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DECOMPOSITION OF A SPECIAL TYPE OF MACHINES

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تحليل بعض الحاسبات الجبرية الخاصة

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في هذا البحث نقدم مفهوم الحاسبة الدورية منتهية الحالات A_n ، ومن ثم نعطي خصائصا تميز هذه الحاسبات الجبرية تمييزاً تاماً ، وذلك باستخدام تحليل العدد الصحيح الموجب n إلى عوامله الأولية .

وعلى وجه الخصوص ، فإننا نبرهن على صحة الآتي : «إن كل حاسبة جبرية منتهية الحالات ودورية :

$$A_n = (Q_n, X, Y, \delta_n, \lambda_n)$$

حيث $(n \geq 2)$ ، تشاكل تماما ، حاسبة جبرية أخرى تتكون من مركبات دورية أقل تعقيداً موصلة بمتعدد روابط على التسلسل أو بمتعدد روابط على التوازي .

Key Words: Automata theory, Finite-state Cyclic machines, Algebraic theory of computation.

ABSTRACT

In this paper, the concept of a finite n-state cyclic machine A_n is defined. These machines are then completely characterized using the prime factorization of n. Specifically, it is proved that any finite-state cyclic machine

$$A_n = (Q_n, X, Y, \delta_n, \lambda_n), \text{ for } n \geq 2$$

is isomorphic to a multi-series or multi-parallel composition of simpler cyclic machines.

INTRODUCTION

One of the basic concepts in computer applications is that of a finite-state sequential machine. Informally, this is a device that is served with inputs and generates outputs at discrete instances of time, and whose output at any time depends on the input and the internal condition (called the state) of the device at that time. In addition, the input and the internal state at any time determine the initial condition at the next instant of time. The action of this system consists of specifying the next state as well as the output of the machine.

Of the many sequential finite-state machines that are studied mathematically, we restrict ourselves here to a special type of what is known as Mealy Machines. In this work we investigate some properties of a special case of an abstract Mealy machine (or automaton) which we call finite-state cyclic machines. This notion of a finite-state cyclic machine extends naturally to initialized machines (i.e. machines in which a particular member of the set of states is designated as the initial state.) Thus we shall deal with both arbitrary, and initialized finite-state cyclic machines simultaneously.

Section 1 introduces the basic concepts and terminology of a finite state cyclic machine $A_n = (Q_n, X, Y, \delta_n, \lambda_n)$, as well as the serial and parallel composition operations $\ominus\!\!\!\rightarrow$ and |b| which are special algebraic devices for combining these machines. It is proved (Theorems 1.7 and 1.10) that for a pair of cyclic machines A_n, A_m, the composite machines A_n $\ominus\!\!\!\rightarrow$ A_m and for (n, m) = 1, A_n |b| A_m are again finite-state cyclic machines. Similar results (Theorem 1.11) can be obtained for the initialized finite-state cyclic machines $A_n^{(i)}$, $A_m^{(i)}$.

In **Section 2** we present the natural concept of a network of cyclic machines, which results from the iteration of the operations of multi-series and multi-parallel compositions of cyclic machines. Applying the conclusions of section 1, we prove some basic results (Theorems 2.3 and 2.7) on the decomposition of a finite-state cyclic machine A_n into simpler cyclic machines using these operations. The corresponding network diagram for this decomposition is illustrated in Figure 5, (which we call the H.-M. diagram of the machine).

1. BASIC CONCEPTS ON CYCLIC MACHINES

1.1. NOTATION AND TERMINOLOGY

$Q_* = \{q_1, q_2, ...\}$ will denote an infinite universal set of 'states' q_i, and n, m denote natural numbers≥1, On what follows we shall only use the first n or m elements of Q_*. For any n≥2, let $Q_n = \{q_1, \dots, q_n\}$.

A will denote a Mealy machine, which is (cf [1]) a 5-tuple $(Q, X, Y, \delta, \lambda) = A$, where Q is the set of internal states, X is the set of input symbols and Y is the set of output symbols; δ: QxX –> Q represents the transition function of A, while λ: QxX –> Y denotes its output function.

1.2. DEFINITION

Let n≥2. The n-state Mealy machine $[A_n = (Q_n, X, Y, \delta_n, \lambda_n)]$ is called an **n-state cyclic machine** if A_n satisfies the following conditions:

a) Q_n is defined as in 1.1;

b) $X = Y = \{0, 1\}$;

c) The transition function δ_n: Q_nxX –> Q_n is given by:

$\delta_n(q_i, 0) = q_i$, $1 \le i \le n$;

$$\delta_n(q_1, 1) = \begin{cases} q_{i+1}, 1 \le i \le n-1, \\ q_1, i = n. \end{cases}$$

d) the output function λ_n : Q_nxX –> Y is given by :

$$\lambda_n(q_i, x) = \begin{cases} 1 & (i, x) = (n, 1), \\ 0 & otherwise \end{cases}$$

1.3 DEFINITION

For n≥2, the initialized n-state cyclic machine $A_n^{(i)}$ is obtained from the cyclic machine A_n by specifying the initial state q_1, and we write

$A_n^{(i)} = (Q_n, X, Y, \delta_n, \lambda_n, q_1)$.

1.4 EXAMPLE

The cyclic machines A_4 and $A_5^{(i)}$ are represented by the following state diagrams (Fig. 1).

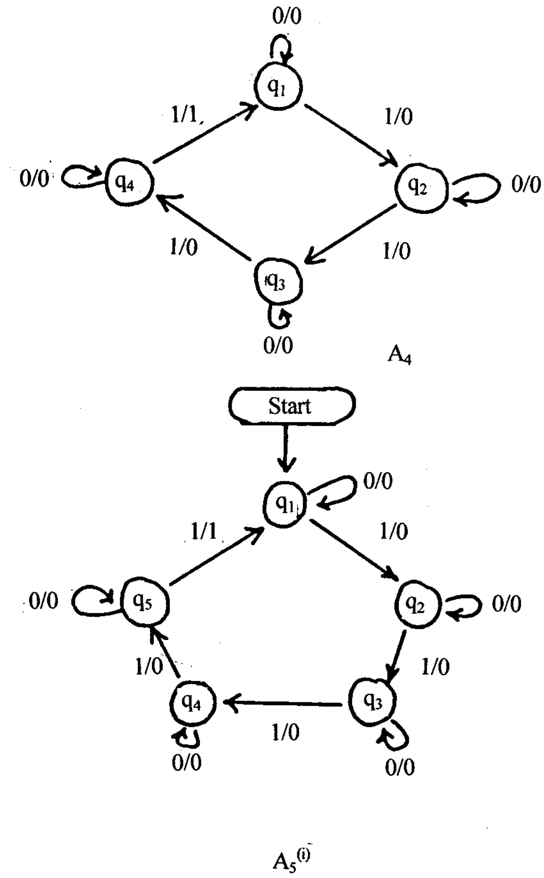

Figure 1

1.5 DEFINITION

The *series composition* of a pair of state machines (both arbitrary or else both initialized) is defined as usual (cf[4]) for the Mealy machines:

For n≥2 and m≥2 we denote the series composition for the two finite state machines A_n, A_m (respectively $A_n^{(i)}$, $A_m^{(i)}$) by $A_n \ominus\!\!\!> A_m$ (respectively $A_n^{(i)} \ominus\!\!\!> A_m^{(i)}$). We then write

$$A_n \ominus\!\!\!> A_m = (Q_n x Q_m, X, Y, \delta, \lambda),$$

(respectively $A_n^{(i)} \ominus\!\!\!> A_m^{(i)} = (Q_n x Q_m, X, Y, \delta, \lambda, (q_1, q_1))$)

where $\delta((q_i, q_j), x) = (\delta_n(q_i, x), \delta_m(q_j, \lambda_n(q_i, x))$ and

$\lambda((q_i, q_j), x) = \lambda_m(q_j, \lambda_n(q_i, x))$, for $q_i \in Q_n$, $q_j \in Q_m$

and $x \in X$.

1.6 REMARK

We recall the fact that a pair of Mealy machines A = (Q, X, Y, δ λ) and A' = (Q', X', Y', δ', λ') are said to be isomorphic if there exist three bijective maps h_Q : Q –> Q', h_X : X —>X' and h_Y:Y –> Y' such that $h_Q(\delta(q_i, x)) = \delta'(h_Q(q), h_X(x))$ and $h_Y(\delta(q_i, x)) = \lambda'(h_Q(q), h_x(x))$ for all $q \in Q$, $x \in X$. That is, the two machines A and A' are isomorphic if the following map diagram (Fig. 2) is commutative.

For the initialized state machines $A^{(i)}$, $A'^{(i)}$ we require the further condition that $h_Q(q_1) = q_1'$. The triple (h_Q, h_X, h_Y) of bijections is said to constitute an isomorphism between the state machines A and A'.

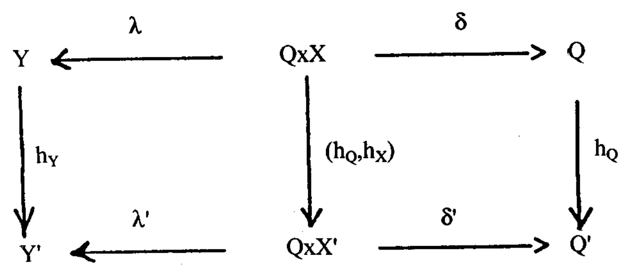

Figure 2

1.7 THEOREM

For n, m ≥2 let A_n, A_m be two finite-state cyclic machines (respectively $A_n^{(i)}$, $A_m^{(i)}$ be two initialized finite-state cyclic machines). Then the series composition $A_n \,(\to)\, A_m$ (respectively $A_n^{(i)} \,(\to)\, A_m^{(i)}$) is isomorphic to the cyclic machine A_{nm} (respectively $A_{nm}^{(i)}$).

Proof: Put $A_n = (Q_n, X, Y, \delta_n, \lambda_n)$ and $A_m = (Q_m, X, Y, \delta_m, \lambda_m)$ (respectively $A_n^{(i)} = (Q_n, X, Y, \delta_n, \lambda_n, q_1)$ and $A_m^{(i)} = (Q_m, X, Y, \delta_m, \lambda_m, q_1)$), and let

$$A_n \,(\to)\, A_m = (Q_n x\, Q_m, X, Y, \delta, \lambda) \text{ (respectively}$$

$$A_n^{(i)} \,(\to)\, A_m^{(i)} = (Q_n x\, Q_m, X, Y, \delta, \lambda, (q_1, q_1)))$$

be their series composition. We shall show that $A_n \,(\to)\, A_m$ (respectively $A_n^{(i)} \,(\to)\, A_m^{(i)}$) is equivalent to a cyclic machine (respectively an initialized cyclic machine).

Consider the cyclic machine $A_{nm} = (Q_{nm}, X, Y, \delta_{nm}, \lambda_{nm})$ (respectively $A_{nm}^{(i)} = (Q_{nm}, X, Y, \delta_{nm}, \lambda_{nm}, q_1)$), and define the map $h_Q : Q_n x Q_m \to Q_{nm}$ by the rule : $h_Q (q_i, q_j) = q_{(j-1)n+i}$ for $1 \le i \le n$, $1 \le j \le m$. For the maps h_X, h_Y take the identity functions $h_X = 1_X$, $h_Y = 1_Y$. We shall now show that h_Q is injective: For $1 \le i \le n$ and $1 \le j \le k \le m$ it follows that

$$h_Q (q_i, q_j) = q_{(j-1)n+i} \neq q_{(k-1)n+i} = h_Q (q_i, q_k).$$

Also, for $1 \le i < j \le n$ and $1 \le k, t \le m$ we deduce that

$$h_Q (q_i, q_k) = q_{(k-1)n+i} \neq q_{(t-1)n+j} = h_Q (q_j, q_t).$$

Thus h_Q is indeed injective. To verify that h_Q is surjective let $q_k \in Q_{nm}$. Then there exsists $q_i \in Q_n$, $q_j \in Q_m$ such that $h_Q (q_i, q_j) = q_k$ (by taking $j = [(k-1)/n] + 1$ and $i = k- (j-1)\, n$).

For the verification of the isomorphism condition it will be necessary to prove, for $1 \le i \le n$, $1 \le j \le m$, and $x \in \{0,1\}$, that

$$h_Q (\delta ((q_i, q_j), x)) = \delta_{nm} (q_{(j-1)n+i}, x) \qquad (1)$$

and

$$\lambda ((q_i, q_j), x) = \lambda_{nm} (q_{(j-1)n+i}, x) \qquad (2)$$

Case 1 : Let x = 0. Then, for $1 \le i \le n$, $1 \le j \le m$ we have:

$h_Q (\delta ((q_i, q_j), 0)) = h_Q (\delta_n (q_i, 0), \delta_m (q_j, \lambda_m (q_j, \lambda_n (q_i, 0))))$

$= h_Q (q_i, q_j) = q_{(j-1)n+i}$.

But $\lambda_{mn} (h_Q (q_i, q_j), 0) = \delta_{mn} (q_{(j-1)n+i}, 0) = q_{(j-1)n+i}$.

Thus, $h_Q (\delta ((q_i, q_j), 0)) = \delta_{nm} (q_{(j-1)n+1}, 0)$.

Moreover, $\lambda ((q_i, q_j), 0) = \lambda_m (q_j, \lambda_n (q_i, 0)) = 0$,

and $\lambda_{nm} (h_Q (q_i, q_j), 0) = \lambda_{mn} (q_{(j-1)n+i}, 0) = 0$.

Hence $\lambda ((q_i, q_j), 0) = \lambda_{mn} (q_{(j-1)n+i}, 0)$.

Case 2: Let x= 1. In a similar fashion it can be seen that the equations (1) and (2) hold. The proof is facilitated by easily verifying it for the following three situations : a) for i= n, j= m; b) for i= n, $1 \le j \le m-1$; and c) for $1 \le i \le n-1$, $1 \le j \le m$.

In the case of initialized finite state cyclic machines, we note that $h_Q (q_1, q_1) = q_1$, which completes the proof.

1.8 COROLLARY

For any three finite-state cyclic machines A_n, A_m, A_p, the following holds :

a) $A_n \,(\to)\, A_m \cong A_m \,(\to)\, A_n$.

b) $(A_n \,(\to)\, A_m) \,(\to)\, A_p \cong A_n \,(\to)\, (A_m \,(\to)\, A_p)$.

We shall now present, for the cyclic machines, the corresponding notion of parallel composition, as described for Mealy machines (cf [2 & 3]).

1.9 DEFINITION

For n,m≥2 consider the two cyclic machines

$$A_n = (Q_n, X, Y, \delta_n, \lambda_n),\ A_m = (Q_m, X, Y, \delta_m, \lambda_m)$$

(respectively the two initialized machines

$$A_n^{(i)} = (Q_n, X, Y, \delta_n, \lambda_n, q_1),\ A_m^{(i)} = (Q_m, X, Y, \delta_m, \lambda_m, q_1))$$

and let b denote a binary operation on Y. Then the *parallel compositon* of the two machines A_n, A_m (respectively $A_n^{(i)}$, $A_m^{(i)}$) with respect to the map b is defined as follows :

$$A_n | b | A_m = (Q_n x Q_m, X, Y, \delta, \lambda),$$

(respectively $A_n^{(i)} | b | A_m^{(i)} = (Q_n x Q_m, X, Y, \delta, \lambda, (q_1, q_1)))$

where $\delta ((q_i, q_j), x) = (\delta_n (q_i, x), \delta_m (q_j, x))$,

and $\lambda ((q_i, q_j), x) = b (\lambda_n (q_i, x), \lambda_m (q_j, x))$.

In other words, if we put $\alpha ((q_i, q_j), x) = ((q_i, x), (q_j, x))$ then the functions δ and λ for the parallel composition machine will be defined so that the following two map diagrams (Figure 3a and Figure 3b) are commutative.

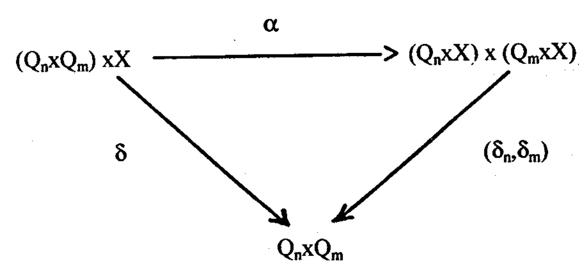

Figure 3a

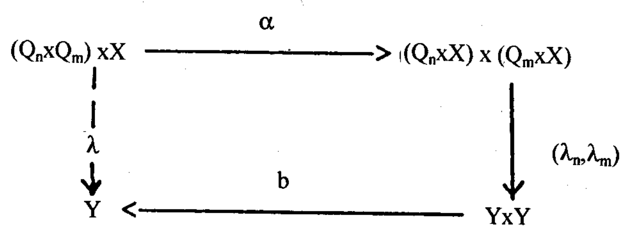

Figure 3b

1.10 REMARK

The series and parallel composition (as defined in 1.5 and 1.9) of two cyclic machines A_n, A_m can be represented diagrammatically as follows (Figure 4) : (cf [4 & 5]).

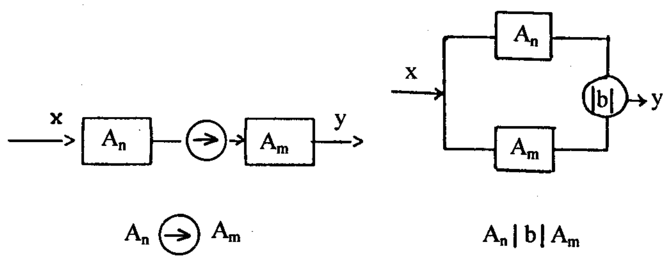

Figure 4

1.11 THEOREM

Let m, n≥2 be two relatively prime integers, and consider the two arbitrary finite-state cyclic machines

$A_n = (Q_n, X, Y, \delta_n, \lambda_n)$, $A_m = (Q_m, X, Y, \delta_m, \lambda_m)$ (respectively the initialized cyclic machines $A_n^{(i)}$ and $A_m^{(i)}$, with initial states q_o). Define the binary operation b on Y via logical conjunction; i.e.

$$b(y_i, y_2) = \begin{cases} 1 & if\, y_1 = 1 = y_2, \\ 0 & otherwise. \end{cases}$$

Then $A_n | b | A_m$ (respectively $A_n^{(i)} | b | A_m^{(i)}$) is isomorphic to the finite-state cyclic machine A_{nm} (respectively the initialized finite-state cyclic machine $A_{nm}^{(i)}$).

Proof : Let $A_n | b | A_m = (Q_n x Q_m, X, Y, \delta, \lambda)$

(respectively $A_n^{(i)} | b | A_m^{(i)} = (Q_n x Q_m, X, Y, \delta, \lambda, (q_1, q_1)))$

be the parallel composition of the two cyclic machines A_n, A_m

(respectively $A_n^{(i)}$, $A_m^{(i)}$). It is easy to see that $A_n | b | A_m$

(respectively $A_n^{(i)} | b | A_m^{(i)}$) has nm distinct states and behaves as the cyclic machine $A_{n\,m}$ (respectively $A_{n\,m}^{(i)}$), as we show below.

Case 1. Let x = 0. Then for 0≤i≤n-1 and 0≤j≤m-1 we have (using definition 1.9):

$$\delta ((q_i, q_j),0) = (\delta_n (q_i, 0),\delta_m (q_j,0)) = (q_i, q_j),$$

and

$$\lambda ((q_i, q_j),0) = b (\lambda_n (q_i, 0),\lambda_m (q_j,0)) = (q_i, q_j) = b (0,0) = 0.$$

Case 2. Let x = 1. Then:

a) For i = n-1 and j = m-1 we have

$$\delta ((q_{n-1}, q_{m-1}),1) = (\delta_n (q_{n-1}, 1),\delta_m (q_{m-1},1)) = (q_0, q_0),$$

and

$$\lambda ((q_{n-1}, q_{m-1}),1) = b (\lambda_n (q_{n-1}, 1),\lambda_m (q_{m-1},1)) = b (1,1) = 1.$$

b) For i = n-1 and 0≤j≤m-1 we have

$$\delta ((q_{n-1}, q_j), 1) = ((q_0, q_{j+1}), 1),$$

and

$$\lambda ((q_{n-1}, q_j), 1) = b (1, 0) = 0.$$

c) For 0≤i≤n-2 and j = m-1 we have

$$\delta ((q_i, q_{m-1}), 1) = (q_{i+1}, q_0);\ \lambda ((q_i, q_{m-1}), 1) = b (0, 1) = 0.$$

d) For 0≤i≤n-2, 0≤j≤m-2 we have

$$\delta ((q_i, q_j), 1) = (q_{i+1}, q_{j+1}), \text{ and } \lambda ((q_i, q_j), 1) = b (0, 0) = 0.$$

We deduce that the number of states reached by an arrow (for value 1) as in a), b), c) and d) above will be respectively 1, m-1, n-1 and (n-1) (m-1), and these states are all distinct. Thus the total number of these states is equal to nm. It follows that the machine $A_n | b | A_m$ (respectively $A_n^{(i)} | b | A_m^{(i)}$ has nm states, each of which emits one arrow and is targeted by one arrow.

Consider now a cycle *C* (of states) of $A_n | b | A_m$ for which

$$\delta ((q_i, q_j), 1) = (q_{(i+1)mod n}, q_{(j+1)mod m}) \text{ for } 0\le i<n,\ 0\le j<m.$$

It is obvious that this cycle has N(n, m) distinct states, with N (n, m) denoting the least common multiple of n and m. But as n and m are relatively prime integers, N (n, m) = nm. Thus the cycle *C* spans all the states of the machine $A_n | b | A_m$ (respectively

$A_n^{(i)} | b | A_m^{(i)}$). We also note that for each $q_k \in Q_{n\,m}$ there corresponds the element $(q_{k mod n}, q_{k mod m})$ of $Q_n x Q_m$ for 0≤k≤nm-1. Hence we get that the machine $A_n | b | A_m$ (respectively $A_n^{(i)} | b | A_m^{(i)}$) is indeed isomorphic to the cyclic machine A_{nm} .

The following corollary and its proof are immediate.

1.12 COROLLARY

For cyclic finite-state machines A_n, A_m, A_p with n, m, p relatively prime in pairs, we have :

a) $A_n | b | A_m \cong A_m | b | A_n$;

b) $(A_n | b | A_m) | b | A_p \cong A_n | b | (A_m | b | A_p)$.

1.13 REMARKS

The condition that n and m are relatively prime is essential in 1.11. If n and m are not relatively prime, then the state diagram for the machine $A_n | b | A_m$ may not be connected.

For x = 0, case 1 of Theorem 1.11 holds.

For x = 1, the cycle *C* will be as in case 2 of Theorem 1.11, and *C* has N (n, m) states. Since N (n, m) <nm, the cycle

C will notgenerate all the states of the machine $A_n|b|A_m$. This implies that $A_n|b|A_m$ cannot be cyclic.

To examine the parallel composition for the initialized finite state machines, we recall the following definitions, (cf [2 & 6] & 7]).

1.14 DEFINITIONS

1) Let $A = (Q, X, Y, \delta, \lambda, q_o)$ be an initialized Mealy machine. The state $q \in Q$ is said to be accessible if there exists a directed path from q_o to q in the state diagram of A. A state for which there is no directed path from it to q_o is called inaccessible.

2) We shall denote by Re (A) the initialized Mealy machine which is obtained from the machine A by deleting from Q the subset of all states (of the machine A) inaccessible from q_o. We shall call Re (A) the *reduced initialized* Mealy machine of A with respect to q_o.

1.15 THEOREM

For $n, m \geq 2$ let $A_n^{(i)} = (Q_n, X, Y, \delta_n, \lambda_n, q_o)$ and $A_m^{(i)} = (Q_m, X, Y, \delta_m, \lambda_m, q_o)$ be two initialized finite-state cyclic machines, and suppose that the map b: $Y \times Y \rightarrow Y$ is given as in 1.11. Then the reduced machine Re $(A_n^{(i)}|b|A_m^{(i)})$ is isomorphic to the finite-state machine $A_{N(n,m)}^{(i)}$, where N (n, m) denotes the least common multiple of n and m.

Proof: Consider the parallel composition $A_n^{(i)}|b|A_m^{(i)}$ of the pair of initialized machines $A_n^{(i)}$, $A_m^{(i)}$. From 1.9 we infer that

$$\delta((q_i, q_j), 1) = (q_{(i+1)\text{mod}n}, q_{(j+1)\text{mod}m}), \text{ for } 1 \leq i \leq n, 1 \leq j \leq m.$$

In the above parallel composition state machine, the various states

$$q = (q_{(i+1)\text{mod}n}, q_{(j+1)\text{mod}m}) \quad (3)$$

are accessible, as explained in the following steps.

(i) The state q can be accessed from (q_1, q_1) by employing the string 1^i for $1 \leq i \leq N(n, m)$.

(ii) For an arbitrary state q as in (3) we have $\delta(q, 0) = q$.

(iii) For $i = N(n, m)$, we have $\delta((q_1, q_1), 1^i) = (q_1, q_1)$.

It follows that for all i, j there exists an integer k such that

$$\delta((q_{(i+1)\text{mod}n}, q_{(i+1)\text{mod}m}), 1^k) = (q_{(j+1)\text{mod}n}, q_{(j+1)\text{mod}m}).$$

Thus, any state from (3) above can be accessed from any other state of (3), by using the string 1^k for any integer k.

It can also be easily inferred that

$$\lambda((q_i, q_j), 0) = 0, \; 1 \leq i \leq n, \; 1 \leq j \leq m;$$

and

$$\lambda((q_i, q_i), 1) = \begin{cases} 1 & for\, i = 1, j = m, \\ 0 & otherwise. \end{cases}$$

This completes the proof that the reduced machine

Re $(A_n^{(i)}|b|A_m^{(i)})$

is isomorphic to the initialized finite state cyclic machine $A^{(i)}_{N(n,m)}$.

2. DECOMPOSITION OF CYCLIC MACHINES

In this section we shall present a method of decomposing an arbirary finite-state cyclic machine A_n (for n composite) to simpler cyclic machines. We shall prove that A_n is equivalent to a network of machines joined under the series and parallel compositions.

2.1 DEFINITION

For $k \geq 1$, let

$$m_1, \ldots, m_k; \; i_1, \ldots, i_k, \quad (1)$$

be two sequences of k non-negative integers, with $m_j \geq 2$ for $1 \leq j \leq k$. (For typographical convenience we shall put $m_j = m(j)$, etc.). Then the construction

$$\Big(\overset{i_1}{\underset{r_1=1}{\bigodot}} A_{m(1)} \Big) \, |b| \ldots |b| \, \Big(\overset{i_k}{\underset{r_k=1}{\bigodot}} A_{m(k)} \Big) \quad (2)$$

formed by the application of the (multi-) series and (multi-) parallel composition operations (as in 1.5 and 1.9), on the finite-state cyclic machines $A_{m(j)}$, for $1 \leq j \leq k$, is called a *network of cyclic machines belonging to the pair of integer sequences* (1). This network is schematically illustrated in Figure 5. The resulting machine (2) will be denoted here by the symbol

$$B^{i_1 \ldots\ldots i_k}_{m(1) \ldots\ldots m(k)}. \quad (3)$$

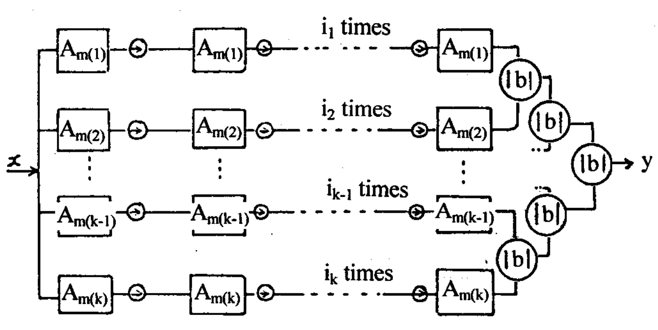

Figure 5

2.2 REMARKS

1. By the corollaries of Theorems 1.7 and 1.11, the construction (3) can easily be shown to be well-defined.

2. It follows from the construction (3) that we can naturally extend it to initialized Mealy machines.

3. If k = 1 then there is no "parallel" operation in $B^{i_1}_{m(1)}$ which is therefore just the series composition for the cyclic machine $A_{m(i)}$ composed i_1 times.

2.3 THEOREM

Let $n\geq 2$ be a natural number and $n = p_1^{i_1} \ldots\ldots p_k^{i_k}\ (k \geq 1)$ the standard factorization of n, with $p_1, \ldots, p_k$ distinct primes. Then the arbitrary finite-state cyclic machine A_n is equivalent to the composite

$$B^{i_1 \ldots\ldots i_k}_{p(1)\ldots\ldots p(k)}$$

Proof : This follows immediately from the results of 1.7, 1.11, and 2.1

2.4 COROLLARY

Other assumptions being as in 2.3, let s be a permutation of the set {1, 2, .., k}, and, for $1\leq r\leq k$, let $t(r) = i_{s(r)}$, $q(r) = p_{s(r)}$ and $v(r) = q_r$. Then

$$(a)\ B^{t(1)\ldots\ldots t(k)}_{q(1)\ldots q(k)} \cong A_n$$

$$(b)\ A_n \cong B^{1\ldots\ldots 1}_{v(1)\ldots v(k)}\ .$$

Proof: Both results follow from 2.3 and the facts that

(i) $A_r \ominus\!\!\rightarrow A_s \cong A_s \ominus\!\!\rightarrow A_r$

and

(ii) $(A_r \ominus\!\!\rightarrow A_s) \ominus\!\!\rightarrow A_t \cong A_r \ominus\!\!\rightarrow (A_s \ominus\!\!\rightarrow A_t)$,

for any positive integers r, s, t, together with the corresponding results for the operation of parallel-composition, but with the added condition that r, s, t are pairwise relatively prime, as well as the following easily proved relation :

(iii) $A_r | b | (A_s \ominus\!\!\rightarrow A_t) \cong (A_r | b | A_s) \ominus\!\!\rightarrow (A_r | b | A_t)$

2.5 EXAMPLE

If some of the integers $m_1, m_2, \ldots, m_k$ are not relatively prime, then the network

$$B^{i_i \ldots\ldots i_k}_{m(1)\ldots m(k)}$$

may have a non-connected state-diagram, as is illustrated by the following example.

Consider the network

$$B^{1,1}_{2,4} = A_2 |b| A_4$$

which is the parallel composition of A_2 and A_4; the state-tables and diargam of these machines are depicted below in Figures 6a, 6b and 6c.

	δ_2 0	δ_2 1	λ_1 0	λ_1 1
q_1	q_1	q_2	0	0
q_2	q_2	q_1	0	1

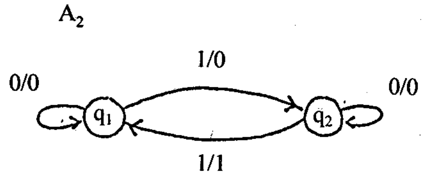

Figure 6a

	δ_4 0	δ_4 1	λ_4 0	λ_4 1
q_1	q_1	q_2	0	0
q_2	q_2	q_3	0	0
q_3	q_3	q_4	0	0
q_4	q_4	q_1	0	1

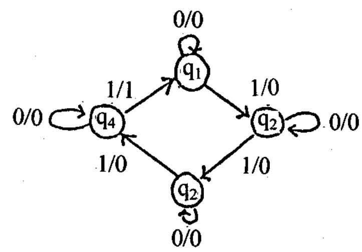

Figure 6b

	δ 0	δ 1	λ 0	λ 1
q_1q_1	q_1q_1	q_2q_2	0	0
q_2q_1	q_2q_1	q_3q_2	0	0
q_3q_1	q_3q_1	q_4q_2	0	0
q_4q_1	q_4q_1	q_1q_2	0	0
q_1q_2	q_1q_2	q_2q_1	0	1
q_2q_2	q_1q_3	q_3q_1	0	0
q_3q_2	q_3q_2	q_4q_1	0	0
q_4q_2	q_4q_2	q_1q_1	0	1

Figure 6c

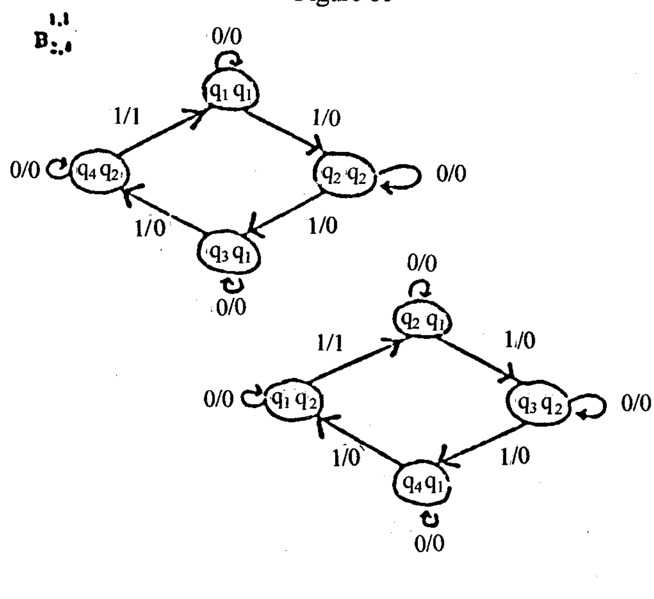

Figure 6c

2.6 REMARK

In a similar vein the concept of multi-series/multi-parallel composition network can be easily adapted to initialized finite-state cyclic machines. The symbol

$$^{(i)}B^{i_1 \ldots i_k}_{m(1)\ldots m(k)}$$

will denote the multi-series/multi-parallel network defined by the pair of integer sequences of 2.1 (1).

The next theorem gives the corresponding result in 2.3 for the initialized finite state cyclic machines

$$A^{(i)}_{m(j)}, \text{ for } 1 \leq j \leq k.$$

2.7 THEOREM

Let n be a natural number ≥2 and $n = p_1^{i_1} \cdots p_k^{i_k}$ be the standard factorization of n, with $p_1, .., p_k$ distinct primes, Then:

$$A_n^{(i)} \cong {}^{(i)}B^{i_1 \ldots i_k}_{p(1) \ldots p(k)}$$

Proof: Follows immediately from 1.7 and 1.11, in conjunction with 2.3.

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