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SOME PROPERTIES OF CLASS Z_w

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ABSTRACT

In paper [5] it was established the majorant w. But examples of a function $f \in Z_w$ were not constructed. In this paper, in the case of a smooth closed curve, an example of such a function is constructed.

Historical Approach and Main Results:

The Premeli Privalov theorem [8], [9], [10] is the classical result of the behaviour of a singular operator in the space of continuous functions.

By H_α we denote the class of functions defined on a piecewise smooth closed curve Υ and satisfying Hölder condition with index α .

After that the Premeli Privalov theorem was proved for k−curves [3], [6]. [A closed rectifiable Jordan curve is called k−curve if there exists a constant $k \geqslant 1$ such that for any $t_1, t_2 \in \Upsilon$, $s(t_1, t_2) \leqslant k\,|t_1 - t_2|$].

On the other hand, at 1924, Zygmund A. [14] established the following relationship between continuity modules of the singular integrals with Hilbert Kernels and the continuity modulus of the density (in the case of a circle):

$$w_g(\delta) \leqslant c\left(\int_0^\delta \frac{w_f(\xi)}{\xi}\, d\xi + \delta \int_\delta^\pi \frac{w_f(\xi)}{\xi^2}\, d\xi\right),$$

where $$g(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi)\, \mathrm{ctg}\, \frac{(\xi - t)}{2}\, d\xi,$$

and $$w_f(\delta) = \sup_{|\tau_1 - \tau_2| \leqslant \delta} |f(\tau_1 - f(\tau_2)|.$$

In particular from this inequality the Premeli Privalov theorem follows:

Later this estimation was proved [7] for the case of integrals with smooth curves Cauchy Kernel.

In [1] Zygmund type results were proved for the case of arbitrary closed rectifiable Jordan curves in terms of the characteristic metrics $\propto(\delta)$, $\beta(\delta)$ of the curve, where

$$\propto(\delta) = \inf_{s(t,\,T) \geqslant \delta} |t - \tau| \quad , \delta \in (0, \ell/2)$$

where ℓ is the length of the curve γ,

$$\beta(\delta) = \sup_{|t-\tau| \leqslant \delta} s(t, \tau) \,, \delta \in (\,o, d = \max_{\tau,\, t \in \gamma} |\,\tau - t\,|\,]$$

in the form

$$w_{\tilde f}(\delta) \leqslant c\left(\int_o^{\beta(\delta)} \frac{w_f[\propto(\xi)]}{\propto(\xi)}\, d\xi + \delta \int_{\beta(\delta)}^{\ell} \frac{w_f[\propto(\xi)]}{\propto^2(\xi)}\, d\xi\right),$$

where $$\tilde f(t) = \frac{1}{\pi i} \int_{\gamma} \frac{f(\xi) - f(t)}{\xi - t}\, d\xi + f(t),\ t \in \gamma.$$

From this inequality, in particular, the Premeli Privalov theorem for the case of a k−curve follows. In [13] new characteristics of curves were introduced:

$$\Theta(\delta) = \sup_{\tau \in \gamma} \Theta_t(\delta)$$

where $$\Theta_t(\delta) = \text{mes}\,\{\,\tau \in \gamma,\ |t - \tau| \leqslant \delta\,\}\ ,\ \delta \in (\,o, d\,).$$

For any arbitrary closed rectifiable Jordan curve the following inequality was proved in [13]:

$$w_{\tilde{f}}(\delta) \leqslant c[\,{}_{o}\int^{\delta} \frac{w_f(\xi)}{\xi}\, d\Theta(\xi) + \delta\, {}_{\delta}\int^{\ell} \frac{w_f(\xi)}{\xi^2}\, d\Theta(\xi)),$$

$$\equiv cz(\delta, w)$$

Hence, the Premeli Privalov theorem follows for the curves satisfying $\Theta(\delta) \sim \delta$. (i.e. there exists $c_1, c_2 > o$ such that for any $\delta \in (o,d]$, $c_1 \delta \leqslant \Theta(\delta) \leqslant c_2 \delta$. Notice that for all curves $\Theta(\delta) \geqslant \delta$. So we can take $c_1 = 1$). The class of these curves is larger than the class of piecewise smooth curves and the class of k−curves.

In [11] the following inequality of the Zygmund type, was obtained

$$w_{\tilde{f}}(\delta) \leqslant c \int_{o}^{\delta} \frac{\Theta(\xi)}{\xi^2} w_f\left(\frac{\xi^2}{\Theta(\xi)}\right) d\xi + \delta \int_{\delta}^{2d} \frac{\Theta(\xi)}{\xi^2}$$

$$w_f\left(\frac{\xi^2}{\Theta(\xi)}\right) d\xi).$$

Zygmund (or Zygmund type) estimations allow us to study the behaviour of singular integrals in generalized Hölder spaces:

$$H_w = \{f \in C_\gamma \mid w_f(\delta) = \Theta[w(\delta)]\},$$

where $w(\delta)$ is a continuity modulus such that $w(\delta) > o$,

$$\lim_{\delta \to o} w(\delta) = o,\ w(\delta) \uparrow,\ w(\delta_1 + \delta_2) \leqslant w(\delta_1) + w(\delta_2).$$

Define a norm in H_W as follows:

$$\| f \|_{H_w} = \| f \|_{C_j} + \sup \frac{w_f(\delta)}{w(\delta)}$$

It is clear that H_W is a B−space

THEOREM 1. [13]: Let γ be a curve with $\Theta(\delta) \sim \delta$ and let

$$\int_0^d \frac{w(\xi)}{\xi}\, d\xi < \infty.$$

Then the operator $Af = \tilde{f}$ maps H_w into H_{w_1}, it is bounded, and

$$Z(\delta, w) = O(w_1(\delta)), \text{ where } w_1(\delta) = \int_0^\delta \frac{w(\xi)}{\xi}\, d\xi.$$

On the other hand in [2] for the case of a circle (and in [13] for the case of curves such that $\Theta(\delta) \sim \delta$ and at any point of which the tangent is continuous) and in [12] an inequality was obtained which is the inverse of Zygmund's inequality in some sense. These results gave necessary and sufficient conditions for the existence of a singular operator from H_w to H_{w_1}. Hence we have shown the following:

THEOREM 2. ([2], [13], [12])

Let γ be a smooth closed curve. Then the operator $Af = \tilde{f}$ maps H_w into H_{w_1} iff

$$\int_0^d \frac{w(\xi)}{\xi}\, d\xi < \infty \quad \text{and} \quad Z(w) = O(w(\delta)).$$

In [4,5] the invariance of the class Z_W with respect to the characteristics of Ω was discussed where

$$\Omega_f(\delta) = \sup_{\substack{t \in \gamma \\ \varepsilon \leq \delta}} \left| \int_{\gamma_\varepsilon(t)} \frac{f(\xi) - f(t)}{\xi - t}\, d\xi \right|, \quad \delta \in (o, d).$$

and

$$Z_w = \{ f \in C_\gamma \mid w_f(\delta) = \Theta(w(\delta)),\ \Omega_f(\delta) = \Theta(w(\delta)) \}$$

We notice that Z_w is a B-space with respect to the norm

$$\| f \|_{Z_w} = \| f \|_{H_w} + \sup_{\gamma} \frac{\Omega_f(\delta)}{w(\delta)}$$

It is easy to see that $Z_w \subset H_w$ and for any

$$f \in Z_W . \| f \|_{H_W} \leqslant \| f \|_{Z_W}$$

i.e. the imbedding of Z_W into H_W is continuous. If

$$\int_o^d \frac{w(\xi)}{\xi} d\xi < \infty$$

then $Z_W = H_W$ their norms are equivalent.

THEOREM 3. ([4], [5])

Let Υ be a smooth closed curve,such that

$$\delta \int_\delta^d \frac{w(\xi)}{\xi^2} d\xi = \Theta (w(\delta)) , \delta \in (o,d].$$

Then the operator $A\dot{f} = \tilde{f}$ is a mapping from $Z_W$to Z_W and is bounded.

It can be easily seen that the condition of theorem 2 is not weaker than the condition of theorem 3.

In [5] a majorant w was established for which

$$\int_o^\delta \frac{w(\xi)}{\xi} d\xi = \infty \qquad \delta \int_\delta^d \frac{w(\xi)}{\xi^2} d\xi = O (w(\delta)),$$

however, an example of a function $f \in Z_W$ was not constructed.

In this paper, in the case of a smooth closed curve, an example of such a function is constructed.

Let Υ be a closed smooth curve and $t = t(s)$, $o \leqslant s \leqslant \ell$,

where ℓ is the length of the curve Υ, and the equation of the curve in the arc coordinate has the form $t(s) = x(s) + iy(s)$.

Put $t(o) = t_o$, and $t(-s) = t(\ell - s)$. Without losing generality we can take $\ell \geqslant 2$.

Consider the function:

$$w(\delta) = \begin{cases} \dfrac{2}{\operatorname{Ln} 2}(1-\delta) & , \delta \in [1 \;\; ; \;\; l/2] \\ & , \delta \in [1/e \;\; ; \;\; l/2] \\ \dfrac{1}{\operatorname{Ln} 1/\delta} & , \delta \in [o \;\; ; \;\; l/e] \end{cases}$$

First we prove that the function $w(\delta)$ satisfies the following properties:

1) $w(\delta) > o$;

2) $w(\delta)$ is a non –decreasing function of δ .

3) $\lim_{\delta \to o} w(\delta) = o$;

4) $\dfrac{w(\delta)}{\delta} \downarrow$.

The proof of preperties 1, 2, 3 is easy.

Now we prove the 4th property. Since $w(\delta)$ is constant on $[1/e , l/2]$, then it is enough to prove that $\dfrac{w(\delta)}{\delta} \downarrow$ on $[o, 1/e]$, by calculating the following derivative.

$$\frac{(w\delta)'}{\delta} = \left(\frac{1}{\delta \operatorname{Ln}}\right)' = \frac{1-\operatorname{Ln}(1/\delta)}{\delta^2 \operatorname{Ln}^2(1/\delta)} \leqslant o.$$

Now we shall prove that the function $w(\delta)$ satisfies the conditions of theorem 3.

Consider the expression

$$A(\delta) = \frac{\delta \displaystyle\int_{\delta}^{l/2} \frac{w(\xi)}{\xi^2} d\xi}{w(\delta)}$$

By using L Hospital's rule, we have.

$$\lim_{\delta \to o} A(\delta) = \lim_{\delta \to o} \frac{\delta \int_{\delta}^{\ell/2} \frac{w(\zeta)}{\zeta^2} d\zeta}{\frac{w(\delta)}{\delta}} = \lim_{\delta \to o} \frac{-\frac{w(\delta)}{\delta^2}}{\frac{1-\mathrm{Ln}(1/\delta)}{\delta^2 \mathrm{Ln}^2(1/\delta)}}$$

$$= \lim_{\delta \to o} \frac{\mathrm{Ln}(1/\delta)}{\mathrm{Ln}(1/\delta) - 1} = 1$$

Therefore

$$\delta \int_{\delta}^{\ell/2} \frac{w(\zeta)}{\zeta^2} d\zeta = O(w(\delta)).$$

Therefore by theorem 3, the class Z_w is invariant under the considered singular operator.

Fix the above curve γ and consider the following function:

$$f(t(s)) = \begin{cases} o & , \ s \in [1 ; \frac{\ell}{2}], \\ \frac{2}{\mathrm{Ln}\, 2}(1-s) & , \ s \in [\frac{1}{e} ; \frac{\ell}{2}], \\ \frac{1}{\mathrm{Ln} \frac{1}{|s|}} & , \ s \in [-\frac{1}{e} ; \frac{1}{e}], \\ \frac{2}{\mathrm{Ln}\, 2}(1+s) & , \ s \in [-1, ; -\frac{1}{e}], \\ o & , \ s \in [-1 ; -\frac{\ell}{2}], \end{cases}$$

Now we prove that

$$w_f(\delta) = \sup_{|s_1 - s_2| \leqslant \delta} \left| f(t(s_1)) - f(t(s_2)) \right| \sim w(\delta)$$

Actually, it is sufficient to show the last relation for any small δ . On the other hand outside the segment $[-\frac{1}{e}, \frac{1}{e}]$ we consider a function which satisfies Lipschitz condition on the segment $[-\frac{1}{e}, \frac{1}{e}]$ and for

$o < s_2 - s_1 < s_1 < s_2 \leqslant \frac{1}{e}$ we have:

$$f(t(s_2)) - f(t(s_1)) = \frac{s_2 - s_1}{\xi \operatorname{Ln}^2 \frac{1}{\xi}}$$

where $\xi \in [s_1, s_2]$, i.e. $\dfrac{1}{\operatorname{Ln}\frac{1}{\xi}} \uparrow$. Then $\dfrac{1}{\operatorname{Ln}\frac{1}{\xi}} \leqslant 1$.

On the other hand, $\dfrac{1}{\xi \operatorname{Ln}\frac{1}{\xi}} \downarrow$ and therefore

$$\frac{1}{\xi \operatorname{Ln}\frac{1}{\xi}} \leqslant \frac{1}{(s_2 - s_1) \operatorname{Ln}\left(\frac{1}{(s_2 - s_1)}\right)}.$$

Hence,

$$o \leqslant f(t(s_2)) - f(t(s_1)) \leqslant \frac{s_2 - s_1}{\xi \operatorname{Ln}^2 \frac{1}{\xi}} \leqslant \frac{1}{\operatorname{Ln}\frac{1}{s_2 - s_1}}$$

For $s_1 \leqslant s_2 - s_1$ we have:

$$o \leqslant f(t(s_2)) - f(t(s_1)) \leqslant f(t(s_2)) = \frac{1}{\operatorname{Ln}\frac{1}{s_2}} =$$

$$= \frac{1}{\operatorname{Ln}\frac{1}{s_1 + (s_2 - s_1)}} \leqslant \frac{1}{\operatorname{Ln}\frac{1}{2(s_2 - s_1)}} \leqslant \frac{c}{\operatorname{Ln}\frac{1}{(s_2 - s_1)}}$$

Therefore, for $s_1, s_2 \in [o, \frac{1}{e}]$ and $o < s_2 - s_1 < \delta$, we have:

$$|f(t(s_2)) - f(t(s_1))| < \frac{c}{Ln \frac{1}{\delta}}$$

For $s_1, s_2 \in [-\frac{1}{e}, \frac{1}{e}]$. In the same way, we obtain

$$w_f(\delta) \leqslant \frac{c}{Ln \frac{1}{\delta}}$$

On the other hand,

$$w_f(\delta) = \sup |f(t(s_2)) - f(t(s_1))| \geqslant f(t(\delta)) - f(t(o)) = \frac{1}{Ln \frac{1}{\delta}} .$$

From this we obtain that

$$w_f(\delta) \sim w(\delta).$$

Now we show that $f \in Z_w$.

Let $t(s)$ be any fixed point on the curve Υ. Consider the integral:

$$\int_{t(s-\varepsilon)\,t(s+\varepsilon)} \frac{f(\tau) - f(t)}{\tau - t} d\tau = \int_{t(s-\varepsilon)\,t(s+\varepsilon)} \frac{f(\tau(\xi)) - f(t(s))}{\tau(\xi) - t(s)} d\,\tau(\xi).$$

Let δ_o be any positive number sufficiently small and $o < s < \delta_o < \frac{1}{2e}$.

Consider the following cases:

1) If $\varepsilon < \frac{s}{2}$ then

$$f(\tau(\xi)) - f(t(s)) = \frac{1}{Ln \frac{1}{\xi}} - \frac{1}{Ln \frac{1}{s}} = - \frac{1}{\tau Ln^2 \frac{1}{\tau}} (\xi - s)$$

where $\quad s+\varepsilon \leqslant \tau \leqslant s-\varepsilon \qquad$ i.e. $\frac{s}{2} < \tau < \frac{3}{2}s$.

Therefore

$$|f(\tau(\zeta)) - f(t(s))| \leqslant c \frac{|\zeta - s|}{s \operatorname{Ln} \frac{1}{s}}$$

For the integral we have.

$$\left| \int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{f(\tau(s)) - f(t(S))}{\tau(s) - t(s)} \, d\tau(s) \right| \leqslant \frac{c}{s\operatorname{Ln} \frac{1}{s}}$$

$$\int_{t(s-\varepsilon)}^{t(s+\varepsilon)} |d\tau(s)| = \frac{c}{s\operatorname{Ln}\frac{1}{s}} \leqslant \frac{c}{\varepsilon \operatorname{Ln}\frac{1}{\varepsilon}} = \frac{c}{\operatorname{Ln}\frac{1}{\varepsilon}} = c\,w(\varepsilon).$$

2) If $\frac{s}{2} \leqslant \varepsilon \leqslant s$ then

$$\int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{f(\tau) - f(t)}{\tau - t} \, d\tau = \left[\int_{t(\frac{s}{2})}^{t(\frac{3s}{2})} + \int_{t(s-\varepsilon)}^{t(\frac{s}{2})} \right.$$

$$\left. + \int_{t(\frac{3s}{2})}^{t(s+\varepsilon)} \right] \frac{f(\tau) - f(t)}{\tau - t} \, d\tau = A_1 + A_2 + A_3$$

A_1 is estimated as in case 1 with $\varepsilon = \frac{s}{2}$. Therefore we get

$$|A_1| \leqslant c \frac{1}{\operatorname{Ln}\frac{1}{\varepsilon}} = c\,w(\varepsilon).$$

Since on smooth curves, $|\tau - t| \leqslant s(t,\tau) \leqslant k\,|t-\tau|$

where $k \geqslant 1$ is a constant, we have

$$|A_2| = \left| \int_{t(s-\varepsilon)\, t(\frac{s}{2})} \frac{f(\tau)-f(t)}{\tau - t}\, d\tau \right| \leqslant$$

$$\int_{t(s-\varepsilon)\, t(\frac{s}{2})} \frac{|f(\tau)-f(t)|}{|\tau - t|}\, |d\tau|$$

$$\leqslant k \int_{t(s-\varepsilon)\, t(\frac{s}{2})} \frac{|f(\tau)-f(t)|}{s(t,\tau)}\, |d\tau| \leqslant$$

$$\frac{2k}{s} \int_{t(s-\varepsilon)\, t(\frac{s}{2})} \left(\frac{1}{Ln\frac{1}{\xi}} + \frac{1}{Ln\frac{1}{s}} \right) d\xi \leqslant$$

$$\leqslant \frac{4k}{sLn\frac{1}{s}} \int_{t(s-\varepsilon)\, t(\frac{s}{2})} d\xi = \frac{4k(\varepsilon - \frac{s}{2})}{sLn\frac{1}{s}} \leqslant$$

$$\leqslant \frac{4k}{Ln\frac{1}{s}} \leqslant \frac{c}{Ln\frac{1}{\varepsilon}} = cw(\varepsilon)$$

For the 3rd integral A_3 we have:

$$|A_3| = \left| \int_{t(\frac{3}{2})\, t(\varepsilon + s)} \frac{f(\tau)-f(t)}{\tau - t}\, d\tau \right| \leqslant$$

$$\int_{t(\frac{3}{2}s)\, t(s+\varepsilon)} \frac{|f(\tau)-f(t)|}{s(t,T)}\, |d\tau| \leqslant \frac{c}{Ln\frac{1}{\varepsilon}} = cw(\varepsilon).$$

Therefore

$$\left| \int_{t(s-\varepsilon)\,t(s+\varepsilon)} \frac{f(\tau)-f(t)}{\tau - t}\, d\tau \right| \leqslant cw(\varepsilon).$$

3) If $\delta_0 > \varepsilon > s$ then

$$\int_{t(s-\varepsilon)\,t(s+\varepsilon)} \frac{f(\tau)-f(t)}{\tau - t}\, d\tau = \Big[\int_{t(o)\,t(2s)} +$$

$$\int_{t(s-\varepsilon)\,t(o)} + \int_{t(2s)\,t(s+\varepsilon)} \Big] \frac{f(\tau)-f(t)}{\tau - t}\, d\tau$$

$$= B_1 + B_2 + B_3 .$$

B_1 is estimated as in case (2) with $\varepsilon = s$. Therefore, we get

$$| B_1 | < c \frac{1}{Ln \frac{1}{s}} < c \frac{1}{Ln \frac{1}{\zeta}} = cw(\varepsilon).$$

Consider now $B_2 + B_3 =$

$$\int_{t(s-\varepsilon)\,t(o)} \frac{f(\tau)\zeta))}{\tau(\zeta) - t(s)}\, d\tau(\zeta) +$$

$$\int_{t(2s)\,t(s+\varepsilon)} . \frac{f(\tau(\zeta))}{\tau(\zeta) - t(s)}\, d\tau(\zeta) -$$

$$f(t) \int_{t(s-\varepsilon)\,t(o)} \frac{d\tau}{\tau(\zeta) - t(s)} + \int_{t(2s)\,t(s+\varepsilon)} \frac{d\tau}{\tau(\zeta) - t(s)}$$

$$= B_1' - B_2'$$

where $B_2'' = f(t)\tau$, and

$$\mathcal{J} = \operatorname{Ln} \frac{(t(o) - t(s))}{t(s-\varepsilon) - t(s)} + \operatorname{Ln} \frac{t(s+\varepsilon) - t(s)}{t(2s) - t(s)}$$

$$= \operatorname{Ln} \frac{|t(o) - t(s)|}{w\, t(s-\varepsilon) - t(s)|} + i \arg \frac{t(o) - t(s)}{(t(s-\varepsilon) - t(s)} +$$

$$+ \operatorname{Ln} \frac{|t(s+\varepsilon) - t(s)|}{t(2s) - t(s)|} + i \arg \frac{t(s+\varepsilon) - t(s)}{t(2s) - t(s)} =$$

$$= \operatorname{Ln} \frac{|t(o) - t(s)| \, . \, |t(s+\varepsilon) - t(s)|}{|t(s-\varepsilon) - t(s) \, . \, |t\,(2s) - t(s)|} + i(\alpha + \beta)$$

Since the curves is smooth, then $\frac{|t(o) - t(s)|}{|t(2s) - t(s)|} \to 1$

as $s \to o$ and $\frac{|t(s+\varepsilon) - t(s)|}{|(s-\varepsilon) - t(s)|} \to 1$ as $s \to o$.

Therefore, the logarithmic part in the last equation is bounded when s is small.

In the case of smooth curve we have

$$\alpha = \arg \frac{t(o) - t(s)}{t(s-\varepsilon) - t(s)} \to o \qquad \text{when } s \to o,$$

$$\beta = \arg \frac{t(s+\varepsilon) - t(s)}{t(2s) - t(s)} \to o \text{ when } s \to o,$$

because α and β are bounded for small s.

Therefore, if we take δ_o sufficiently small, and $o < s < \delta_o$ we find that $|\mathcal{J}| \leqslant$ constant.

Hence,

$$|B_2'| = |f(t(s))| \, . \, |\mathcal{J}| \leqslant \text{const} \, |f(t))| = \frac{c}{\text{Ln}\,\frac{1}{s}} \leqslant$$

$$\leqslant \frac{c}{\text{Ln}\,\frac{1}{\varepsilon}} = c\, w(\varepsilon)$$

For

$$|B_1'| = \int_{s-\varepsilon}^{o} \frac{f(\tau)\xi))}{\tau(\xi) - t(s)} \dot{\tau}(\xi)\, d\tau + \int_{2s}^{s+\Sigma} \frac{f(\tau(\xi))}{\tau(\xi) - t(s)} \tau(\xi)\, d\xi =$$

$$= I_1 + I_2$$

Now, in the integral I_1, let $y = s - \xi$ and in the integral I_2, $y = \xi - s$. Then we have:

$$|B_1'| = \int_{\varepsilon}^{s} \frac{f(\tau(s-y))}{\tau(s-y) - t(s)} \dot{\tau}(s-y)dy +$$

$$\int_{s}^{\varepsilon} \frac{f(\tau(y+s))}{\tau(y+s) - t(s)} \dot{\tau}(y+s)dy =$$

$$= \int_{s}^{\varepsilon} \left(\frac{f(\tau(y+s))}{\tau(y+s) - t(s)} \dot{\tau}(y+s) - \frac{f(\tau(s-y))}{\tau(s-y) - t(s)} \dot{\tau}(s-y)\right) dy =$$

$$= \int_{s}^{\varepsilon} \left(\frac{f(\tau(y+s)) - f(\tau(s-y))}{\tau(y+s) - t(s)} \dot{\tau}(s-s)\right) dy +$$

$$+ \int_{s}^{\varepsilon} \left(\frac{f(\tau(s-y) - f(t(s))}{\tau(s-y) - t(s)} \dot{\tau}(y+s)\, dy -\right.$$

$$- \int_s^{\varepsilon} \left(\frac{f(\tau(s-y)) - f(t(s))}{\tau(s-y) - t(s)} \tau(s-y)\, sy + \right.$$

$$+ f(t) \int_s^{\varepsilon} \left(\frac{\tau(y+s)}{\tau(y+s) - t(s)} - \frac{\tau(y+s)}{\tau(y-s) - t(s)} \right) dy =$$

$$= A_1 + A_2 + A_3 + A_4 .$$

$$| A_1 | \leqslant \int_s^{\varepsilon} \frac{f(\tau(y+s)) \quad f(\tau(s-y))}{\tau(y+s) - t(s)} dy \leqslant k \int_s^{\varepsilon} \frac{\dfrac{1}{\operatorname{Ln} \frac{1}{y+s}} - \dfrac{1}{\operatorname{Ln} \frac{1}{y-s}}}{y} dy$$

If $\varepsilon \leqslant 2s$ then

$$| A_1 | \leqslant \int_s^{2s} \frac{\dfrac{1}{\operatorname{Ln} \frac{1}{y+s}} - \dfrac{1}{\operatorname{Ln} \frac{1}{y-s}}}{y} dy \leqslant \frac{1}{s} \left(\frac{1}{\operatorname{Ln} \frac{1}{3s}} \right) s = \frac{1}{\operatorname{Ln} \frac{1}{3s}}$$

$$\leqslant c \frac{1}{\operatorname{Ln} \frac{1}{\varepsilon}}$$

If $\varepsilon > 2s$; then

$$| A_1 | \leqslant \left(\int_s^{2s} + \int_{2s}^{\varepsilon} \right) \frac{\dfrac{1}{\operatorname{Ln} \frac{1}{y+s}} - \dfrac{1}{\operatorname{Ln} \frac{1}{y-s}}}{y} dy \leqslant c \frac{1}{\operatorname{Ln} \frac{1}{\varepsilon}}$$

$$+ \int_{2s}^{\varepsilon} \frac{\dfrac{1}{\operatorname{Ln} \frac{1}{y+s}} - \dfrac{1}{\operatorname{Ln} \frac{1}{y-s}}}{y} dy$$

$$\left| \frac{1}{\operatorname{Ln} \frac{1}{y+s}} - \frac{1}{\operatorname{Ln} \frac{1}{y-s}} \right| = \frac{\left| \operatorname{Ln} \frac{1}{y+s} - \operatorname{Ln} \frac{1}{y-s} \right|}{\operatorname{Ln} \frac{1}{y+s} - \operatorname{Ln} \frac{1}{y-s}} \leqslant$$

$$\leqslant \frac{\operatorname{Ln} \frac{y+s}{y-s}}{\operatorname{Ln} \frac{1}{2y} . \operatorname{Ln} \frac{1}{y}} \leqslant c \frac{\operatorname{Ln} \frac{y+s}{y-s}}{\operatorname{Ln}^2 \frac{1}{y}.} \leqslant \frac{c}{\operatorname{Ln}^2 \frac{1}{y}}$$

for $y \in [2s, \varepsilon]$

$$\mathrm{Ln}\,\frac{y+s}{y-s} = \mathrm{Ln}\,(1+\frac{2s}{y-s}) \leqslant \frac{2s}{y-s} \leqslant 2\,.$$

Therefore,

$$|A_1| \leqslant C\frac{1}{\mathrm{Ln}\,\frac{1}{\varepsilon}} + \int_{2s}^{\varepsilon}\frac{dy}{y\mathrm{Ln}^2\frac{1}{y}} = C\frac{1}{\mathrm{Ln}\,\frac{1}{\varepsilon}} - \int_{2s}^{\varepsilon}\frac{dy}{\mathrm{Ln}^2\frac{1}{y}}\,d\mathrm{Ln}(\frac{1}{y})$$

$$= C\frac{1}{\mathrm{Ln}\,\frac{1}{\varepsilon}} - C\int_{2s}^{\varepsilon}\frac{dt}{t^2} = C\frac{1}{\mathrm{Ln}\,\frac{1}{\varepsilon}} + C\left|\frac{1}{t}\right|_{2s}^{\varepsilon}$$

$$= C\frac{1}{\mathrm{Ln}\,\frac{1}{\varepsilon}} + C\left|\frac{1}{\mathrm{Ln}\,\frac{1}{y}}\right|_{2s}^{\varepsilon} = C\frac{1}{\mathrm{Ln}\,\frac{1}{\varepsilon}} +$$

$$C\left(\frac{1}{\mathrm{Ln}\,\frac{1}{\varepsilon}} - \frac{1}{\mathrm{Ln}\,\frac{1}{2s}}\right) \leqslant C\frac{1}{\mathrm{Ln}\,\frac{1}{\varepsilon}} = CW(\varepsilon)\,.$$

Estimating the integrals A_2 and A_3 similarly, we find that $|A_1| + A_2 + A_3| \leqslant CW(\varepsilon)$.

For A_4 :

$$|A_4| \leqslant |f(t)| \left|\int_{s}^{\varepsilon}\left(\frac{1}{\tau(y+s)-t(s)} - \frac{1}{\tau(s-y)-t(s)}\right)dy\right| =$$

$$= |f(t)| \left|\int_{s}^{\varepsilon}\frac{\tau(s-y)-\tau(s+y)}{(\tau(y+s)-t(s))\,.\,(\tau(s-y)-t(s))}\,dy\right|$$

$$\leq |f(t)|\,.\,2sk^2 \int_s^{\varepsilon} \frac{dy}{y^2} = \frac{1}{\text{Ln}\,\frac{1}{s}}\,.2sk^2\left(\frac{1}{s}-\frac{1}{\varepsilon}\right) \leq$$

$$C\frac{1}{\text{Ln}\,\frac{1}{s}} \leq C\frac{1}{\text{Ln}\,\frac{1}{\varepsilon}} = CW(\varepsilon).$$

This ends the proof.

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بعض خواص فصل الدوال z_w

رؤوف عيسى

في بحث سابق تم تعريف فصل جديد من الدوال Z_w وقد تم إيجاد قيمة حدًا أعلى w لأي دالة من هذا الفصل ولكن لم يعطي أي مثال على دوال هذا الفصل .

وفي هذا البحث تم وضع مثال أي إيجاد دالة $\wp$ من دوال هذا الفصل والمعرفة على المنحنيات المغلقة الملساء . ثم أثبت أن هذه الدالة تنتمي إلى الفراغ Z_w .