Qatar Univ. Sci. Bull. (1985), 5: 29-32

A SHORT NOTE ON THE POINT-WISE SUMMABILITY OF THE CONJUGATE SERIES OF A FOURIER SERIES IN THE NORLUND SENSE

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Key words: Conjugate Fourier series. Nerlund summability.

ABSTRACT

In this note we consider an analogous theorem of [1] in the sense of point-wise summability of the conjugate series of a Fourier series in the Norlund sense

INTRODUCTION

1. Let f(t) be a periodic function with period 2π, integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let $\tilde{S}[f]$ denote the conjugate series of the Fourier series of f. Let $\phi(t) = f(x+t) + f(x-t) - 2f(x)$, $\psi(t) = f(x+t) - f(x-t)$,

$$\Psi(t) = \int_0^t |\psi(u)|\,du, \text{ and } \Phi(t) = \int_0^t |\phi(u)|\,du$$

In [1] we have :

Theorem 1.1 : Let $q_n = \frac{1}{(n+1)^\alpha}$, $0 \leqslant \alpha < 1$. Then S [f] is summable $S(N, q_n)$ to f(x) at each point x where $\Phi(t) = o(t)$.

(The case $\alpha = o$ is Lebesgue's theorem [5])

In this note we prove the following analogous theorem of theorem 1.1 above.

Theorem 1: Let $q_n = \frac{1}{(n+1)^\alpha}$, $0 \leqslant \alpha < 1$. Then $\tilde{S}[f]$ is summable $S(N,q_n)$ to $\frac{1}{\pi}\int_0^\pi \frac{\psi(t)}{2\tan\frac{1}{2}t}$ at each point x where $\Psi(t) = o(t)$.

Proof: Let $\tilde{S}_n(x)$ denote the sequence of partial sums of the conjugate series of a Fourier series, and let $t_n(x)$ be the corresponding Norlund mean. Then clearly ([3] , and [5])

$$t_n(x) = \sum_{k=o}^{n} q_k \frac{1}{2\pi Q_n} \int_o^{\pi} \psi(t) \frac{\cos \frac{t}{2} \cos (n-k+\frac{1}{2})\, t}{\sin \frac{t}{2}}$$

Hence

$$t_n(x) - \frac{1}{\pi} \int_o^{\pi} \frac{\psi\,(t)}{2 \tan \frac{t}{2}}\, dt = -\int_o^{\pi} \psi(t)\, K_n\,(t)\, dt, \text{ where}$$

$$K_n(t) = \frac{1}{2\,\pi\, Q_n} \sum_{k=o}^{n} \frac{\cos\,(n-k+\frac{1}{2})\, t}{\sin \frac{t}{2}} .$$

In order to prove the theorem we show that :

$$\int_o^{\pi} \psi\,(t)\, K_n\,(t)\, dt = o(1) \qquad \text{as} \qquad n \to \infty .$$

Now

$$\int_o^{\pi} \psi\,(t)\, Kn(t)\, dt = \left(\int_o^{\frac{1}{n}} + \int_{\frac{1}{n}}^{1} + \int_1^{\widehat{\pi}}\right) \psi\,(t)\, K_n(t)\, dt$$

$$= \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3, \text{ say}$$

First by[2] above we have:

$$\tilde{I}_1 = \int_o^{\frac{1}{n}} \psi\,(t)\, K_n(t)\, dt$$

$$= o\left(n \int_o^{\frac{1}{n}} |\, \psi\,(t)\, |\, dt\right)$$

$$= o\,(1) \text{ as } n \to \infty .$$

Second, clearly the moethod S(N, $\frac{1}{(n+1)^{\alpha}}$) is regular. Hence by

the Riemann Lebesgue theorem, and the regularity of the method S(N, $\frac{1}{(N+1)^{\alpha}}$) we have :

$$\tilde{I}_3 = \int_{\pi}^{1} \psi(t)\, K_n(t)\, dt = o\,(1) \qquad \text{as } n \to \infty.$$

Third by [4] above we have:

$$\tilde{I}_2 = \int_{\frac{1}{n}}^{1} \psi(t) \quad K_n(t) \quad dt = o\left(\frac{1}{Q_n} \int_{\frac{1}{n}}^{1} |\psi(t)| \frac{Q_\tau}{t} dt\right).$$

Now

$$\frac{1}{Q_n} \int_{\frac{1}{n}}^{1} |\psi(t)| \frac{Q_\tau}{t} dt = \left(\int_{\frac{1}{n}}^{\frac{1}{n-1}} + \int_{\frac{1}{n-1}}^{\frac{1}{n-2}} + \dots + \int_{\frac{1}{2}}^{1}\right) |\psi(t)| \frac{Q_\tau}{t} dt$$

Hence by integration by parts and simplyfying we obtain:

$$\frac{1}{Q_n} \int_{\frac{1}{n}}^{1} \psi(t) \frac{Q_\tau}{t} dt - o(1) = \frac{1}{Q_n}\left\{\left[\Psi(t) \frac{Q_\tau}{t}\right]_{\frac{1}{n}}^{1} + \psi(t) \frac{Q_\tau}{t^2} dt\right\}$$

Now

$$\frac{1}{Q_n} \Psi(t) \frac{Q_\tau}{t} \text{ldt'} = O\left(\frac{1}{Q_n}\right) + o\left(\frac{1 \,.\, 1.\, Q_n}{Q_n \,.\, n \,.\, 1}\right) = o(1) \text{ as } n \to \infty$$

$$\frac{1}{Q_n} \Psi(t) \frac{Q_\tau}{t^2} dt = \frac{1}{Q_n} \int_1^n \Psi\left(\frac{1}{u}\right) Q(u) \, du$$

$$= o\left(\frac{1}{Q_n} \int_1^n \frac{Q(u)}{u} du\right)$$

$$= o(1) \text{ as } n \to \infty \text{ by } [1.]$$

This completes the proof of theorem 1.

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