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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A FUNCTIONAL DIFFERENTIAL EQUATION

By

I. A. GOMA

Faculty of Science, University of Qatar.

INTRODUCTION

The interest towards the theory of functional differential equations in the last decades is great, due to the increasing circle of applications in various fields of science and technology. A detailed survey of the literature that reflects tthis theory is done in (1), (5), (6) and others. In this paper we are concerned with the problem:

$$\frac{dx}{dt} = F(t, x, x(h(t, x))), x(0) = X_O, \tag{1}$$

where x is an element of Banach space E, and $t \in [o, T]$. Problem (1) has been investigated in (2). However, in the present paper an existence and uniqueness theorem is proved for weaker hypeothesis than that needed in (2). The operator F is assumed to satisfy a generalized Lipschitz condition (9) of the types used by authr in (3). We also prove the convergence of the successive approximations:

$$x'_{n+1}(t) = (t, x_n(t), x_n(h(t, x_n(t)))),$$
$$x_o(t) = x_o \quad o \leq t \leq T \tag{2}$$

to the unique solution of (1) and determine the rate of convergence. We now prove the following lemma which will be used in our subsequent theorem.

Lemma

Suppose that $\mathcal{L}(t, u, v)$ $[o \leq t \leq T, o \leq u \leq 2r, o \leq v \leq 2Cr]$ is a nonnegative continuous function of the totally of its arguments and non-decreasing function of u, v and that the problem

$$\frac{du}{dt} = \mathcal{L}(t, u, Cu), u(o) = o \tag{3}$$

has only the trival solution, moreover

$$\max_{o \leq t \leq T} \mathcal{L}(t, 2r, 2Cr) \leq 2r, \ r\,T \leq r \tag{4}$$

Then the function sequence

$$\mathcal{E}'_1(t) = \varphi\,(t, 2r\,2cr) \qquad o \leq t \leq T \tag{5}$$

$$\mathcal{E}'_{n+1}(t) = \varphi\,(t, \mathcal{E}_n(t), C\mathcal{E}_n(t)) \quad o \leq t \leq T \tag{6}$$

$$\mathcal{E}_n(o) = o \;,\; n = 1, 2 \ldots$$

satisfies the conditions

$$1.\; o \leq \mathcal{E}_{n-1}(t) \leq \mathcal{E}_n(t) \; o \leq t \leq T \tag{7}$$

$$2.\; \mathcal{E}_n(t) \to o \quad \text{when } n \to \infty \tag{8}$$

uniformly w.r.t. $t \in [o, T]$

Proof:

It follows from (4), (5), (6) and the fact that φ (t, u. v) is a nondecreasing function, that

$$\mathcal{E}'_2(t) = \varphi(t, \mathcal{E}_1(t), c\,\mathcal{E}_1(t)) \leq \varphi(.T, 2R, 2Cr) = \mathcal{E}_1(t)$$

and $\mathcal{E}_2(o) = \mathcal{E}_1(o) = o$, hence $\mathcal{E}_2(t) \leq \mathcal{E}_1(t)$ $o \leq t \leq T$

We assume that $\mathcal{E}_n(t) \leq \mathcal{E}_{n-1}(t)$ $\quad o \leq t \leq T$

Since $\varphi(t, u, v)$ is nondecreasing we get

$$\mathcal{E}_{1+1}(t) \leq \varphi(t, \mathcal{E}_{n-1}(t), C\,\mathcal{E}_{n-1}(t)) = \mathcal{E}'_n(t)$$

and $\mathcal{E}_{n+1}(o) = \mathcal{E}_n(o) = o$, hence (7) is proved

$$\therefore \lim_{n \to \infty} \mathcal{E}_n(t) = \mathcal{E}(t) \text{ uniformly w.r.t. } t \in [o, T]$$

taking the limit in

$$\mathcal{E}_{n-1}(t) = \int_o^t \varphi\,(S, \mathcal{E}_n(S), C\mathcal{E}_n(S))\,ds$$

and using the fact that the problem (3) has only the trivial solution we conclude that $\mathcal{E}(t) \equiv o$ $o \leq t \leq T$

Theorem

Suppose that the following conditions are satisfied

A - The operator F (t, x, y) is continuous with respect to the totally of its arguments on $Q = \{ [o, T], \| x - x_o \| \leq r, \| y - x_o \| \leq r \}$
and satisfying in Q the condition

$$\| F(t, x, y) - F(t, x, y) \| \leq \varphi(t, \| x - x \|, \| y - u \|) \tag{9}$$

$$\text{and } \| F(t, x, y) \| \leq r \tag{10}$$

B - The function h(t, x) is continuous with respect to $t \in [o, T]$ and $\|x-x_o\| \leq r$ into te closed interval $[o, T]$ and satsifying

$$\| h(t, x) - h(t, \bar{x}) \| \leqslant L \| x - \bar{x} \| \quad (11)$$

$$o \leqslant h(t, x) \leqslant t \quad (12)$$

C - The function $\mathscr{C}$ (t, u, v) satisfies the conditions of the lemma for $C = 1 + Lr'$

Then the problem (1) has a unique solution $x^*(t)$ and the sequence of abstract functions determined by (2) Coverges to this solution. Moreover the rate of convergence is determined by

$$\| x_O(t) - x^*(t) \| \leqslant \mathcal{E}_n(t) \qquad o \leqslant t \leqslant T \quad (13)$$

Proof:

I. We prove that $\| x'_n(t) \| \leqslant r', \| x_n(t) - x_O \| \leqslant r, n = 1,2,......$ (14)

Let $\| x'_n(t) \| \leqslant r'$ and $\| x_n(t) - x_O \|$ r then follows from (10) that

$\| x'_{n+1}(t) \| = \| F(t, x_n(t), x_n(h(t, x_n(t)))) \| \leqslant r'$

$\therefore \| x_{n+1}(t) - x_o \| \leqslant r$

Since $\| x'_1(t) \| \leqslant r'$ and $\| x_1(t) - x_O \| \leqslant r$ hence (14) is proved.

II. We prove that

$$\| x_n(t) - x_m(t) \| \leqslant \mathcal{E}_n(t), o \leqslant t \leqslant T, n \leqslant m, n = 1, 2, ... \quad (15)$$

From (9), (14), (11), (12) and the fact that $\mathscr{C}(t, u, v)$is a nondecreasing function we get

$\|x'_1(t) - x'_m(t)\| = \| F(t, x_O, x_O) - F(t, x_{m-1}(t), x_{m-1}(h(t,x_{m-1}(t)))) \|$
$\leqslant \mathscr{C}$ t(, 2r, (1 + Lr')2r) $= \mathcal{E}'_1(t)$

and $\| x_1(o) - x_m(o) \| = o$ hence

$\| x_1(t) - x_m(t) \| \leqslant \mathcal{E}_1(t) \qquad o \leqslant t \leqslant T$

We assume that (15) holds for $m \geqslant n$ and prove that it holds for $m \geqslant n + 1$

Let $\| X_n(t) - X_m(t) \| \leqslant \mathcal{E}_n(t) \qquad o \leqslant t \leqslant T$

From (9),(14), (11), (12) and the fact that $\mathscr{C}(t, u, v)$ is nonnegative and nondecreasing it follows that

$\| x'_{n+1}(t) - x_m(t) \| = \|F(t, x_n(t), x_n(h(t, x_n(t)))) -$

$F(t, x_{m-1}(t), x_{m-1}(h(t, x_{m-1}(t)))) \| \leqslant$

$\leqslant \mathscr{C}(t, \| x_n(t) - x_{m-1}(t) \|, \|x_n(h(t, x_n(t))) - x_n(h(t,x_{m-1}(t))) \| +$

$\| x_n(h(t, x_{m-1}(t))) - X_{m-1}(h(t,x_{m-1}(t))) \|)$

$\leqslant \mathscr{C}(t, \mathcal{E}_n(t), \mathcal{E}_n(t)(1 + Lr')) = \mathcal{E}'_{n-1}(t) \qquad o \leqslant t \leqslant T$

and $\| x_{n-1}(o) - x_m(o) \| = o$

$\therefore \| x_{n-1}(t) - x_m(t) \| \leq \varepsilon_{n-1}(t) \qquad o \leq t \leq T$

Relation (15) now proved by induction and (8) implies that x_n (t) x^* (t)

III. Conversion to the limit in (2) as $n \to \infty$ confirms that x^* (t) is a solution of (1)

IV. To prove that the solution is unique let y^* (t) be another solution then by the same method as (15) was proved it can be shown that

$\| x_n(t) - y^*(t) \| \leq \varepsilon_n (t) \qquad o \leq t \leq T , n = 1, 2, \ldots$

hence letting $n \to \infty$ we conclude that $x^*(t) = y^*(t)$ $o \leq t \leq T$

V. Now taking the limit in (15) for $m \to \infty$, we obtain the estimate (13) of the rate of convergence; this completes the proof.

Remark:

A sufficient condition for problem (3) to have only the trivial solution is that $\mathscr{U}$ (t, u, cu) = $\psi(t)\omega(u)$ and $\omega(u)$ is an Osguda function (see (4), (7)), and this include the case when $\mathscr{U}$ (t, u, v) =

L_1 (t) u + L_2(t) v

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حول وجود الحل ووحدته لاحدى المعـدلات التفاصلية الداليـــة

إبراهيم أحمد جمعه

في هذا البحث درست في فراغ بناخ الشروط الكافية لوجود الحل ووحدته للمسألة الابتدائية .

$$\frac{d\times(t)}{dt} = F\,(t,x(t),\ x\quad h(t,x(t))\quad)\ ,\ x(o) = x_o$$

كذلك انشئت متتابعة من الدوال التي تتقارب إلى هذا الحل واعطيت معادلة لحساب معدل التقارب .