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THE ONE DIMENSIONAL INVERSE SPECTRAL PROBLEM OF A GENERALIZED STATIONARY SCHRÖDINGER EQUATION

F. A. MEGAHED and A.A. DARWISH
Department of Mathematics, Faculty of Science, University of Qatar, Doha, Qatar.

مسألة الطيف العكسية لمعادلة سباتية عامة لشرودنجر في بعد واحد

في هذا البحث قمنا بدراسة المسألة العكسية على نصف خط الأعداد لمعادلة عامة لشرودنجر لاتعتمد على الزمن . ولقد اعتبرنا حلول المعادلة ودرسنا خواصها . وبمعرفة دالة التوزيع الطيفي أوجدنا حلا للمسألة العكسية .

ABSTRACT

In this paper we study the inverse problem on the half line for a generalized stationary one dimensional Schrödinger equation. We consider the solutions of the differential equation and study its properties. Given the spectral distribution function, we solve the inverse problem.

Key Words: Inverse problem, stationary Schrödinger equation, spectral function, Parseval's equality, discontinuous coefficient..

INTRODUCTION

Consider, in the space $L_2(0, \infty; \rho(x))$, the generalized stationary Schrödinger equation

$$\left[\frac{d^2}{dx^2} - q(x) + \lambda\rho(x)\right]u = 0 \qquad (1)$$

and

$$u'(0) = 0 \qquad (2)$$

We assume that

$$\rho(x) = \begin{cases} a_n^2, & a_{n-1} \le x \le a_n;\ n = 1, \ldots, m \\ 1 & a_m < x < \infty \end{cases} \qquad (3)$$

where $a_n > 0$, $a_0 = 0$, $\alpha_n \neq \alpha_{n+1}$, $\alpha_n \neq 1$ and λ is a constant. Let W_n be the set of functions $\rho(x)$. Also denote with $L_{1,1}$ the class of potentials

$$L_{1,1}=\left\{ q \mid q=\bar{q}, \int_0^\infty x \mid q(x) \mid dx<\infty \right\} \quad (4)$$

The inverse problem can be stated as follows: knowing the spectral distribution function of (1)-(2), can we reconstruct equation (1), i.e., can we determine the potential q(x) and the density function ρ(x)? The boundary value problem (1), (2) was discussed in [1, 2, 5, 8] for the case ρ≡1 and u(0) = 0, on the coefficients was investigated in [4, 9, 11]. In [12} the inverse problem of (1), (2) was investigated using the scattering data by employing the technique of the spectral distribution function.

Preliminaries

In this section, we give some results for the case $\rho(x) \in W_n$ which will be used in the subsequent sections. Denote with $\phi(x,\lambda)$, $x \in [0, a_1]$ the solution of equation (1) which satisfy

$$\phi(0,\lambda) = 1 , \phi'(0,\lambda) = 0 \quad (5)$$

and with $\psi(x,\lambda)$ the solution of (1) which satisfy

$$\psi(0,\lambda) = 0 , \psi'(0,\lambda) = 1 \quad (6)$$

To obtain these solutions, we shall use the results of [7, 8]. Let $\rho(x) \in W_1$; $\alpha_1 = \alpha, a_0 = 0$, and $q \in L_{1,1}$ then we have

$$\phi(x,\lambda)=\cos\sqrt{\lambda}\alpha x)+\int_0^x A(x,t)\cos(\sqrt{\lambda}\alpha t)\,; 0\le x\le a_1 , \quad (7)$$

where the kernel A(x,t) has summable derivatives A_x, A_t, and

$$\frac{dA(x,x)}{dx}=\frac{1}{2}q(x) \text{ and } \frac{\partial A(x,t)}{\partial x}\Big|_{t=0}=0$$

This solution $\phi(x,\lambda)$ is an entire function of $\sqrt{\lambda}$ for any fixed α. Moreover

$$\phi(x,\lambda)=\cos(\sqrt{\lambda}\alpha x)\left(1+O(\frac{1}{\sqrt{\lambda}})\right) \text{as } \operatorname{Im}\sqrt{\lambda}\ge 0, |\sqrt{\lambda}|\to\infty$$

Uniformly with respect to $\sqrt{\lambda}$ on $[0,a_1]$

The second solution $\psi(x,\lambda)$ of equation (1) is given by

$$\psi(x,\lambda)=\frac{\sin(\sqrt{\lambda}\,\alpha x)}{\sqrt{\lambda}\,\alpha}+\int_0^x B(x,t)\frac{\sin(\sqrt{\lambda}\,\alpha t)}{\sqrt{\lambda}\,\alpha}dt\,; 0\le x\le a_1 , \quad (8)$$

where the kernel B(x,t) has summable derivatives B_x, B_t, and

$$\frac{dB(x,x)}{dx}=\frac{1}{2}q(x) \text{ and } B(x,0)=0$$

This solution $\psi(x,\lambda)$ has also the property that it is an entire function of $\sqrt{\lambda}$ for $\operatorname{Im}\sqrt{\lambda}\ge 0, \left|\sqrt{\lambda}\right|\to\infty$.

Moreover

$$\psi(x,\lambda)=\frac{\sin(\sqrt{\lambda}\alpha x)}{\sqrt{\lambda}\alpha}\left(1+O(\frac{1}{\sqrt{\lambda}})\right)$$

uniformly with respect to $\sqrt{\lambda}$ on $[0,a_1]$.

The proof of the following lemma has been done in [7].

Lemma 1

If condition (4) is satisfied for any λ from the upper half plane then equation (1) has a solution $F(x,\lambda)$ which can be written in the form

$$F(x,\lambda)=\exp(i\sqrt{\lambda}x)+\int_x^\infty K(x,t)\exp(i\sqrt{\lambda}\,t)\,dt\,; a_1<x<\infty , \quad (9)$$

The kernel K(x,t) is twice differentiable and satisfies the equation

$$\frac{\partial^2 K(x,t)}{\partial x^2}-\frac{\partial^2 K(x,t)}{\partial t^2}=q(x)K(x,t), \quad (10)$$

and the conditions

$$\frac{dK(x,x)}{dx}=-\frac{1}{2}q(x)\,, \quad \lim_{x+t\to\infty}\frac{\partial K(x,t)}{\partial t}=0 \quad (11)$$

The solutions $F(x,\lambda)$ is analytic in the upper half plane $\operatorname{Im}\sqrt{\lambda}>0$ and continuous on the real line. The asymptotic behaviour of the solution is

$$F(x,\lambda)=\exp(i\sqrt{\lambda}\,x)[1+O(1)]\,;$$
$$F_x(x,\lambda)=i\sqrt{\lambda}\exp(i\sqrt{\lambda}x)\left[i\sqrt{\lambda}+o(1)\right]$$
as $x\to\infty$ for all $\operatorname{Im}\sqrt{\lambda}\ge 0, \sqrt{\lambda}\ne 0$.

The proof of the following lemma could be obtained from [12].

Lemma 2

The boundary value problem (1)-(2) has a finite number of negative eigenvalues and they are all simple.

The main results

Lemma 3

Let f(x) be a finite function which has a continuous derivative in L_2 (0, ∞; ρ(x)) and satisfies the boundary condition (2). Then

$$\int_0^\infty f^2(x)\rho(x)dx = \frac{1}{\pi}\int_0^\infty \frac{\left|F(\sqrt{\lambda})\right|^2\sqrt{\lambda}}{W(\sqrt{\lambda})\,W(-\sqrt{\lambda})}d\lambda + \sum_{n=1}^{\ell} c_n F^2(\sqrt{\lambda n}) ,$$

where

$$F(\sqrt{\lambda}) = \int_0^\infty f(x)\,\phi(x,\sqrt{\lambda})\rho(x)dx,\ c_n = \frac{2\sqrt{\lambda_n}}{f(0,\sqrt{\lambda_n})};$$

and $w(\sqrt{\lambda}) = f'(0,\sqrt{\lambda})$.

Proof:

The resolvent R_λ of (1)-(2) can be written in the form

$$R_\lambda(x,t) = \frac{-1}{F^{l}(0,\sqrt{\lambda})}\begin{cases} f(x,\sqrt{\lambda})\,\phi(t,\sqrt{\lambda}) & ,t\le x \\ f(t,\sqrt{\lambda})\,\phi(x,\sqrt{\lambda}), & t\ge x\end{cases} \qquad (12)$$

Thus we have

$$R_{\lambda+i0} - R_{\lambda-i0} = \frac{u(t,\sqrt{\lambda})\,2i\sqrt{\lambda}u(x,\sqrt{\lambda})}{w(\sqrt{\lambda})\,w(-\sqrt{\lambda})} ,$$

where

$$u(x,\sqrt{\lambda}) = \frac{1}{2i\sqrt{\lambda}}[f'(0,\sqrt{\lambda})f(x,-\sqrt{\lambda}) - f(x,\sqrt{\lambda})f^{l}(0,-\sqrt{\lambda})]$$

and $w(\sqrt{\lambda}) = f'(0,\sqrt{\lambda})$.

Upon implementing the method of Titchmarch [10], hence we obtain

$$f(x) = \frac{1}{2\pi i}\int_0^\infty d\lambda\int_0^\infty \{R_{\lambda+i0}(x,t) - R_{\lambda-i0}(x,t)\}\rho(t)f(t)dt$$

$$+\sum_{n=1}^{l}\mathrm{Res}\left(2\sqrt{\lambda}\int_0^\infty R(x,t,\lambda)\rho(t)f(t)dt\right)$$

$$= \frac{l}{\pi}\int_0^\infty \frac{u(x,\sqrt{\lambda})F(-\lambda)\sqrt{\lambda}}{w(\sqrt{\lambda})w(-\sqrt{\lambda})}d\lambda + \sum_{n=1}^{l} c_n\phi(x,\lambda_n)F(\lambda_n)$$

Multiplying both sides by f(x)·ρ(x) and then integrating from 0 to ∞ with respect to x, we obtain

$$\int_0^\infty f^2(x)\rho(x)dx = \frac{1}{\pi}\int_0^\infty \frac{F(\sqrt{\lambda})F(-\sqrt{\lambda})\sqrt{\lambda}}{w(\sqrt{\lambda})w(-\sqrt{\lambda})}d\lambda + \sum_{n=1}^{l} c_n F(\sqrt{\lambda_n})\int_0^\infty \phi(x,\lambda_n)\rho(x)f(x)dx$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\left|F(\sqrt{\lambda_n})\right|^2\sqrt{\lambda}}{w(\sqrt{\lambda})w(-\sqrt{\lambda})}d\lambda + \sum_{n=1}^{l} c_n F^2(\sqrt{\lambda_n}),$$

which completes the proof of the lemma.

The following lemma could be proved using the previous one.

Lemma 4

The following Parseval's equation holds

$$\frac{1}{\pi}\int_0^\infty \frac{u(x,\sqrt{\lambda})\sqrt{\lambda}u(t,-\sqrt{\lambda})}{w(\sqrt{\lambda})w(-\sqrt{\lambda})}d\lambda + \sum_{n=1}^{l} c_n u(x,\sqrt{\lambda_n})u(t,\sqrt{\lambda_n})$$

$$= \int_{-\infty}^{\infty} u(x,\sqrt{\lambda})u(t,-\sqrt{\lambda})d\sigma(\lambda) ,$$

where

$$\sigma(\lambda) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^\lambda \frac{d\lambda}{w(\sqrt{\lambda})w(-\sqrt{\lambda})} & , \lambda \ge 0 \\ -\displaystyle\sum_{0>\lambda_n>\lambda} c_n & , \lambda < 0 \end{cases}$$

and

$$c_n = \frac{-4\lambda_n}{\left|w(\sqrt{\lambda_n})\right|^2}$$

Corollary 1:

As q(x) ≡ 0, and using formula (12), we get

$\sigma_0(\lambda) =$

$$= \begin{cases} \frac{1}{\pi}\int_0^{\lambda} \frac{1}{\sqrt{\lambda}}[\alpha_1^2 \sin^2(\sqrt{\lambda}\alpha_1 a_1) + \cos^2(\sqrt{\lambda}\alpha_1 a_1)]^{-1}, & \lambda \geq 0 \\ 0, & \lambda < 0. \end{cases}$$

Lemma 5

(i) In formula (7), the kernel A(x,t) satisfies the fundamental equation

$$F(x,t) + A(x,t) + \int_0^x A(x,s)F(s,t)dt = 0, \quad 0 \leq t \leq x < a_1, \tag{13}$$

where

$$F(x,t) = \frac{\partial^2}{\partial x \partial t}\int_{-\infty}^{\infty} \frac{\sin(\alpha_1 x\sqrt{\lambda})\sin(\alpha_1 t\sqrt{\lambda})}{\lambda} d\tau(\lambda),$$

$$\tau(\lambda) = \begin{cases} \sigma(\lambda) - \sigma_0(\lambda) & , \lambda \geq 0 \\ \sigma(\lambda) & , \lambda < 0 \end{cases}$$

and

$$\sigma_0(\lambda) = \begin{cases} \frac{1}{\pi}\int_0^{\lambda} \frac{1}{\sqrt{\lambda}}[\alpha_1^2 \sin^2(\alpha_1 a_1\sqrt{\lambda}) + \cos^2(\alpha_1 a_1\sqrt{\lambda})]^{-1} d\lambda, & \lambda \geq 0 \\ 0, & \lambda < 0. \end{cases}$$

(ii) The integral equation (13) has one and only one solution A(x,t) defined on $0 \leq t \leq x < a_1$

Lemma 6

(i) The kernel B(x,t) of formula (8) satisfies Gelfand-Levitan equation.

$$H(x,t) + B(x,t) + \int_{a_1}^{x} B(x,s)H(s,t)ds = 0, \quad a_1 \leq t \leq x < \infty \tag{14}$$

where

$$H(x,t) = \frac{\partial^2}{\partial x \partial t}\int_{-\infty}^{\infty} \frac{\sin(\sqrt{\lambda}\alpha_1(x-a_1))\sin(\sqrt{\lambda}\alpha_1(t-a_1))}{\lambda} d\tau_a(\lambda),$$

and

$$\tau_a(\lambda) = \begin{cases} \sigma_a(\lambda) - \frac{2\sqrt{\lambda}}{\pi} & , \lambda \geq 0 \\ \sigma_a(\lambda) & , \lambda < 0. \end{cases}$$

(ii) Equation (14) has a unique solution B(x,t) as $a_1 < t \leq x < \infty$

Proof

Since $\phi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of equation (1) together with the initial conditions (2), hence we take $\phi_{a_1}(x,\lambda)$ and $\psi_{a_1}(x,\lambda)$ as the solutions at $x = a_1$. Also, denote by m_0 the Wely's function [5,7,8] of (1)-(2) and m_{a_1} the Wely's function of (1) together with the initial condition $y'(a_1) = 0$. Thus, we have

$$F(x,\lambda) = \phi(x,\lambda) + \psi(x,\lambda)\, m_0(\lambda)$$

$$f_{a_1}(x,\lambda) = \phi_{a_1}(x,\lambda) + \psi_{a_1}(x,\lambda)\, m_{a_1}(\lambda)$$

Since $f(x,\lambda)$ and $f_{a_1}(x,\lambda)$ are independent solutions as $x > a_1$, hence we have $f_{a_1}(x,\lambda) = f(x,\lambda)\,\gamma(\lambda)$. Thus

$$m_{a_1}(\lambda) = [\phi'(a_1,\lambda) + \psi'(a_1,\lambda)m_0(\lambda)]\gamma(\lambda)$$

and

$$1 = [\phi(a_1,\lambda) + \psi(a_1,\lambda)m_0(\lambda)]\gamma(\lambda).$$

Hence

$$m_{a_1}(\lambda) = [\phi'(a_1,\lambda) + \psi'(a_1,\lambda)m_0(\lambda)]$$

$$[\phi(a_1,\lambda) + \psi(a_1,\lambda)m_0(\lambda)]^{-1}$$

This function is to be used to find the spectral function of equation (1) through the relation

$$\sigma_{a_1}(\lambda) = \lim_{\varepsilon \to 0}\int_0^{\lambda} \operatorname{Im}(m_{a_1}(s + i\varepsilon))\, ds$$

Thus equation (14) can be obtained as in [5, 6, 7]. One can also prove the uniqueness of B(x,t) using [8].

Theorem 1: (Uniqueness Theorem)

If the condition (3) is satisfied and $\rho(x) \in w_n$, then by using the spectral function $\sigma(\lambda)$ of (1)-(2), the potential function q(x) and $\rho(x)$ can be defined uniquely.

Proof

It is evident that if $a_1 \neq a_2$ and $\alpha_1 \neq \alpha_2$ then the function

$$\sigma_0(\lambda, a_1, \alpha_1)\, \sigma_0^{-1}(\lambda, a_2, \alpha_2)$$

has no limit as $\lambda \to \infty$.

Therefore the asymptotic behaviour of $\sigma_0(\lambda, a_1, \alpha_1)$ as $\lambda \to \infty$ determines a and α uniquely. Hence the function $\rho(x)$ can be reconstructed uniquely. Here it should be mentioned that this case is true for $\rho(x) \leq w_n$. From lemma (3) we have already deduced the fundamental equation (13), $0 \leq x < a_1$, by using the spectral function $\sigma(\lambda)$. Moreover, and in view of lemma (3), equation (13) has the unique solution A(x,t) as $0 \leq x \leq a_1$ on the form

$$q(x) = 2\frac{dA(x,x)}{dx}$$

Thus, the function q(x) is defined uniquely as $0 \leq x \leq a_1$. From lemma (4) we have

$$q(x) = 2\frac{dB(x,x)}{dx} \text{ as } a_1 < x < \infty.$$

Hence, we conclude that the equation (1) can be reconstructed on the interval $(0, \infty)$ and the theorem is now completely proved.

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