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FUNCTIONS NEAR OF NA-CONTINUITY

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ABSTRACT

In 1986, the concept of Na-continuity was initiated by Chae et al., this leads us to introduce two classes of functions, namely strongly Na-continuity and weakly Na-continuity. We characterize these new types. Several properties of them are studied. Also, their relationship with other types of functions are investigated. Finally, the effects of these functions on some topological spaces are established.

INTRODUCTION

Throughout the present paper, (X, τ) and Y, σ (or simply X and Y) denote topological spaccs on which no separation axiom is assumed unless explicitly stated. A subset S of X is said to be semi-open [11] (resp. α -set [15] , β-open [1]) if $S \subset cl(int(S))$ (resp. $S \subset int(cl(int(S)))$, $S \subset cl(int(cl(S)))$, where Cl(S) (resp. int (S)) denotes the closure (resp. interior) of S. The complement of a semi-open is semi-closed [8] . The intersection of all semi-closed sets that contain S is called the semi-closure of S [8] and is denoted by s-cl (S). A set S is said to be feebly open [12] if there exists an open set G such that $G \subset S \subset$ s-cl (G). A set S is called δ -open [20] if for each $x \in S$, there exists a regular open set H such that $x \in H \subset S$, or equivalently, if S is expressible as an arbitrary union of regular open sets. δ -closure and feebly closure are defined as semi-closure, previously and denoted by δ-cl (S) and f.cl (S), respectively. By RO(X) (resp. SO(X), $\beta O(X)$, $\delta O(X)$, FO(X)) we denote the family of all regular open (resp. semi-open, β-open, δ -open, feebly open) of X. Maheshwari, et al. [12] showed that feebly openness and α-sets are equivalent. A function $f:X \rightarrow Y$ is called semi-continuous [11] (resp. δ -continuous (δ-C) [17] , super continuous (SC) [14] , almost continuous in the sense of Singal (ACS) [18] , feebly continuous (F.C.) [12] , feebly irresolute (F.I.) [6], strongly semi-continuous (S.S.C.) [2], Na-continuous (Na-C.) [7]) if the inverse image of each open (resp. δ -open, open, regular open, open, feebly open, semi-open, feebly open) set in Y is semi-open (resp. δ -open, δ -open, open, feebly open, feebly open, open, δ -open) set in X. f is called semi-open [5] (resp α -open [13]) if the image of each open set in X is semi-open (resp. α -set). Recall that a space X is called extremely disconnected (E.D.) if the closure of every open set is open and called submaximal if all dense subsets of X are open. A space X is called semi-compact [10] (resp. S. Closed [19] , S-closed [9]) if for every semi-open cover $\{U_i:i \in I\}$ of X, there exists a finite subfamily I_o of I such that:$X= \cup\{U_i:i \in I_o\}$ (resp. $X = \cup\{cl(U_i):i \in Io\}$, $X = \cup\{s\text{-}cl\,(U_i) : i \in I_o\}$). A space X is called nearly compact (Singal and Mathur 1974) (resp. α -compact [13] if every δ -open (resp. α -open) cover of X has a finite subcover, X is called nearly Lindelof [4] if each regular open cover of X has a countable subcover.

STRONGLY NA-CONTINUITY

Definition (2.1): A function $f:X\rightarrow Y$ is said to be strongly Na-continuous (abbreviated as S. Na-C.) if $f^{1}(V) \in \delta O(X)$, for each $V \in SO(Y)$ Next result characterizes S. Na-C. In various ways.

Theorem (2.1): For a function $f:X\rightarrow Y$, the following statements are equivalent.

(i) F is S.Na-C.

(ii) For each $x \in X$ and each $V_{f(x)} \in SO(Y)$, there exists $U_x \in \delta O(X)$ such that $f(U_x) \subset V_{f(x)}$.

(iv) For each semi-closed set F of Y, f^{-1} (F) is -closed.

(v) $f(\delta\text{-cl}\,(A)) \subset s\text{-cl}\,(f(A))$, for each $A \subset X$.

(vi) $\delta\text{-cl}\,(f^{-1}(B))$ C f^{-1} (s-cl(B)), for each $B \subset Y$.

Proof: (i)$\Rightarrow$(ii) : Left to the reader.
(ii)$\Rightarrow$(iii) : Let $x \in X$ and $V_{f(x)} \in SO(Y)$, by (ii), there exists $(U_o)_x \in \delta O(X)$ such that $f\,(U_O)_X \subset V_{f(x)}$, by the meaning o δ -openness, there exists $U_X \in RO(X)$ such that $U_X \in (U_O)_x$, thus $f(x) \in F(U_X) \subset f(U_O)_X \subset V_{f(x)}$.

(iii)$\Rightarrow$(iv): Let F be a semi-closed set of Y, then $Y\text{-}F \in SO(Y)$. For each $X \in f^{-1}(Y\text{-}F)$, there exists $U_x \in RO(X)$ such the
$x \in U_X \subset f^{-1}(Y\text{-}F)$, and for all $x \in f^{-1}(Y\text{-}F)$,
$f^{-1}(Y\text{-}F) = \cup\{U_X : x \in f^{-1}(Y\text{-}F)\} \in \delta\, O\,(X)$.
Hence $f^{-1}(F)$ is δ -closed.

(iv)→(v): Since $A \subset f^{-1}(\text{s-cl}(f(A)))$. for $A \subset X$, then by (iv) $\delta\text{-cl}(A) \subset f^{-1}(\text{s-cl}(f(A)))$. Hence $f(\delta\text{-cl}(A)) \subset \text{s-cl}(f(A))$.
(v)⇒(vi) : Put $f^{-1}(B)$ instead of A in (v).
(vi⇒(i) : Take V SO(Y), then Y-V is semi-closed, by (vi) $\delta\text{-cl}(f^{-1}(Y-V))$ $f^{-1}(\text{s-cl}(Y-V)) = f^{-1}(Y-V)$. Thus $f^{-1}(V)\} \in \delta O(X)$

Theorem (2.2): If $f:X \to Y$ is S.Na-C. and $A \in \tau(X)$. Then $f/A:A \to Y$ is S.Na-C.

Proof: Assume $V \in SO(Y)$. Then $f^{-1}(V)\} \in \delta O(X)$ which is a union of regular open sets V_i of X. Since $A \in \tau(X)$, then $V_i \cap A$ RO(A) [20, Theorem (4)]. Therefore, $(f/A)^{-1}(V)$ is the union of $f^{-1}(Vi) \cap A$, and hence $(f/A)^{-1}(V) \in \delta O(A)$.

The following Lemma is very useful in the sequel.

Lemma (2.1): Let $\{X_\lambda / \lambda \in D\}$ be a family of spaces and $U_{\lambda i}$ be a subset of $X_{\lambda i}$ for each i=1,2,......,n. Then $U = \prod_{i=1}^{n} U_{\lambda i} \times \prod_{\lambda \neq \lambda i} X_{\lambda i}$ is semi-open [16] (resp. δ-open [7], feebly open [7]) in $\prod_{\lambda \in D} X_\lambda$ iff $U_{\lambda i}$ $SO(X_{\lambda i})$ (resp. $U_{\lambda i} \in O(X_{\lambda i})$, $U_{\lambda i}$ $FO(X_{\lambda i})$) for each i=1, 2,......,n.

Theorem (2.3): Let $f:X \to Y$ be a function and $g:X \to X \times Y$ be the graph function of f defined by $g(x) = (x,f(x))$, for each $x \in X$, Then f is S.Na-C., if g is S.Na-C.

Proof: Let $x \in X$ and $V_{f(x)} \in SO(Y)$, Then by Lemma (2.1), $X \times V \in SO(X \times Y)$, containing g(x). Since g is S.Na-C., by Theorem (2.1), there exists $U_X \in \delta O(X)$ such that : $g(U) \subset X \times V$.

Hence $f(U_X) \subset V_{f(x)}$

Theorem (2.4): Let $\{X_\alpha : \alpha \in \nabla\}$ be any family of spaces and $f:X \to \prod X_\alpha$ be S.Na-C. Then P_α of: $X \to X_\alpha$ is S.Na-C. for each $a\alpha \in \nabla$. where P_α is the projection of $\prod X_\alpha$ onto X_α, for each $\alpha \in \nabla$.

Proof: Let $V_j \in SO(X_j)$, for a fixed j of ∇, from projection properties, $P_j^{-1}(V_j) = (V_j) X \prod X_\alpha \in SO(\prod X_\alpha)$ (Lemma 2.1). so, $(P_j \text{ of})^{-1}(V_j) = f^{-1}(P_j^{-1}(V_j)) = f^{-1}(V_j$

$x \prod X_\alpha) \in \delta O(X)$.

Therefore P_j of is S.Na-C., for each $j \in V$.

Theorem (2.5): Let $f_\lambda : X_\lambda \to Y_\lambda$ be a function for each $\lambda \in D$ and $f: \prod X \to \prod Y_\lambda$ a function defined by $f(\{x_\lambda\}) = \{f_\lambda(x_\lambda)\}$ for each $\{x_\lambda\} \in \prod X$. If f is S.Na-C., then f_λ is S.Na-C. for each $\lambda \in D$.

Proof: Let $\beta \in D$ and $V_\beta \in SO(Y_\beta)$. Then, by Lemma (2.1) $V = V_\beta \times \prod_{\lambda \neq \beta} Y_\lambda$ is semi-open in $\prod Y_\lambda$ and $f^{-1}(V) = f_\beta^{-1}(V_\beta) \times \prod_{\lambda \neq \beta} X_\lambda$ is δ-open in $\prod X_\lambda$. From Lemma (2.1), $f_\beta^{-1}(V_\beta) \in \delta O(X)$.

Therefore, f_β is S.Na-C.

WEAKLY NA-CONTINUITY

Definition (3.1): A function $g:X\to Y$ is said to be weakly Na-continuous (abb. W.Na-C.) if $g^{-1}(V) \in FO(X)$, for each $V \in \delta O(Y)$

The following result provides some characterizations of W.Na-C.

Theorem (3.1): For a function $g:X\to Y$, the following statements are equivalent.

(i) g is W.Na.-C

(ii) For each $x \in X$ and each $V_{g(x)} \in \delta O(Y)$ there exists $U_x \in FO(X)$ such that $g(U) \subset V$.

III) For each δ-closed set F of Y, $g^{-1}(F)$ is feebly closed

(iv) $g(f.cl(A)) \subset \delta\text{-}cl(g(A))$, for each $A \subset X$.

(v) $f.cl(g^{-1}(B)) \subset g^{-1}(\delta\text{-}cl(B))$, for each $B \subset Y$.

Proof (i)⇒(ii)⇒(iii) Are obvious

(iii)⇒(iv) : Since $A \subset g^{-1}(\delta\text{-}cl\, g(A))$ for each $A \subset X$. Then $f.cl(A) \subset g^{-1}(\delta\text{-}cl\, g(A))$. Therefore we obtain, $g(f.cl(A)) \subset \delta\text{-}cl(g(A))$

(iv)⇒(v) : Clear.

(v)⇒(i) : Let $V \in \delta O(Y)$, then $(Y-V)$ is δ-closed and $f.cl(g^{-1}(Y-V)) \subset g^{-1}(8\text{-}cl(Y-V) = g^{-1}(Y-V)$. Thus $g^{-1}(V) \in \delta O(X)$.

Theorem (3.2): The restriction function of a W.Na-C. function by any β-open set is also W.Na-C.

Proof: Let $g:X\to Y$ be W.Na-C. and $A \in \beta O(X)$. To show that g/A is W.Na-C., let $V \in \delta O(Y)$, then $g^{-1}(V) \in FO(X)$, $g^{-1}(V) \cap A = (g/A)^{-1}(V) \in FO(A)$. Therefore g/A is W.Na-C.

A corresponding results of S.Na-C. have been established nextly for W.Na-C. which stated without proof for it is similarly by using Lemma (2.1).

Theorem (3.3): Let $f:X\to Y$ be a function and $g:X\to XxY$ be the graph function of f defined by $g(x) = (x,f(x))$, for each $x \in X$. If g is W.Na-C., then f is So.

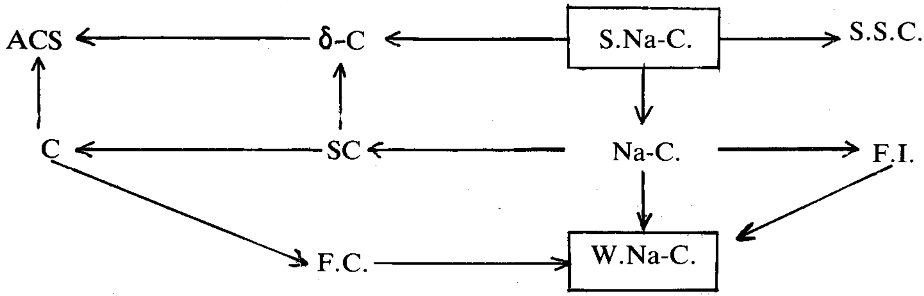
Theorem (3.4): Let $\{X_\alpha : \alpha \in \nabla\}$ be any family of spaces and $g:X\to \Pi X_\alpha$ be W.Na-C., then $P_\alpha \, o g:X\to X_\alpha$ is W.Na-C., for each $\alpha \in \nabla$ where P_α is the projection of ΠX_α onto X_α, for each $\alpha \in \nabla$

Theorem (3.5): Let $g_\alpha : X_\alpha \to Y_\alpha$, $\alpha \in \nabla$ be a family of functions and $g:\Pi X_\alpha$ defined as $g(\{x_\alpha\}) = \{g_\alpha(x_\alpha)\}$, for each $\{x_\alpha\} \in \Pi X_\alpha$. Then g_α is W.Na-C. if g is W.Na-C. for each $\alpha \in \nabla$.

CONNECTIONS WITH OTHER TYPES

In this section we investigate relations between S.Na-C., W.Na-C. and several forms of continuity.

Remark (4-1): From the previous definitions it follows immediately that, we have the following implications



Where C= Continuous and other abbreviations are the shown in the introduction. We now show that none of these implications is reversible.

Example (4.1): For a set X = {a,b,c}, we have the following.

(i) If τ ={ϕ {a}, {b}, {a,b}, then f: (X, τ)→ (X, τ) define by f(a) = f(b) = a, and f(c) = c. is Na-C., but it is not S.Na-C.

(ii) If τ = {ϕ {a}, {b}, {a,b}, {a,c}, X}, and σ = {ϕ {a}, X} Then f:(X, τ)→(X, σ) defined as f(a) = a = f(b) and f(c) = b is S.S.C. but it is not S.Na-C.

Example (4.2): For a set X = {a,b,c,d} , we have the following.

(i) If τ = {ϕ, {a,b} , {a,b,c} , X} , σ ={ϕ,{a,c}, {a,b,c} ,X} and we define f:(X, τ)→(X, σ) as follows f(a) = f(b)=a, f(c)=b and f(d) =c. It is clear that f is δ -C. while it is not S.Na-C.

(ii) if τ = {ϕ, {c} , {a,b,c}, X} , σ ={ϕ,{a},{c}, {a,c} , {a,b} , {a,b,c}, {a,b,c,d} ,X} . Then f:(X, τ)→(X, σ) defined as follows: f(a) =f(b)=b, and f(c) =f(d) = a. is W.Na.C. but it is not F.C.

Remark (4.2): Na-C. and S.S.C. are independent of each other, (i) and (ii) of example (4.1) show this fact.

For any space X, the collection of all δ-open sets forms a topology τ_δ on X. A basic fact which we shall exploit is that $\tau_\delta = \tau_\delta$.

The following Theorems do not require proof.

Theorem (4.1): For a function $f:(X,\tau)\to(Y,\sigma)$, the following are true:

(i) If Y is E.D., then f is S.Na-C. iff f is Na-C.
(ii) If Y is submaximal, E.D., then f is S.Na-C. iff f is SC.
(iii) f is S.S.C. iff $f:(X, \tau_2)\to (Y,\sigma)$ is S.Na-C.
(iv) f is W.Na-C. iff $f:(X\tau)\to(Y, \ )$ is F.C.

Theorem (4.2): Let $f:X\to Y$ and $g: Y\to Z$ be two functions, then the following statements hold.

(i) gof is S.Na-C. if one of the next verified.
(1) Both f and g are S.Na-C.
(2) f is S.Na-C. and g is I.
(3) f is -C. and g is S.Na-C.
(4) f is SC and g is S.S.C.
(ii) gof is W.Na-C. if one of the following holds:
(1) f is W.Na-C. and g is -C.
(2) f is F.I. and g is W.Na-C.
(iii) gof is SC , if f is S.Na-C. and g is semi-continuous.
(iv) gof is F.C., if f is W.Na-C. and g is SC.
(v) gof is F.I., if f is W.Na-C. and g is Na-C.
(vi) f is super open, if g is S.Na-C. and gof is semi-open
(vii) f is δ -open, if g is W.Na-C. and gof is super open.

Remark (4.3): $f:(X,\tau)\to(Y,\sigma)$ is super open if $f:(X,\tau)\to(Y,\sigma_S)$ is open

NEW TYPES OF FUNCTIONS AND SOME TOPOLOGICAL SPACES

Theorem (5.1): The image of a nearly compact space under S.Na-C. surjection is semi-compact.

Proof: Let $f:X\to Y$ be S.Na-C. surjection , and X is nearly compact. To prove that Y is semi-compact , let $\{V_\alpha : \alpha\in\nabla\}$ be a semi-open cover of Y, then $\{ f^{-1}(V \ )$ $: \ \in\nabla\}$ is δ-open cover of X, but X is nearly compact, then there exists a finite subfamily ∇_o of ∇ such that : $X= \cup \{ f^{-1} (V_\alpha) : \alpha\in\nabla\}$. Thus $Y=f(X) = \cup \{(V_\alpha): \alpha\in\nabla\}$. therefore Y is semi-compact.

Corollary (5.1): Let $f:X\to Y$ be S.Na-C. surjection , then Y is s-closed if X is nearly compact.

Corollary (5.2): The image of nearly compact under S.Na-C. surjection is S-closed.

Theorem (5.2): Let f:X→ Y be a S.Na-C. surjection function and X is compact , then Y is sime-compact.

Proof: Follows directly from the fact that each δ-open cover is an open cover.

Theorem (5.3): if fX→ Y is S.Na-C. surjection , and X is a semi-regular space , then Y is Lindelof if X is nearly Lindelof.

Proof: Assume $\{(V_\alpha):\alpha\in\nabla\}$ be an open cover of Y , then it is semi-open cover , so , $\{f^{-1}(V_\alpha):\alpha\in\nabla\}$ is δ-open cover of X, from semi-regularization of X , $\{f^{-1}(V_\alpha):\alpha\in\nabla\}$ is regular open cover, but from hypothesis , there exists a countable subcover with $X=\cup\{f^{-1}(V_\alpha):\alpha\in\nabla\}$. Hence $Y= f(X) = \cup\{(V_\alpha):\alpha\in\nabla\}$.

Theorem (5.4): The image of a-compact space under W.Na-C. surjection is nearly compact.

Proof: Obvious.

REFERENCES

[1] **Abd El-Monsef M.E., El-Deeb S.N.,** and **Mahmoud R.A. 1983 .** β-open Sets and β-Continuous Mappings, Bull Fac. Sci. Assuit Univ., 12 (1) , 77-90.

[2] **Abd El-Monsef M.E., Mahmoud R.A.,** and **Nasef, A.A.** Strongly Semi-continuous Functions (submitted).

[3] **Atia R.H., El-Deeb S.N.,** and **Mashhour A.S.** α-Compactness and α-homeomorphism (Preprint).

[4] **Balasubramanian , G. 1982 :** On Some Generalizations of Compact Spaces. Glas. Math Ser. III 17 (37) 367-380.

[5] **Biswas, N. 1969 :** On Some Mappings in Topological Spaces, Bull. Calcutta Math. Soc., 61 , PP. 127-135.

[6] **Chae, G.I.** and **Lee H.W. 1985 :** Feebly Irresolute Functions Sungshin W.Univ. Report. 21, 273-280.

[7] **Chae, G.I., Noiri T.** and **Lee D.W. 1986 :** On NA-Continuous Cunctions Kungpook Math. J.V. 26.N.(1), 73-79.

[8] **Crossley, S.G.,** and **Hildebrand S.K. 1971 :** Semi-Closure, Texas, J Sci , (2+3), 99-119.

[9] **Di Maio** and **Noiri T. 1987 :** On S-Closed Spaces, Indian J. pure Appl Math. 18(3) , 226-233.

[10] **Dorsett, C. 1981 :** Semi Covergence and Semi Compactness Indian J. Math. Mech. 19 , 11-17.

[11] **Levine, N. 1963 :** Semi-Open Sets and Semi-Continuity in Topological Spaces, Amer. Math. Monthly , (70), 36-41.

[12] **Maheshwari, S.N.** and **Tapi U.D. 1979 :** Note on Some Applications of Feebly Open Sets, M.B. , Jr. University of sauger.

[13] **Mashhour A.A., Hasanien I.A.** and **El-Deeb S.N. 1983 :** α-Continuous and α-Open Mappings, Acta. Math. Acad, Sci. Hungar, vol. (41) (3+4), 213-218.

[14] **Munshi, B.M.** and**Bassan, D.S. 1982 :** Super-Continuous Mappings Indian Jr. Pure Appl. Math., 13 , 229-236.

[15] **Njastad, O. 1965 :** On some classes of nearly open sets , pacific J. Math. 15, 961-970.

[16] **Noiri, T. 1973 :** Remarks on Semi-Open Mappings, Bull. Gal. Math Soc., 4(65) , 197-201. Mr, 53 3984.

[17] **--------- 1980 :** On δ-Continuous Functions. Jr. Korean Math. Soc., 16 . 161-166.

[18] **Singal M.K.,** and **Singal A.R. 1968 :** Almost-Continuous Mappings, Yokohama Math. J. 16 , 63-73.

[19] **Thompson T. 1976 :** S-Closed Spaces, Proc. Amer. Math. Soc. 60, 335-338.

[20] **Velicko N.V. 1968 :** H-Closed Topological Spaces, Amer. Math. Soc. Trans (78) (2) , 103-118.

دلالات إستمراريـــــة قـــرب ـ Na

رمضـــــان عبد العاطي محمـــود - عبـد المنصــــــف
ناصــــــــف

بالنسبة للمفاهيم الخاصة باستمرارية ـ Na تم اقتراحها في هذا البحث . كما تم تقديم خواص هذين النوعين من الدوال وعلاقتهما بأنواع أخرى .

كما تم إيجاد تأثير هذه الدوال الجديدة على بعض الفضاءات التوبولوجية .