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ON A KP CLASS OF PARTIAL DIFFERENTIAL EQUATIONS CONSERVATION LAWS AND SOLITARY WAVE SOLUTIONS

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عن الفصل KP للمعادلات التفاضلية الجزئية قوانين الثبات والحلول الموجبة الانعزالية

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نبرهن أن الفصل KP للمعادلات التفاضلية الجزئية تحتوي على عدد لانهائي من قوانين الثبات بالإضافة إلى ذلك نشتق صراحة الحلول الموجية الانعزالية لهذا الفصل .

Key Words: KP equation, Conservation laws, solitary waves

ABSTRACT

It is shown that a KP class of partial differential equations possesses an infinite number of conservation laws. In addition, exact and explicit solitary wave solutions are constructed for the KP class of equations, which has been obtained recently in many physical systems.

I. INTRODUCTION

Consider a KP class of partial differential equations in the form :

$$(u_t + uu_x + u_{xxx})_x + Au_x + Bu_y + Cu_{yy} + Du_{xy} + Eu_{xx} = 0 \qquad (1.1)$$

or

$$u_t + uu_x + u_{xxx} + Au + B\partial_x^{-1}u_y + C\partial_x^{-1}u_{yy} + Du_y + Eu_x = 0 \qquad (1.2)$$

where A and C are arbitrary functions of y and t,

$$B = \frac{1}{2}C_y\ ,\ D = C^{\frac{1}{2}}(2F+G)\ ,\ E = F^2 + \int(F_y G + C^{-\frac{1}{2}}F_t)dy\ ,$$

$$F = \int AC^{-\frac{1}{2}}dy \text{ and } G = \frac{1}{2}\int C^{-\frac{3}{2}}C_t dy$$

Equations (1.2) can be considered as a generalization of the KP equation [3] with additional terms and variable coefficients and appears in many physical systems [1,4]. In [2] a set of Backlund transformations were determined by applying a singular-point analysis for (1.2). Lin and Chen [5] studied the KP equations

$$u_t + 6uu_x + u_{xxx} - \partial_x^{-1}u_{yy} = 0 \qquad (1.3)$$

$$u_t + 6uu_x + u_{xxx} + \partial_x^{-1}u_{yy} = 0 \qquad (1.4)$$

and showed that these equations possess an infinite number of conservation laws in the form

$$T_t + X_x + u_{xxx} + Y_y = 0 \qquad (1.5)$$

where T, the conserved density, and -X and -Y, the fluxes, are polynomials in u and its derivatives with respect to x and y.

The first three of the conserved densities have the forms :

$$T_1 = (1/2)u,$$

$$T_2 = (i/2)(u_x - b\partial_x^{-1}u_y),$$

$$T_3 = (1/8)(-u_{xx} + 2bu_y - b^2\partial_x^{-2}u_{yy} - u^2), \tag{1.6}$$

where $b = -i/\sqrt{3}$ in the case of (1.3) and $b = -1/\sqrt{3}$ in the case of (1.4).

In this paper, we show that (1.2) possesses an infinite number of conservation laws in the form (1.5). In addition to this, we construct the solitary wave solutions for (1.2). In section II we derive the first three conservation laws of (1.2). In section III we prove that there are an infinite number of conservation laws for (1.2). Generating functions are given. Finally, in section IV, solitary wave solutions for (1.2) are constructed.

II. DERIVATION OF EXPLICIT CONSERVATION LAWS

In this section we derive some conservation laws to show the method for generating them explicitly. Even though we can give an algorithm for deriving the conserved densities and corresponding fluxes, we will see below that difficulties arise in attempting to use this algorithm for obtaining the conserved densities and corresponding fluxes in the general case.

For definiteness, throughout the remainder of this paper we are considering the KP class of equations (1.2).

Clearly (1.2) is in conservation form with

$$T = u\ ,\ X = (1/2)u^2 + u_{xx} + \partial_x^{-1}(Au - B\partial_x^{-1}u_y + Du_y) + Eu \ \text{ and} \quad Y = C\partial_x^{-1}u_y = 0 \tag{2.1}$$

multiplying (1.2) by u yields a second conservation laws

$$(\frac{1}{2}u^2)_t + (uu_{xx} - \frac{1}{2}u_x^2 + \frac{1}{3}u^3 + \partial_x^{-1}(Au - Bu\partial_x^{-1}u_y - Cu_y\partial_x^{-1}u_y + Duu_y)$$

$$+(\frac{1}{2}Eu^2)_x + (Cu\partial_x^{-1}u_y)_y = 0 \tag{2.2}$$

to find a third conservation laws, multiply (1.2) by u^2 giving :

$$u^2u_t + u^3u_x + u^2u_{xxx} + Au^3 + Bu^2\partial_x^{-1}u_y + Cu^2\partial_x^{-1}u_{yy} + Du^2u_y + Eu^2u_x = 0 \tag{2.3}$$

and adding (2.3) to $-2u_x$ (1.1) we obtain :

$$u^2u_t + u^3u_x + u^2u_{xxx} + Au^3 + Bu^2\partial_x^{-1}u_y + Cu^2\partial_x^{-1}u_{yy} + Du^2u_y + Eu^2u_x$$
$$-2u_x(u_{xt} + u_x^2 + uu_{xx} + u_{xxxx} + Au_x + Bu_y + Cu_{yy} + Du_{xy} + Eu_{xx}) = 0$$

which can be rewritten as :

$$(\frac{1}{3}u^3 - u_x^2)_t + (\frac{1}{4}u^4 + u^2u_{xx} - 2uu_x^2 - 2u_xu_{xxx} + u_{xx}^2 + \frac{1}{3}Eu^3 - Eu_x^2 +$$
$$\partial_x^{-1}(Au^3 - Bu^2\partial_x^{-1}u_y - \frac{1}{3}D_yu^3 - 2Au_x^2 - 2Bu_xu_y - 2Cu_xu_{yy} + D_yu_x^2$$
$$-2Cuu_y\partial_x^{-1}u_y))_x + (Cu^2\partial_x^{-1}u_y + \frac{1}{3}Du^3 - Du_x^2)_y = 0 \tag{2.4}$$

which is also a conservation law. Since the algorithm for generating conservation laws developed in this section does not easily generalize, we seek a different method for obtaining them.

III. EXISTENCE OF AN INFINITE NUMBER OF CONSERVATION LAWS

In this section, we generalize the results in [5,6] to prove that there exists an infinite number of conservation laws in the form (1.5) of (1.2). We present a nonlinear transformation relating solutions of (1.2) and a similar modified nonlinear equation. For this purpose, we need a lemma.

Lemma 1. The KP class of equations (1.2) is satisfied by u

$$u = -2\sqrt{3}\varepsilon C^{\frac{1}{2}}\partial_x^{-1}w_y - 2\sqrt{3}\varepsilon Fw - 6\varepsilon w_x - 6w - 6\varepsilon^2w^2 \tag{3.1}$$

if w satisfies the equation :

$$w_t + w_{xxx} - 6(w + \varepsilon^2w^2)w_x - 2\sqrt{3}\varepsilon C^{\frac{1}{2}}w_x\partial_x^{-1}w_y - 2\sqrt{3}\varepsilon Fww_x$$

$$+Aw + B\partial_x^{-1}w_y + C\partial_x^{-1}w_{yy} + Dw_y + Ew_x = 0 \tag{3.2}$$

where ε is any real parameter.

Proof : Substitution from (3.1) into (1.2) yields :

$$u_t + uu_x + u_{xxx} + Au + B\partial_x^{-1}u_y + C\partial_x^{-1}u_{yy} + Du_y + Eu_x =$$
$$(-6 - 6\varepsilon\partial_x - 12\varepsilon^2w - 2\sqrt{3}\varepsilon C^{\frac{1}{2}}\partial_x^{-1}\partial_y - 2\sqrt{3}\varepsilon F)(w_t + w_{xxx} - 6(w + \varepsilon^2w^2)w_x$$
$$-2\sqrt{3}\varepsilon C^{\frac{1}{2}}w_x\partial_x^{-1}w_y - 2\sqrt{3}\varepsilon Fww_x + Aw + B\partial_x^{-1}w_y + C\partial_x^{-1}w_{yy} + Dw_y + Ew_x)$$

Hence u, given by (3.1), is a solution of (1.2) if w is a solution of (3.2) but of course not necessarily vice versa, and we complete the proof.

It is clear that if we set $\varepsilon = 0$ then (3.1) reduces to $u = -6w$ and, correspondingly, (3.2) becomes (1,2). Note that (3.2), for all ε, has a conservation law of the form :

$$w_t + (w_{xx} - 3w^2 - 2\varepsilon^2w^3 - 2\sqrt{3}\varepsilon C^{\frac{1}{2}}w_x\partial_x^{-1}w_y - 2\sqrt{3}\varepsilon Fw^2$$

$$-\frac{1}{2}\sqrt{3}C^{\frac{1}{2}}C_y\partial_x^{-1}w_t^2A\partial_x^{-1}w - B\partial_x^{-2}w_y + D\partial_x^{-1}w_y + Ew)_x$$

$$+(C\partial_x^{-1}w_y + \sqrt{3}\varepsilon C^{\frac{1}{2}}w^2)_y = 0 \tag{3.1}$$

and so :

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w\,dx\,dy = \text{constant}$$

In order to generate conservation laws for (1.2), we take advantage of the arbitrary parameter ε. Since $w \to -\frac{1}{6}u$ as $\varepsilon \to 0$ we choose to represent w by an asymptotic expansion in ε :

$$w(x,y,t,\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n w_n(x,y,t) \quad \text{as } \varepsilon \to 0 \tag{3.4}$$

if we treat the constant in (3.3) similarly as a power series in ε, then by writing w as its asymptotic expansion, we obtain :

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_n \, dx\, dy = \text{constant} \tag{3.5}$$

for each $n = 0, 1, 2, \ldots$

Finally, we use the asymptotic expansion, (3.4), in (3.1), and equate coefficients of ε^n for each n = 0, 1, 2, ...thus from :

$$\begin{aligned}\sum_{n=0}^{\infty}\varepsilon^n w_n &= -\frac{1}{6}u - 3^{-\frac{1}{2}}\varepsilon C^{\frac{1}{2}} \sum_{n=0}^{\infty} \varepsilon^n \partial_x^{-1} w_{ny} - 3^{-\frac{1}{2}}\varepsilon F \sum_{n=0}^{\infty}\varepsilon^n w_n \\ &\quad - \varepsilon \sum_{n=0}^{\infty}\varepsilon^n w_{nx} - \varepsilon^2\left(\sum_{n=0}^{\infty}\varepsilon^n w_n\right)^2\end{aligned} \tag{3.6}$$

we see that

$$w_0 = -\frac{1}{6}u \tag{3.7}$$

$$w_1 = \frac{1}{6}(3^{-\frac{1}{2}} C^{\frac{1}{2}} \partial_x^{-1} u_y + 3^{-\frac{1}{2}} Fu + u_x) \tag{3.8}$$

$$\begin{aligned}w_2 = -\frac{1}{6}(&2(3)^{-\frac{1}{2}} C^{\frac{1}{2}} u_y + \frac{1}{3} C\partial_x^{-2} u_{yy} + \frac{1}{3} B\partial_x^{-2} u_y + \frac{2}{3} C^{\frac{1}{2}} F\partial_x^{-1} u_y \\ &+ \frac{1}{3} A\partial_x^{-1} u + 2(3^{-\frac{1}{2}}) Fu_x + \frac{1}{3} F^2 u + u_{xx} + \frac{1}{6}u^2)\end{aligned} \tag{3.9}$$

$$\begin{aligned}w_3 = \frac{1}{18}(&2uu_x + 3u_{xxx} + 3B\partial_x^{-1}u_y + 3C\partial_x^{-1}u_{yy} + 2(3^{-\frac{1}{2}})C^{\frac{1}{2}}\partial_x^{-3}u_{yy} \\ &+ 3^{-\frac{1}{2}} C^{\frac{1}{2}} B\partial_x^{-3} u_{yy} + 3^{-\frac{1}{2}} C^{\frac{3}{2}} \partial_x^{-3} u_{yyy} + 3^{-\frac{1}{2}} B_y C^{\frac{1}{2}} B\partial_x^{-3} u_y \\ &+ 4(3^{-\frac{1}{2}}) C^{\frac{1}{2}} A\partial_x^{-2} u_y + 2(3^{-\frac{1}{2}}) BF\partial_x^{-2} u_y + 2(3^{-\frac{1}{2}}) CF\partial_x^{-2} u_{yy} \\ &+ 3^{-\frac{1}{2}} C^{\frac{1}{2}} A\partial_x^{-2} u + 3Au + 6C^{\frac{1}{2}} Fu_y + 3(3^{-\frac{1}{2}}) FA\partial_x^{-1} u \\ &+ 2(3^{-\frac{1}{2}}) C^{\frac{1}{2}} F^2 \partial_x^{-1} u_y + 9(3^{-\frac{1}{2}}) C^{\frac{1}{2}} u_{xy} + 3^{-\frac{1}{2}} C^{\frac{1}{2}} \partial_x^{-1} uu_y \\ &+ 3^{-\frac{1}{2}} CF\partial_x^{-1} u_{yy} + 3^{-\frac{1}{2}} BF\partial_x^{-1} u_y + 2(3^{-\frac{1}{2}}) C^{\frac{1}{2}} F\partial_x^{-1} u_y \\ &+ 3F^2 u_x + 3^{-\frac{1}{2}} F^3 u + 9(3^{-\frac{1}{2}}) Fu_{xx} + \frac{3}{2}(3^{-\frac{1}{2}}) Fu^2 \\ &+ 3^{-\frac{1}{2}} C^{\frac{1}{2}} u\partial_x^{-1} u_y)\end{aligned} \tag{3.10}$$

From (3.7) - (3.10), we see that w_n (n = 0, 1, 2, ...) are conserved densities. Thus there will be infinity of conserved densities. Then, we have the following theorem.

Theorem 1. The KP class of equations (1.2) possesses an infinite number of conserved densities obtained from (3.6) by equating coefficients of powers of ε to zero.

Now we can equate coefficients of ε^n in (3.3) for n = 0, 1, 2, ... to find more conservation laws for (1.2). Then (3.3) becomes :

$$\begin{aligned}&(\sum_{n=0}^{\infty}\varepsilon^n w_n)_t + [\sum_{n=0}^{\infty}\varepsilon^n w_{nxx} - 3(\sum_{n=0}^{\infty}\varepsilon^n w_n)^2 - 2\varepsilon^2(\sum_{n=0}^{\infty}\varepsilon^n w_n)^3 - \\ &2\sqrt{3}\varepsilon C^{\frac{1}{2}}(\sum_{n=0}^{\infty}\varepsilon^n w_n)(\sum_{n=0}^{\infty}\varepsilon^n \partial_x^{-1} w_{ny}) - \sqrt{3}\varepsilon F(\sum_{n=0}^{\infty}\varepsilon^n w_n)^2 + \\ &\sqrt{3}\varepsilon C^{-\frac{1}{2}} B\partial_x^{-1}(\sum_{n=0}^{\infty}\varepsilon^n w_n)^2 + A\sum_{n=0}^{\infty}\varepsilon^n\partial_x^{-1} w_n - B\sum_{n=0}^{\infty}\varepsilon^n \partial_x^{-2} w_{ny} + \\ &D\sum_{n=0}^{\infty}\varepsilon^n\partial_x^{-1} w_{ny} + E\sum_{n=0}^{\infty}\varepsilon^n w_n]_x + [C\sum_{n=0}^{\infty}\varepsilon^n \partial_x^{-1} w_{ny} - \sqrt{3}\varepsilon C^{\frac{1}{2}}(\sum_{n=0}^{\infty}\varepsilon^n w_n)^2]_y = 0\end{aligned} \tag{3.11}$$

Thus, for ε^0 ,

$$u_t + (u_{xx} + \frac{1}{2}u^2 + A\partial_x^{-1}u - B\partial_x^{-2}u_y + D\partial_x^{-1}u_y + Eu)_x + (C\partial_x^{-1}u_y)_y = 0 \tag{3.12}$$

For ε^1 ,

$$\begin{aligned}&(u_x + 3^{-\frac{1}{2}} C^{\frac{1}{2}} \partial_x^{-1} u_y + 3^{-\frac{1}{2}} Fu)_t + (u_{xxx} + 3^{-\frac{1}{2}} C^{\frac{1}{2}} u_{xy} + 3^{-\frac{1}{2}} Fu_{xx} + uu_x \\ &+ \frac{1}{2} 3^{-\frac{1}{2}} Fu^2 + \frac{1}{6} 3^{\frac{1}{2}} C^{-\frac{1}{2}} B\partial_x^{-1} u^2 + Au + 3^{-\frac{1}{2}} C^{\frac{1}{2}} A\partial_x^{-2} u_y + 3^{-\frac{1}{2}} FA\partial_x^{-1} u \\ &- B\partial_x^{-1} u_y - 3^{-\frac{1}{2}} C^{\frac{1}{2}} B\partial_x^{-3} u_{yy} - 3^{-\frac{1}{2}} C^{\frac{1}{2}} B^2 \partial_x^{-3} u_y - 3^{-\frac{1}{2}} BF_y \partial_x^{-2} u - 3^{-\frac{1}{2}} FB\partial_x^{-2} u_y \\ &+ Du_y + 3^{-\frac{1}{2}} C^{\frac{1}{2}} D\partial_x^{-2} u_{yy} + \frac{1}{2} 3^{-\frac{1}{2}} C^{-\frac{1}{2}} C_y D\partial_x^{-2} u_y + 3^{-\frac{1}{2}} F_y D\partial_x^{-1} u + \\ &3^{-\frac{1}{2}} FD\partial_x^{-1} u_y + Eu_x + 3^{-\frac{1}{2}} C^{\frac{1}{2}} CE\partial_x^{-1} u_y + 3^{-\frac{1}{2}} FEu)_x + \\ &(Cu_y + 3^{-\frac{1}{2}} C^{\frac{3}{2}} \partial_x^{-2} u_{yy} + 3^{-\frac{1}{2}} C^{\frac{1}{2}} B\partial_x^{-2} u_y + 3^{-\frac{1}{2}} C^{\frac{1}{2}} A\partial_x^{-1} u + \\ &3^{-\frac{1}{2}} CF\partial_x^{-1} u_y - \frac{1}{2} 3^{-\frac{1}{2}} C^{\frac{1}{2}} u^2)_y = 0\end{aligned} \tag{3.13}$$

and so on. Thus there will be infinity of conservation laws.

IV. EXISTENCE OF SOLITARY WAVE SOLUTIONS

In this section we consider exact solitary wave solutions of (1.1) or (1.2). In [3], solitary wave solutions of the KP equation were obtained. Also by using the Hirota method in soliton theory, solitary wave solutions of KP equation [8] were obtained. Exact solutions of nonlinear differential equations (NDEs) are of importance in physical problems. So far there exists no general method for finding solutions of NDEs. Generally, a relevant nonlinear trans-formation is a powerful method for solving NDEs. Through a dependent variable transformation, two-dimensional solitons of the KP equation were obtained [8]. Here we present the solitary wave solutions of (1.1) via the introduction of certain transformations.

In order to obtain the solitary wave solutions of (1.2), we make the transformation of the form :

$$u(x,y,t) = w(\xi,\theta,\tau) \quad (4.1)$$

$$\xi(x,y,t) = x - \int FC^{-\frac{1}{2}}dy \ , \ \theta(y,t) = \int C^{-\frac{1}{2}}dy \ , \ \tau(t) = t$$

Equation (1.1) becomes :

$$w_{\xi\xi}[-\int(F_t C^{-\frac{1}{2}} - \frac{1}{2}FC^{-\frac{3}{2}}C_t)dy] + w_{\xi\theta}(\int -\frac{1}{2}C^{-\frac{3}{2}}C_t dy) + w_{\xi\tau} + w_\xi^2$$

$$ww_{\xi\xi} + w_{\xi\xi\xi\xi} + Aw_\xi + B(-FC^{-\frac{1}{2}}w_\xi + C^{-\frac{1}{2}}w_\theta) + C[-F_y C^{-\frac{1}{2}}w_\xi +$$

$$\frac{1}{2}FC^{-\frac{3}{2}}C_y w_\xi - FC^{-\frac{1}{2}}(-FC^{-\frac{1}{2}}w_{\xi\xi} + C^{-\frac{1}{2}}w_{\xi\theta} - \frac{1}{2}C^{-\frac{3}{2}}C_y w_\theta +$$

$$C^{-\frac{1}{2}}(-FC^{-\frac{1}{2}}w_{\xi\theta} + C^{-\frac{1}{2}}w_{\theta\theta})] + [C^{-\frac{1}{2}}(2F+G)][-FC^{-\frac{1}{2}}w_{\xi\xi} +$$

$$C^{-\frac{1}{2}}w_{\xi\theta}] + (F^2 + \int(F_y G + C^{-\frac{1}{2}}F_t)dy)w_{\xi\xi} = 0 \quad (4.2)$$

using :

$$\int FG_y dy = FG - \int F_y G dy$$

Equation (4.2) becomes :

$$w_{\xi\tau} + w_\xi^2 + ww_{\xi\xi} + w_{\xi\xi\xi\xi} + w_{\theta\theta} = 0 \quad (4.3)$$

Then let $w(\xi,\tau,\theta) = w(\xi - c\tau + k\theta) = \phi(\eta)$; so (4.3) becomes

$$-c\phi_{\eta\eta} + (\phi\phi_\eta)_\eta + \phi_{\eta\eta\eta\eta} + k^2\phi_{\eta\eta} = 0 \quad (4.4)$$

Integration of equation (4.4) gives :

$$-c\phi_\eta + \phi\phi_\eta + \phi_{\eta\eta\eta} + k^2\phi_\eta = 0 \quad (4.5)$$

where the constant of integration is set equal to zero which is equivalent to imposing the boundary conditions $\phi, \phi', \phi'', \phi''' \to 0$ as $\eta \to \pm\infty$ which describe the solitary wave. Equation (4.5) may be integrated once to yields :

$$-c\phi + \frac{1}{2}\phi^2 + \phi_{\eta\eta} + k^2\phi = c_1 \quad (4.6)$$

where c_1 is an arbitrary constant. Using ϕ_η as an integra-tion factor, we get:

$$-\frac{1}{2}c\phi^2 + \frac{1}{6}\phi^3 + \frac{1}{2}\phi_\eta^2 + \frac{1}{2}k^2\phi^2 = c_1\phi + c_2 \quad (4.7)$$

with c_2 as arbitrary. For simplicity, we continue with $c_1 = c_2 = 0$, which is equivalent to imposing the boundary conditions mentioned above. Thus equation (4.7) becomes :

$$\phi_\eta^2 = \phi^2(-\frac{1}{3}\phi + c - k^2) \quad (4.8)$$

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Integration of equation (4.8) gives :

$$\phi = 3(c-k^2)\text{Sech}^2[\frac{1}{2}(c-k^2)^{\frac{1}{2}}(\eta+\delta)] \quad (4.9)$$

where δ is an arbitrary constant of integration. Coming back to equation (4.1), we get the solitary wave solutions of the KP class of equation (1.1)

$$u(x,y,t) = 3(c-k^2)\text{Sech}^2[\frac{1}{2}(c-k^2)^{\frac{1}{2}}(x - \int c^{-\frac{1}{2}}(F+k)dy - ct + \delta)] \quad (4.10)$$

Therefore we obtain the following theorem.

Theorem 2. For the KP class of equations (1.1) there exist solitary wave solutions. These solitary wave solutions have the form :

$$u(x,y,t)_8 = 3(c-k^2)\text{Sech}^2\frac{1}{2}(c-k^2)^{\frac{1}{2}}\psi \quad (4.11)$$

where

$$\psi = x - \int c^{-\frac{1}{2}}(F+k)dy - ct + \delta$$

c, k and δ are constants.

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