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CONCERNING SEQUENTIAL SPACES

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ABSTRACT

In this paper, important notions concerning sequential spaces are clarified new ideas, one of which is a complete and direct proof of the fundamental theorem characterizing sequential spaces, are introduced.

INTRODUTION

Sequential spaces are topological spaces in which sequentially open sets are open (or equivalently, sequentially closed sets are closed). They have been studied by many mathematicians, but they have been studied most extensively by S.P. Franklin. He termed such spaces. "spaces in which sequences suffice".

Franklin gave two proofs of this fundamental theorem characterizing sequential spaces as the quotients of metric spaces. However, the first (Franklin, 1965), while very short, is so cryptic as to obscure exactly what is intended, and the second proof (Franklin, 1967) comes out of a theory of covering spaces which is not essential to the theorem at hand.

In part 1 of this paper we give a proof which is complete, and which on the other hand, we think, exposes the nature of the relationships involved.

In part 2 of the paper we present an example, again due to Franklin (Franklin, 1967) which is more of a counterexample than an example. It is a countable, sequential, compact (and therefore sequentially compact) space with unique sequential limits. However, it has four properties which make it qualify as a counterexample to some possible conjectures about sequential spaces. These four properties are listed at the end of section 2.

In section 3 we prove a theorem which contrasts with the previous example. It says that the product of two sequential spaces must be sequential if one of the factors is locally compact. In particular, if one of the factors is a compact Hausdorff space the product must be sequential. This theorem was presented by one of the present authors at the Symposium on Convergence Structures at the University of Oklahoma, 1965, but it has not been published seperately.

1. Franklin's Theorem

Let X be a topological space, let S be a subset of X, and suppose (X_n: n=1, 2, ...) is a sequence of X. We say x_n is eventually in S if there is an N such that n > N implies x_n lies in S.

A subset U of X is said to be sequentially open for T if every sequence which converges in the topology T to a point of U is eventually in U.

X is said to be a sequential space if every sequentially open set of X is open.

Note that in any space with a topology T the open sets are sequentially open. If T_s is the collection of sets which are sequentially open for the topology T, it is easy to see that T_s is also a topology for X and thus T_s is a larger topology than T. The topology T_s is called the sequential topology generated by T. Thus X is a sequential space if T_s = T.

Some of the properties of sequential spaces are given by the following two theorems (1).

Theorem 1. If X is a sequential space, Y is any topological space, and f is a function on X to Y, then f is continuous if and only if f is sequentially continuous.

Theorem 2. If x is a sequential space then compactness, sequential compactness, and countable compactness are all equivalent for X; that is, if X has one of these properties it has the other two.

Of course, every metric, or even first countable space is sequential. By the use of quotients we shall see that even more is true.

If X is a topological space, f a mapping of X onto Y, and T_Q is given by $T_Q = \{ \psi\ Y :$ $f^{-1}(\psi)$ is open in X} the T_Q forms a topology for Y. When Y has this topology we say is the quotient f X under the mapping f.

Franklin has pointed out the following important property of sequential spaces.

Theorem 3. The quotient of a sequential space is sequential space. For completeness sake we give the simple proof.

Proof. Let ψ be a sequentially open set in Y. Let $x_n \rightarrow x$ as $n \rightarrow \infty$, where x lies in $f^{-1}(\psi)$. Since a quotient map is continuous, $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$, and since ψ is sequentially open $f(x_n)$ is eventually in ψ. Thus x_n is eventually in $f^{-1}(\psi)$. It follows that $f^{-1}(\psi)$ is sequentially open, and therefore open in X. Thus ψ is open in the quotient topology for Y.

Since metric space are sequential spaces, we have a corollary.

Corollary 1. The quotient of a metric space is a sequential space.

The important theorem of Franklin which we want contains the converse of this statement.

Let X be a topological space, and let J be the positive integers. Then X^J is the set of all sequences on X.

We shall assume the following terminology. Let X be a topological space. Then

I) $A = A(T) = \{ a=(f,x): f \in X^J$ and $f(n) \to x$ in T as $n \to \infty \}$
II) $s = \{ 0 \} \cup \{ 1/n: n=1,2... \}$
III) $S = s \times A$.

We define a metric, p, on S by

IVa) $p((y,a), (z,b)) = 1$ if $a = b$
IVb) $p((y,a), (z,a)) = | z-y |$.

We define a mapping, F, of S onto X by

Va) $F((1/n, (f,x)) = f(n)$

Vb) $F((0,(f,x)) = x$.

In what follows T is the initial topology on X, T_Q is the topology obtained when X is the quotient of S under the mapping F given in V) above. The topology T_s is the sequential topology generated by T.

Lemma 1. For any topological space X, we have $T_Q \subset T_s$.

Proof of Lemma 1. Take $U \subset X$ such that $F^{-1}(U)$ is open in S. To show U is sequentially open for T, pick x in U and suppose that $f(n) \to x$ in T. We need to show f(n) is eventually in U. Now $a = (f,x)$ is in A, $(0,(f,x))$ is in $F^{-1}(U)$, and $(1/n, (f,x)) \to (0,(f,x))$ in S as $n \to \infty$. It follows that $(1/n, (f,x))$ is eventually in the open set $F^{-1}(U)$, and therefore $F((1/n,(f,x)) = f(n)$ is eventually in U. Thus U is sequentially open for T, which is to say U belong to T_s.

Lemma 2. For any topological space X we have $T \subset T_Q$.

Proof of Lemma 2. We need to show that if ψ is open in T then ψ is open in T_Q. This will be true if $F^{-1}(\psi)$ is open in S for each ψ in T i.e. if F is a continuous function on S to (X,T). Since S is metric it is sufficient to show F is sequentially continuous.

Suppose $z_k = (y_k, a_k)$ is a sequence of S which converges to $z= (y, a_o)$ in S as $k \to \infty$. Because of the definition of the metric p, $a_k = a_o$ for all sufficiently large k, and $y_k \to y$ in s when s has the usual topology induced from the real line. we can thus assume that $a_k = a_o$ for all k, and there are two cases to consider. First, the case where y =the reciprocal of an integer and second the case where y = 0.

Now, if $y = i/n_o$ or some positive integer n_o, then $\{ z \} = \{ (y,a) \}$ is an open set in S and it must be that $z_k = z$ for sufficiently large k. Thus $F(z_k) \to F(z)$.

In the second case, y = 0. It is only necessary to consider those values of k such that $y_i \neq 0$. Thus we shall suppose $y_k = 1/n_k$ for some positive integer n_k, k = 1, 2, ... Since $y_k \to 0$ in s, we have

(1) $\quad n_k \to \infty \qquad$ as $k \to \infty$.

Now $F(z_k) = F(1/n_k, (f,x)) = f(n_k)$ and $F(z)\ F(O,(f,x)) = x$ where

$f(n) \to x$

In T as $n \to \infty$. We want to show $f(n_k) \to x$ in T as $k \to \infty$. We shall use the Urysohn Property to do this.

Urysohn Property. If (X,T) is a topological space and $(x_j: j=1, 2, ...)$ is a sequence in x with the property that every subsequence has a subsequence which converges to x, then $x_j \to x$ as $j \to \infty$.

In view of (1). every subsequence of the sequence n_k has an subsequence n_j which increases with j. thus $f(n_j)$ is a subsequence of f(n). Since $f(n) \to \infty$ x in T, the subsequence $f(n_j) \to x$ in T as $j \to \infty$. By the Urysohn Property, the sequence $f(n_k) \to x$ in T as $k \to \infty$. This proves Lemma 2.

Franklin's Theorem. A topological space is sequencial if and only if it is the quotient of a metric space.

Proof of Theorem. By Lemmas 1 and 2, for any topological space with topology T we have $T \subset T_Q \subset T_s$. If T is sequential then T = T thus $T = T_Q = T_s$, which proves the theorem.

For any topology T on x it is easy to see that $A(T) = A(T_s)$. Thus the class of convergent sequence of any topology on X is also the class of convergent sequence for a sequential topology on X. Therefore we have te following corollary.

Corollary 2. For any topology T on X the set of convergent sequence is the same as the convergent sequences for a topology on X which is the quotient topology obtained from a metric space.

2. An Interesting Example

We shall give an example of a compact, countable, sequential space, X with four important properties.

In the construction of X we shall utilize five disjoint subsets of R^2 which we call $S_{1,1}$, $S_{1,2}$, S_2, S_3, and S_4.

Let $S = S_{1,1} \cup S_{1,2} \cup S_2 \cup S_3 \cup S_4$ where:

$S_{1,1} = \{ (1/n, 1/m): n,m = 1,2, \dots \}$
$S_{1,2} = \{ (2+1/n, 1/m): n,m = 1,2, \dots \}$
$S_2 = \{ (1/n,y): n=1,2, \dots, -1 < y \leqslant 0 \}$
$S_3 = \{ (2,Y): 0 \leqslant y \leqslant 1 \}$
$S_4 = \{ (0,-1) \}$

It is important for the example that S_3 is a compact interval.

Give S the topology induced from R^2. Thus S is a metric space.

Let x be the imagine of S under the map f defined as follows:

on $S_{1,1}$ $f(1/n, 1/m) = (1/n, 1/m)$	Define $X_1 = f(S_{1,1})$
on $S_{1,2}$ $f(2+1/n,1/m) = (1/n, 1/m)$	Then also $X = f(s_{1,2})$
on S_2 $f(1/n,y) = (1/n,o)$	Define $X_2 = f(S_2)$
on S_3 $f(2,y) = (0,1)$	Define $X_3 = f(S_3)$
on S_4 $f(0,-1) = (0,0)$	Define $X_4 = f(S_4)$.

Thus $x = X_1 \cup X_2 \cup X_3 \cup X_4$ where $X_1 = S_{1,1}$, $X_2 = \{ (1/n,0): n = 1, 2, \dots \}$, $X_3 = \{ (0.1) \}$, and $X_4 = \{ (0,0) \}$.

Give X the quotient topology from S under the mapping f.

For each z = (x,y) in X a basis of neighbourhood, V(z), in the quotient topology can be seen to be:

for z = (1/n, 1/m) in X_1

{ z } is an open set in X_1
for z = (1/n,O) in X_2

$V(z) = \{ V(z, \varepsilon): V(z, \varepsilon) = \{ (1/n,y) \in X: 0 \leqslant y \leqslant \varepsilon \}, \varepsilon > 0 \}$

for z = (0,1) in X_3

$V(z) = \{ V(z,N)\ V(z,N) = \{ (0,1) \} \cup \{ (1/n,y) \in X: y > O, n \geqslant N \}, N \geqslant 1 \}$

for z = (O,O) in X_4
$V(z) = \{ V(z) : V(z) = \cup V((1/n,O), \varepsilon_n) \cup \{ (0,0) \}, \varepsilon_n > 0, N \geqslant 1 \}$.

From the description of the neighbourhood bases it can be seen that any open covering of X has a finite subcovering. Thus, X is compact. We can also obtain the following properties of the space X.

Property 1. X is a sequential space, but it is not a first countable space.

Since X is the quotient if the metric space S it is a sequential space; however the point (0,0) does not possess a countable neighborhood base therefore X is not first countable.

Property II. X has unique sequential limits, but X is not Hausdorff.

Given any pair of points z and w in X there exist disjoint neighborhoods of z and w unless { z,w } = { (0,0), (0,1) }. But from the above characterization of the neighborhood base at (0,0) it can be seen that any sequence which converges to (0,0) must eventually lie in X_2 U X_4. Since (0,1) has a neighborhood disjoint from X_2 U X_4, X has unique sequential limits even though X is not Hausdorff.

Property III. X has a subspace, namely, Y = X - X_2, which is not sequential in the topology induced by X.

In fact the set { (0,0) } is sequentially open in Y since no non-constant sequence from Y converges to (0,0). However the set { (0,0) } is not open in the induced topology.

Property IV. The product space X × X is not sequential in the product topology.

Let Δ = { (x,x): x in X } be the diagonal in X x X. By a well known theorem, (Bouvbeki 1951), a topological space is Hausdorff if and only if the diagonal is a closed subset of the product space. Thus the diagonal is not closed in X × X. However, since X has unique sequential limits it is easy to see that the diagonal is sequentially closed in the product space. Since X × X contains a set which is sequentially closed but not closed, X × X not a sequential space.

3. A Theorem on Products

In contrast to the previous example we shall prove a theorem which shows, in particular, that the product of two sequential spaces, one of which is a compact Hausdorff space, must be sequential.

Theorem 4. If X and Y are sequential spaces and Y is locally compact then X × Y is sequential.

Recall that by Theorem 2, compactness is equivalent to sequential compactness since Y is a sequential space.

Lemma 3. If X and Y are sequential spaces, W is a sequentially open subset of X × Y, and (x,y) lies in W, then there is a neighborhood U of y such that

$$(x,U) = \{ (x,z): z \text{ in } U \} \subset W.$$

Proof of Lemma 3. The mapping $y \to (x,y)$ is sequentially continuous on Y to X × Y and thus it is continuous when X × Y is given the sequential topology associated with the product topology on X × Y. Thus the inverse image of the sequentially open set W is an open set U.

Proof of Theorem 4. Let W be a non-empty sequentially open in X × Y which contains the point (x,y). We shall show that W contains a set of the form (V,U) where V is a neighborhood of x and U is a neighborhood of y. By Lemma 3, W contains (x,U) for some compact neighborhood U of y. Let V be the largest set such that (V,U) is contained in W. If V is not open there exists a sequence x_n in the complement of V such that $x_n \to x_o$ for some x_o in V.

For each n = 1,2,... , since x_n lies in the complement of V, there exists y_n in U such that

(2) $(x_n,y_n) \notin W$,

and since U is sequentially compact, with no loss of generality, we can suppose that $Yn \to y_o$ for some Y_o in U. Thus $(x_n,y_n) \to (x_n,y_n) \to (x_o,y_o)$ in X × Y, since W is sequentially open in X × Y, we must have

(3) $(X_n,Y_n) \in W$

for sufficiently large n.

The contradiction between (2) and (3) shows that V must have been open.

Corollary 3. If X and Y are sequential spaces, and Y is a compact Hausdorff space, then X × Y is sequential.

Since a comact Hausdorff space is locally compact, the corollary follows from Theorem 4.

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