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A STUDY ON $|N, p_n, q_n|_k$ SUMMABILITY FACTORS OF INFINITE SERIES

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دراسة في معاملات تجميع لمتسلسلات لا نهائية

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يتضمن البحث برهنة نظرية جديدة تتـعلق بمعاملات التجميع من نوع $|N, p_n, q_n|_k$ لمتسلسلات لانهائية ، نتائج أخرى مستنتجة أيضاً .

Key Words : Summability, Series, Sequences

ABSTRACT

A new theorem concerning $|N, p_n q_n|_k$ summability factors of infinite series $\sum_1^\infty a_n$ is proved. Some other results are deduced.

1. INTRODUCTION

Let $\sum_1^\infty a_n$ be an infinite series with partial sums s_n. Let σ_n^δ and η_n^δ denote the nth Cesaro mean of order δ ($\delta > -1$) of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series $\sum a_n$ is said to be summable (C, δ) with index k, or simply summable $|C,\delta|_k$, $k \geq 1$ if

$$\sum_{n=1}^\infty n^{k-1}\left|\sigma_n^\delta - \sigma_{n-1}^\delta\right|^k < \infty$$

or equivalently

$$\sum_{n=1}^\infty n^{-1}\left|\eta_n^\delta\right|^k < \infty \quad .$$

Let $\{p_n\}$ be a sequence of real or complex numbers with $P_n = p_0 + p_1 + \ldots + p_n$, $P_n \to \infty$ as $n \to \infty$, $p_{-1} = P_{-1} = 0$

The series $\sum a_n$ is said to be summable $|N, p_n|$, if

$$\sum_{n=1}^\infty |t_n - t_{n-1}| < \infty \tag{1}$$

where

$$t_n = P_n^{-1}\sum_{v=1}^\infty p_{n-v}s_v \qquad (t_{-1} = 0)$$

We write $p = \{p_n\}$ and

$$M = \left\{p : p_n > 0 \;\&\; \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1\,,\; n = 0, 1, \cdots\right\}$$

It is known that for $p \in M$, (1) holds if and only if (Das [4])

$$\sum_{n=1}^\infty \frac{1}{nP_n}\left|\sum_{v=1}^n p_{n-v} v a_v\right| < \infty \quad .$$

Definition 1 : For $p \in M$, we say that $\sum_1^\infty a_n$ is summable $|N, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^\infty \frac{1}{nP_n^k}\left|\sum_{v=1}^n p_{n-v} v a_v\right|^k < \infty \quad .$$

In the special case in which $p_n = A_n^{r-1}$, $r > -1$, where A_n^r is the coefficient of x^n in the power series expansion of $(1-x)^{-r-1}$ for $|x| < 1$, $|N,p_n|_k$ summability reduces to $|C,r|_k$ summability see [4] .

The series $\sum_1^\infty a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \geq 1$ if.

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_n - T_{n-1}|^k < \infty$$

where

$$T_n = P_n^{-1}\sum_{v=1}^{n} p_v s_v \ .$$

If we take $p_n = 1$, then $|N,p_n|_k$ summability is reduces to $|C,1|_k$ summability. In general, these two summabilities are not comparable.

Let $\{p_n\}$, $\{q_n\}$ be sequences of numbers and denote

$$Q_n = q_0 + q_1 + \cdots + q_n \ , q_{-1} = Q_{-1} = 0$$

$$R_n = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0 \, , R_n \to \infty \text{ as } n \to \infty$$

$$\Delta f_n = f_n - f_{n+1}, \text{ for any sequence } \{f_n\}.$$

Definition 2 :

Let $\{p_n\}$, $\{q_n\}$ be sequences of positive real numbers, such that $q \in M$; We say that $\sum_1^\infty a_n$ is summable $|N,p_n, q_n|_k$, $k \geq 1$, if.

$$\sum_{n=1}^{\infty}\frac{p_n}{P_n R_{n-1}^k}\left|\sum_{v=1}^{n} P_{v-1} q_{n-v} a_v\right|^k < \infty$$

Clearly, $|N,p_n 1|_k$ reduces to $|\bar{N}, p_n|_k$.

The series $\sum a_n$ is said to be bounded $[R,\log n, 1]_k$, $k \geq 1$ if

$$\sum_{v=1}^{n} v^{-1}|s_v|^k = 0 \text{ (long) as } n \to \infty \text{ (Mishra[5])}$$

The series $\sum a_n$ is said to be bounded $[\bar{N},p_n]_k$, $k \geq 1$

$$\sum_{v=1}^{n} v^{-1}\left|s_v\right|^k = o\,(P_n) \text{ as } n \to \infty \text{ (Bor [2])}$$

If we take $k = 1$ (resp. $p_n = n^{-1}$), then $[\bar{N},p_n]_k$ boundedness is the same as $[\bar{N},p_n]_k$(resp. $[R,\log n,1]_k$) boundedness.

Here we give these two new definitions :

Definition 3 :

The series $\sum a_n$ is said to be bounded $[\bar{N},p_n]_k$, $k \geq 1$, if.

$$\sum_{v=1}^{n} q_{n-v}|s_v|^k = O(Q_n) \text{ as } n \to \infty \ .$$

Definition 4 :

The series $\sum a_n$ is said to be bounded $[N,p_n,q_n]_k$ $k \geq 1$, if

$$\sum_{v=1}^{n} p_v q_{n-v}\left|s_v\right|^k = o\,(R_n) \quad \text{as } n \to \infty$$

$[N,p_n,1]_k$ $k \geq 1$ and $[N,1,q_n]_k$, $k \geq 1$ are reduces to $[\bar{N},p_n]_k$ and $[N,q_n]_k$ respectively.

The object of this paper is to prove the following

2. MAIN RESULT

THEORM 1 :

If $\sum_1^\infty a_n$ is bounded $[N,p_n,q_n]_k$, $k \geq 1$ and if $\{p_n\}$, $\{q_n\}$, and $\{\lambda_n\}$ are positive real sequences satisfy the conditions : $q \in M$, $\{ p_n / P_n R^k_{n-1} \}$ nonincreasing for $q_n \neq 0$, and

$$\sum_{n=v+1}^{m+1}\frac{p_n q_{n-v-1}}{P_n R_{n-1}} = O\left(P_v^{-1}\right), m \to \infty \qquad (2)$$

$$\sum_{v=1}^{n} p_n q_{n-v-1}|\lambda_v| = O(1), n \to \infty \qquad (3)$$

$$\sum_{v=1}^{n}|\Delta R_v||\lambda_v|\frac{1}{q_{n-v-1}} = O(1), n \to \infty \qquad (4)$$

$$R_n|\Delta\lambda_n| = O\{|\Delta R_n||\lambda_n|\}, \ n \to \infty \qquad (5)$$

then the series $\sum_1^\infty a_n R_n \lambda_n$ is summable $[N, p_n , q_n]_k$ $k \geq 1$.

We need the following lemmas for our object

LEMMA 1 :

If the sequences $\{p_n\}$, $\{q_n\}$, and $\{\lambda_n\}$ satisfy the conditions (3) - (5) of theorem 1, then $R_n |\lambda_n| = 0\,(1)$ as $n \to \infty$.

Proof : By Abel's transformation, we have

$$\sum_{v=1}^{n} p_v q_{n-v}|\lambda_v| = \sum_{v=1}^{n}\left(\sum_{r=1}^{v} p_r q_{v-r}\right)|\Delta\lambda_v| + R_n|\lambda_n|.$$

This implies

$$R_n|\lambda_n| \leq \sum_{v=1}^{n} p_v q_{n-v}|\lambda_v| + \sum_{v=1}^{n}\left(\sum_{r=1}^{v} p_r q_{v-r}\right)|\Delta\lambda_v|.$$

$$= O(1)\sum_{v=1}^{n} R_v|\Delta\lambda_v| = O(1)\sum_{v=1}^{n}|\Delta R_v||\lambda_v|\frac{1}{q_{n-v-1}} = O(1)$$

LEMMA 2 :

$\{q_n\}$ nonincreasing implies $|\Delta R_n| = O(p_n)$.

Proof : Since $R_n \leq P_n q_0$, then for $D = \frac{d}{dn}$ and a constant k,

$$k \geq \lim_{n\to\infty}\frac{R_n}{P_n} = \lim_{n\to\infty}\left|\frac{DR_n}{DP_n}\right| = \lim_{n\to\infty}\left|\frac{\Delta R_n}{\Delta P_n}\right| = \lim_{n\to\infty}\frac{|\Delta R_n|}{p_n}$$

LEMMA 3 :
let $q \in M$, then for $0 < r \le 1$,

$$\sum_{n=v}^{\infty} \frac{q_{n-v}}{n^r Q_n} = O(v^{-r})$$

Proof of Theorem 1 : Write

$$\phi_n = \sum_{v=1}^{n} P_{v-1} R_v q_{n-v} a_v \lambda_v ,$$

then by Abel's transformation , we have

$$\phi_n = \sum_{v=1}^{n-1} (\sum_{r=1}^{v} a_r) \Delta_v (q_{n-v} P_{v-1} R_v \lambda_v) + $$
$$+ (\sum_{r=1}^{n} a_r) q_0 P_{n-1} R_n \lambda_n$$
$$= \sum_{v=1}^{n-1} \{ \Delta_v q_{n-v} P_{v-1} R_v \lambda_v s_v - p_v q_{n-v-1} R_v \lambda_v s_v +$$
$$+ q_{n-v-1} P_v R_v \Delta\lambda_v s_v$$
$$- q_{n-v-1} P_v \Delta R_v \lambda_{v+1} s_v \} + q_0 P_{n-1} R_n \lambda_n s_n$$
$$= \phi_{n,1} + \phi_{n,2} + \phi_{n,3} + \phi_{n,4} + \phi_{n,5} \quad \text{where}$$

$$\phi_{n,1} = \sum_{v=1}^{n-1} \Delta_v q_{n-v} P_{v-1} R_v \lambda_v s_v$$
$$\phi_{n,2} = \sum_{v=1}^{n-1} - p_v q_{n-v-1} R_v \lambda_v s_v$$
$$\phi_{n,3} = \sum_{v=1}^{n-1} q_{n-v-1} P_v R_v \Delta\lambda_v s_v$$
$$\phi_{n,4} = \sum_{v=1}^{n-1} - q_{n-v-1} P_v \Delta R_v \lambda_{v+1} s_v$$

and

$$\phi_{n,5} = - q_0 P_{n-1} R_n \lambda_n S_n$$

In order to prove the theorem, by Minkowski's inequality, it is therefore sufficient to show that .

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n R_{n-1}^k} |\phi_{n,r}|^k < \infty, \quad r = 1,2,3,4,5$$

Applying Holder's inequality

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |\phi_{n,1}|^k = \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \left| \sum_{v=1}^{n-1} \Delta_v q_{n-v} P_{v-1} R_v \lambda_v s_v \right|^k$$
$$\le \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \sum_{v=1}^{n-1} |\Delta_v q_{n-v}| P_{v-1}^k R_v^k |\lambda_v|^k |s_v|^k \left\{ \sum_{v=1}^{n-1} |\Delta_v q_{n-v}| \right\}^{k-1}$$
$$= O(1) \sum_{v=1}^{m} P_{v-1}^k R_v^k |\lambda_v|^k |s_v|^k \sum_{n=v+1}^{m+1} \frac{p_n |\Delta_v q_{n-v}|}{P_n R_{n-1}^k}$$
$$= O(1) \sum_{v=1}^{m} p_v (\frac{P_{v-1}}{R_{v-1}}) (R_v |\lambda_v|)^{k-1} |\lambda_v| |s_v|^k$$
$$= O(1) \sum_{v=1}^{m} p_v |\lambda_v| |s_v|^k , \qquad \text{by lemma 1}$$

$$= O(1) \sum_{v=1}^{m} p_v q_{m-v} |\lambda_v| |s_v|^k . \frac{|\lambda_v|}{q_{m-v}}$$
$$= O(1) \sum_{v=1}^{m-1} (\sum_{r=1}^{v} p_r q_{m-r} |s_r|^k) \left| \Delta \{ \frac{|\lambda_v|}{q_{m-v}} \} \right| + O(1) R_m \frac{|\lambda_m|}{q_0}$$

, by Abel's Transformation

$$= O(1) \sum_{v=1}^{m} (\sum_{r=1}^{v} p_r q_{v-r} |s_r|^k) \Delta_v (\frac{1}{q_{m-v}}) |\lambda_v| +$$
$$+ \frac{1}{q_{m-v-1}} |\Delta\lambda_v| + O(R_m |\lambda_m|)$$

$$= O(1) \sum_{v=1}^{m} \Delta_v (\frac{1}{q_{m-v}}) R_v |\lambda_v| + O(1) \sum_{v=1}^{m} \frac{1}{q_{m-v-1}} R_v |\Delta\lambda_v|$$
$$+ O(1) \sum_{v=1}^{m} \frac{1}{q_{m-v-1}} |\Delta R_v| |\lambda_v| + O(1)$$
$$= O(1) \sum_{v=1}^{m-1} (\sum_{r=1}^{v} \Delta_v \frac{1}{q_{m-v}}) |R_v \Delta |\lambda_v| + \Delta R_v |\lambda_{v+1}||$$
$$+ O(1) \sum_{v=1}^{m-1} \frac{1}{q_{m-v-1}} |\Delta R_v| |\lambda_v| + O(1)$$

$$= O(1) \sum_{v=1}^{m} \frac{1}{q_{m-v-1}} |\Delta R_v| |\lambda_v| + O(1) \sum_{v=1}^{m} \frac{1}{q_{m-v-1}} |\Delta R_v| |\lambda_{v+1}| +$$
$$+ O(1) = O(1)$$

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |\phi_{n,2}|^k = \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \left| \sum_{v=1}^{n-1} p_v q_{n-v-1} R_v \lambda_v s_v \right|^k$$
$$\le \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} p_v q_{n-v-1} R_v^k |\lambda_v|^k |s_v|^k \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{P_n R_{n-1}} \right\}^{k-1}$$
$$= O(1) \sum_{v=1}^{m} p_v R_v^k |\lambda_v|^k |s_v|^k \sum_{n=v+1}^{m+1} \frac{p_n q_{n-v-1}}{P_n R_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} p_v \frac{R_v}{P_v} (R_v |\lambda_v|)^{k-1} |\lambda_v| |s_v|^k$$
$$= O(1) \sum_{v=1}^{m} p_v |\lambda_v| |s_v|^k \text{ , as } R_v \le q_0 P_v$$
$$= O(1) .$$

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |\phi_{n,3}|^k =$$
$$= \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \left| \sum_{v=1}^{n-1} p_v q_{n-v-1} (\frac{P_v}{p_v}) R_v \Delta\lambda_v s_v \right|^k$$

$$\le \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} p_v q_{n-v-1} (\frac{P_v}{p_v})^k R_v^k |\Delta\lambda_v|^k |s_v|^k$$

$$\text{X} \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{R_{n-1}} \right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m} p_v (\frac{P_v}{p_v})^k R_v^k |\Delta\lambda_v|^k |s_v|^k \sum_{n=v+1}^{m+1} \frac{p_n q_{n-v-1}}{P_n R_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} (\frac{P_v}{p_v})^{k-1} |\Delta R_v|^k |\lambda_v|^k |s_v|^k$$

$$= O(1)\sum_{v=1}^{m} p_v (\frac{P_v}{R_v})^{k-1} (R_v|\lambda_v|)^{k-1} |\lambda_v| |s_v|^k$$

$$= O(1)\sum_{v=1}^{m} p_v |\lambda_v| |s_v|^k ,$$

$$= O(1).$$

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |\phi_{n,4}|^k = \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \left| \sum_{v=1}^{n-1} q_{n-v-1} P_v \Delta R_v \lambda_{v+1} s_v \right|^k$$

$$\le \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} p_v q_{n-v-1} (\frac{P_v}{p_v})^k |\Delta R_v|^k |\lambda_{v+1}|^k |s_v|^k \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{R_{n-1}} \right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m} p_v (\frac{P_v}{p_v})^k |\Delta R_v|^k |\lambda_v|^k |s_v|^k \sum_{n=v+1}^{m+1} \frac{p_n q_{n-v-1}}{P_n R_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} (\frac{P_v}{p_v})^{k-1} |\Delta R_v|^k |\lambda_v|^k |s_v|^k$$

$$= O(1).$$

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |\phi_{n,5}|^k = \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |q_0 P_{n-1} R_n \lambda_n s_n|^k$$

$$= O(1)\sum_{v=1}^{m} p_n (\frac{P_{n-1}}{R_{n-1}})^k (\frac{R_n}{P_n})(R_n|\lambda_n|)^{k-1} |\lambda_n| \, |s_n|^k$$

$$= O(1)\sum_{v=1}^{m} p_n |\lambda_n| |s_n|^k$$

$$= O(1).$$

3. APPLICATIONS

THEOREM 2 : (Bor [3]) :

If $\sum_{1}^{\infty} a_n$ is bounded $[N,p_n]_k$, and the sequences $\{\lambda_n\}$ and $\{P_n\}$ satisfy the conditions :

(i) $\sum_{n=1}^{m} p_n |\lambda_n| = O(1)$

(ii) $P_m |\Delta\lambda_m| = O(p_m |\Delta\lambda_m|)$,

then the series $\sum a_n P_n \lambda_n$ is summable

Proof : Follows from Theorem 1 by putting $q_n = 1$ for all n.

THEOREM 3 :

If $\sum a_n$ is bounded $[N,p_n]_k$, and the sequences $\{\lambda_n\}$ and $\{q_n\}$ satisfy the conditions : $q \in M$, and

(i) $\sum_{v=1}^{m} q_{n-v} |\lambda_v| = O(1)$

(ii) $\sum_{v=1}^{m} \frac{q_{v+1}}{q_{n-v-1}} |\lambda_v| = O(1)$

(iii) $Q_n |\Delta\lambda_n| = O(q_n |\Delta\lambda_n|)$,

then the series $\sum a_n Q_n \lambda_n$ is summable $|N,q_n|_k$, $k \ge 1$.

Proof : Follows from Theorem 1 by putting $p_n = 1$ for all n, and making use of lemma 3, for $r = 1$.

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