nabla}x_{2n+1}, Tx_{2n+1}(x), \\ &\qquad F^{\nabla}x_{2n}, T x_{2n+1}((2-\alpha)x)), F^{\nabla}x_{2n+1}, Sx_{2n}(x)\} - \varepsilon/2^{2n+1} \\ &\geq \min\{Fx_{2n}, x_{2n+1}(x), Fx_{2n}, x_{2n+1}(x), Fx_{2n+1}, x_{2n+2}(x), \\ &\qquad Fx_{2n}, x_{2n+2}((1+q)x)), Fx_{2n+1}, x_{2n+1}(x)\} - \varepsilon/2^{2n+1} \\ &\geq \min\{Fx_{2n}, x_{2n+1}(x), Fx_{2n}, x_{2n+1}(x), Fx_{2n+1}, x_{2n+2}(x), \\ &\qquad t(Fx_{2n}, x_{2n+1}(x), Fx_{2n+1}, x_{2n+2}(qx)), 1\} - \varepsilon/2^{2n+1}. \end{aligned} \quad (1.2)$$

Since t is a continuous T- norm and distribution function **F** is left continuous, letting $q \to 1$, in (1.2), we have

$$Fx_{2n+1}, x_{2n+2}(kx) \geq \min\{Fx_{2n}, x_{2n+1}(x), Fx_{2n+1}, x_{2n+2}(x)\} - \varepsilon/2^{2n+1}. \quad (1.3)$$

Similarly we have also

$F_{x_{2n+2},\, x_{2n+3}}(kx) \geq \min\{ F_{x_{2n+1},\, x_{2n+2}}(x), F_{x_{2n+2},\, x_{2n+3}}(x)\} - \varepsilon/2^{2n+2}$. (1.4)

Thus from (1.3) and (1.4) it follows that

$F_{x_{n+1},\, x_{n+2}}(kx) \geq \min\{ F_{x_n,\, x_{n+1}}(x), F_{x_{n+1},\, x_{n+2}}(x)\} - \varepsilon/2^{n+1}$

for n = 1, 2, … and so for positive integers n, p

$F_{x_{n+1},\, x_{n+2}}(kx) \geq \min\{ F_{x_n,\, x_{n+1}}(x), F_{x_{n+1},\, x_{n+2}}(x/k^p)\} - \varepsilon/2^{n+1}$.

Letting $p \to \infty$, we get

$F_{x_{n+1},\, x_{n+2}}(kx) \geq F_{x_n,\, x_{n+1}}(x) - \varepsilon/2^{2n+1}$.

Since $\varepsilon$ is arbitrary making $\varepsilon \to 0$ we obtain

$F_{x_{n+1},\, x_{n+2}}(kx) \geq F_{x_n,\, x_{n+1}}(x)$.

Therefore by Lemma 1, $\{x_n\}$ is a Cauchy sequence. So it converges to a point $z \in X$.

Now by (1.1) with $\alpha = 1$, we have

$$\begin{aligned} F^{\nabla}_{x_{2n+2},\, Sz}(kx) &\geq F_{\nabla Sz,\, Tx_{2n+1}}(kx) \\ &\geq \min\{F_{z,\, x_{2n+1}}(x), F^{\nabla}_{z,\, Sz}(x), F^{\nabla}_{x_{2n+1},\, Tx_{2n+1}}(x), \\ &\qquad F^{\nabla}_{z,\, Tx_{2n+1}}(x), F^{\nabla}_{x_{2n+1},\, Sz}(x)\} \\ &\geq \min\{F_{z,\, x_{2n+1}}(x), F^{\nabla}_{z,\, Sz}(x), F_{x_{2n+1},\, x_{2n+2}}(x), \\ &\qquad F_{z,\, x_{2n+2}}(x), F^{\nabla}_{x_{2n+1},\, Sz}(x)\}. \end{aligned}$$

Letting $n \to \infty$, we obtain

$F^{\nabla}_{z,\, Sz}(kx) \geq \min\{1, F^{\nabla}_{z,\, Sz}(x), 1, 1, F^{\nabla}_{z,\, Sz}(x)\}$.

This gives

$F^{\nabla}_{z,\, Sz}(kx) \geq F^{\nabla}_{z,\, Sz}(x)$,

which is a contradiction. Thus we have $z \in Sz$. Similarly we can prove that $z \in Tz$.

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