Qatar Univ. Sci. J (2000), 20 : 17 - 20

ELLIPTIC SURFACES OVER A GENUS 1 CURVE WITH THE PAIR (I_3, I_9) OF SINGULAR FIBERS.

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السطوح الناقصية على منحنى من جنس 1 ذات الليفين المنفردين (I_3, I_9)

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نعطي في هذا البحث تصنيفاً للسطوح الناقصية الأصغرية على منحنى من جنس ١ بحيث يكون لها مقطع وليفين منفردين هما (I_3, I_9)

Running Title: Elliptic surfaces with (I_3, I_9) as singular fibers.
Key Words: Elliptic surfaces, J-maps, pull-back, singular fibers.

ABSTRACT

In this paper we give a classication of all minimal elliptic surfaces ($\pi : E \to C$, $g(c) = 1$) with a section and exactly the pair (I_3, I_9) of singular fibers.

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1. INTRODUCTION

In this paper, let C be a genus 1 curve, we classify all minimal elliptic surfaces ($\pi : E \rightarrow C$) with a section and exactly the pair (I_3, I_9) of singular fibers.

The serious study of elliptic surfaces was started by Kodaira [4], he listed all possible types of singular fibers, gave their invariants and analyzed an important invariant called the J-map. His list of singular fibers consists of the following types: I_0^*, I_n, I_n^*, $n \geq$ I, II, III, IV, II^* III^*, IV^*. Beauville [2] has studied the cases in which all singular fibers are of type I_n (i.e., The semi-stables cases), he proved that there are 6 semi-stable cases with the minimal number (=4) of singular fibers, and he wrote Weierstrass equations for these 6 cases.

In 1986 R. Miranda and U. Persson [7] have listed all (=16) extremal ratinal elliptic surfaces, one of the surface in their list is the surface X431. This surface hs exactly 3 singular fibers, a type IV^* fiber over 0, and the pair (I_1, I_3) of singular fibers over ∞.

In 1990 U. Person [8] has listed all possible confinguration of singular fibers on a rational elliptic surface. Also in 1990 R. Miranda [6] has analyzed the same problem by giving a more combinatorial and less geometric analysis. In 1988 Stiller [9] has classified all minimal elliptic surfaces over a curve of genus 1, with exactly one singular fiber necesarily of type I_6^*. Now we write down some preliminaries.

(1.1) To build a minimal elliptic surface $\pi : X \rightarrow C$, it is enough to build the J-map associated to this surface (see [6], p 202], then pull-back via J one of the elliptic surfaces which has J(t) = t, then adjust the fibers of type IV^*, III^*, II^* and I_n^*.

(1.2) The existence of the J-map J: $C \rightarrow \mathbb{P}^1$ which is ramified over 0, 1, and ω is equivalent (see [6], Cor. 3.5) to the eixstence of three permutations σ_0, σ_1 and σ_∞ is S_d (d = deg (J)) representing the monodromy of J about 0, 1, and ∞ respectively, such that these permutations generate a transitive subgroup of S_d, $\sigma_0\sigma_1 = \sigma^1{}_\infty$, and such that the cycle structure of σ_i (i = 0, 1, ∞) corresponds to the ramification of J over the point i (i = 0, 1, ∞). Moreover if these permutations are unique (up to conjugation), then the pair (J, C) is unique up to isomorphism.

(1.3) One of the tools of building a minimal elliptic surface, is to realize this surface as a pull-back of some well known surface, in fact we have: If $\pi : X \rightarrow C$ is a minimal elliptic surface with a section, and if g: $C_1 \rightarrow C$ is a map of curves, then the pull-back $\pi_1 : X_1 = X \times C^C1 \rightarrow C_1$ is a minimal elliptic surface with a section (see [7], Section 7) and the fibers of $\pi_1 : X_1 \rightarrow C_1$ can be calculated as in Table 7.1 page 555 of [7].

(1.4) Given a genus 1 curve C, then there is a degree of 3 map $f : C \rightarrow \mathbb{P}^1$, which is obtained by projecting $C(\leq \mathbb{P}^2)$ from a point q off C in to a line (see [5], p. 1153).

Finally a word about notation, the notation $\|[J^{-1}(x)]\| = (n_1, n_2, \ldots, n_s)$ will be used to mean that $J^{-1}(x)$ consists of s points say $\{x_1, \ldots, x_s\}$ such that the multiplicity of J at x_i [$m_{x_i}(J)$] is n_i, for i = 1, ..., s.

2. MAIN RESULTS

There are the following types of singular fibers (see [4]) I_0^*, I_n, I_n^*, II, IV^*, III, III^*, IV, II^*, $n \geq 1$. Now if e(F) denotes he Euler number of the fiber F, then the Euler numbers of the above list of singular fibers are: 6, n, n+6, 2, 8, 3, 9, 4 and 10 respectively.

Lemma 2.1: Let C genus 1 curve, suppose $\pi : E \rightarrow C$ is a minimal elliptic surface with a section and exactly two singular fibers. If the degree of the line bundle $\mathbb{I}$. Is 1 (i.e. $\mathbb{I}$ is the conormal bundle to the section), then there are twenty five possible pairs (F_1, F_2) of singular fiber types such that the sum of the Euler numbers is 12.

Proof: Immediate from the fact if (F_1, F_2) is a possibel pair of singular fibers, then e(F1) + e(F2) = 12. Q.E.D.

Now the pair (I3, I9) is one of these possible pairs, and this case cannot occur if the genus of the base curve C is 0 (i.e., if $C \cong \mathbb{P}^1$).

Lemma 2.2: Let C be a genus 1 curve, suppose $\pi : E \rightarrow C$ is a minimal elliptic surface with a section and exactly the pair (I_3, I_9) of singular fibers, If J: $C \rightarrow \mathbb{P}^1$ is the J-map associated to this fibration, then degree (J) = 12, and the J-map is ramified as follows:

$\|[J^{-1}(0)\| = (3, 3, 3, 3)$, $\|[^{J-1}(1)\| = (2, 2, 2, 2, 2, 2)$, and $\|[^{J-1}(\infty)\| = 3, 9)$.

Proof: Deg (J) $\sum_{n\geq 1} n$(# of I_n -fibers + # of I_n^* - fibers)
= 12 (see [7], p. 543).

By Hurwitz's formula for the genus of a curve we have:

$24 = \sum_{x\in R} (m_x(J)-1)$, where R = {ramification points of J},

and $m_x(J)$ is (J) is the multiplicity of J at x. Now if x ∈ J-1 (0), then 3 divides m_x (J) (see [6], p. 194). Hence the minimum ramification of J over O is obtained if $|(J)^{-1}(0)]| = 3, 3, 3, 3)$. If $x \in J^{-1}(1)$, then 2 divides $m_x(J)$ (see [6], pp. 194). Hence the minimum ramification of J over 1 is obtained if $|[J]^{-1}(1)]| = (2, 2, 2, 2, 2, 2)$. Moreover $|[J^{-1}(\infty)]| = (3, 9)$ becuase over ∞ we have the pair (I3, I9) of singular fibers; thus

$\sum_{x \in J^{-1}(\{0, 1, \infty\})} (m_x(J) - 1) = 24$, which is the right ramification of J, and hence there is no other ramification of J. Q.E.D.

Theorem 2.3: Under the hypothesis of Lemma 2.2, the curve C is unique, and the J-map ($J: C \to \mathbb{P}^1$) exists and is unique.

Proof: To prove this theorem it is enough to find a triple (σ_0, σ_1, σ_∞) of permutations in S_{12} representing the monodromy of J around 0, 1 and ∞ respectively such that: $\sigma_0\sigma_1 = \sigma^{-1}_\infty$, the triple ($\sigma_0$, σ_1, σ_∞) generates a transitive subgroup of S_{12}, this triple is unique up to conjugation, and such that the cycle structure of σ_0 is (3^4), that of σ_1 is (2^6) and that of σ_∞ is (3, 9).

Assume σ_0 = (1 2 3) (4 5 6) (7 8 9) (10 11 12), and σ_1 = (1 b) (c d) (e f) (g h) (i j) (k l). Since 1 has to appear in one of the 2-cycles of σ_1, we may assume a = 1, hence b ≠ 2, 3 (otherwise the produce $\sigma_0\sigma_1$ would have a fixed point); thus we may assume b = 4. It is clear that we may assume c = 5, hence d = 2 or 3 or we may assume d = 7. If d = 2, assume e = 6, hence f = 3 or we may assume f = 7, if f = 3, then we get a 6-cycle in $\sigma_0\sigma_1$, which is not allowed. So assume f = 7. Now 8 has to appear in one of the 2-cycle, so we may assume g = 8, hence h = 6 or 9, or we may assume h = 10, but it is easy to check that h ≠ 6, 9 and 10, hence f ≠ 7; thus d ≠ 2. Similarly one can check that d ≠ 3, hence we may assume d = 7.

Without loss of generality assume e = 8, and to get a 3-cycle in the product we must have f = 3. Now assume g = 2, hence h = 6 or 9 or we may assume h = 10. If h = 6, then we get the 2-cycle (2 4) in the product, which is not valid, and if h = 9, then we get the cycle (3 9) in the product $\sigma_0\sigma_1$, which is not allowed: thus assume h = 10. Now an easy checks shows that we must have (i j) = (6 12) and (k l) = (9 11) hence σ_1 = (1 4) (5 7) (3 8) (2 10) (6 12) (9 11), and $\sigma_0\sigma_1 - \sigma^{-1}_\infty$ = (1 5 8) (2 11 7 6 10 3 9 12 4).

Moreover, it is clear from our proof above that σ_0, σ_1 and σ_∞ generate a transitive subgroup of S_{12} and they are unique up to conjugation, hence the curve C is unique (up to isomorphism) and the J-map ($J: C \to \mathbb{P}^1$) exists and is unique.

Q.E.D.

To build a minimal elliptic surface $\pi : E \to C$, it is enough to build the J-map ($J: C \to \mathbb{P}^1$) associated to this surface, hence we have:

Corollary 2.4: There exist a unique (up to analytica isomorphism of surfaces) minimal elliptic surface ($\pi : E \to C$) with a section and exactly the pair (I_3, I_9) of singular fibers, where C is the unique genus 1 curve of Theorem 2.3 above.

Proof: This is immediate since the J-map exists, and since the permutations σ_0, σ_1 and s_∞ given in Theorem 2.3 are unique up to conjugation and hence the J-map is unique and so is the surface.

Next we construct our surface, the plan there is to write the J-map ($J: C \to \mathbb{P}^1$) as as composition of two maps $f: C \to \mathbb{P}^1$ and $J_1 : \mathbb{P}^1 \to \mathbb{P}^1$ such that degree (f) = 3 and degree (J_1) = 4. Notice that the existence of a degree 3 map $f: C \to \mathbb{P}^1$ is guaranteed, since every genus 1 curve C is trigonal (i.e., a triple cover of $\mathbb{P}^1$) in dimension 2 ways (see [5], page 1153), in fact f is obtained by projecting C ($\leq \mathbb{P}^2$) from a point q off C to a line.

Theorem 2.5: If $J: C \to \mathbb{P}_1$ is of degree 12 map of Theorem 2.3 and if $J = J_1$ of, where $J_1: \mathbb{P}_1 \to \mathbb{P}_1$ is a degree 4 map, and $f: C \to \mathbb{P}_1$ is a degree 3 map then f must be totally remified over 3 points of $\mathbb{P}_1$ and the ramification of J_1 just be as follows:

$|[J^{-1}_1(0)]| = (1, 3)$, $|[J^{-1}_1(1)]| = (2, 2)$ and $|[J^{-1}_1(\infty)]| = (1,3)$. Moreover the 3 points over which f is ramified are the two points whose J_1 -value is ∞, and the points whose J_1 -value is 0, and at which the multiplicity of J_1 is 1.

Proof: This is an easy consequence of Hurwitz's formula for the genus of a curve, and is required to get the right ramification of J. Q.E.D.

Theorem 2.6: Let C be the unique genus 1 curve of theorem 2.3 and lef $f: C \to \mathbb{P}_1$ be the degree 3 map described in Theorem 2.5, then the curve C must be the Fermat cubic (i.e.: C: $y^2z = X^3 + Z^3$) and $F: C \to \mathbb{P}_1$ is given by f ([x:y:z]) = [y+z:2z].

Proof: The map f is obtained by projecting C ($\leq \mathbb{P}^2$) form a point q off C to a line, therefore, the three points of $\mathbb{P}^1$ over which f is ramified give rise to a three flex points of C, and this gives rise to a three flex lines concurrent at q, hence proposition (9.7) of [5] implies that C must be the Fermant cubic (i.e., C: $Y^2Z = X^3 + Z^3$). Notice that the three flex points of such a curve are collinear, in fact they all lie on the line x = 0, and hence the three flex points of C are ∞ = [0:1:1], r_1

= [0:-1:1] and r_2 = [0:1:1].

To find the exact formula for f, let $F(x,y,z) = y^2z - x^3 - z^3$ and let T_α denote the tangent line at α, then T_∞ : z = 0, T_{r_1} : y = -2, and T_{r_2} : y = z, and clearly $T_\infty \cap T_{r_1} \cap T_{r_2} = 1 =$ [1:0:1]. Let g: C $\rightarrow \mathbb{P}_1$ be projection form q to the line x = 0, then g(∞) = [1:0], g(r_1) = [-1:1] and g(r_2) = [1:1]. Now let β: $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the change of coordinates given by $\beta(x) = \frac{x+1}{2}$, then β([1:01) = [1:0], β([-1, 1]) = [0:1] and β([1:1]) = [1:1]; thus if f = βog, then f is the desired triple cover.

Theorem 2.7: given a genus 1 curve C, which is triple cover of $\mathbb{P}^1$ totally ramified over 3 points of $\mathbb{P}^1$, then we can build the unique (up to analytic isomorphism) minimal elliptic surface π: E $\rightarrow$ C, with a section and exactly the pair (I_3, I_9) of singular fibers.

Proof: Consider the unique minimal elliptic surface X_{431} (see [7] pp. 546). This surface has exactly three singular fibers: a type IV^* -fiber over 0, and the pair (I_1, I_3) of singular fibers over ∞. Moreover, it has a Weierstrass equation given by:

$$Y^2 = X^3 + v^3 (24u - 27v) X + v^4 (16u^2 - 72uv + 54 v^2)$$

and the J-map ($J_1 : \mathbb{P}^1 \rightarrow \mathbb{P}^1$) associated to this fibration is given by:

$$J_1 (u, v) = v(24u - 27v)^3 / 64.27 u^3 (u - v).$$

It is clear that deg(J_1) = 4, and it must be ramified as follows:

1. $|[J_1^{-1} (0)]|$ = (1,3), since over 0 we have the IV^* -fiber and a smooth I_0 -fiber.
2. $|[J_1^{-1} (\infty)]|$ = (1,3), since over ∞ we have the pair (I_3, I_1) of singular fibers.
3. $|[J_1^{-1} (1)]|$ = (2, 2), since if $x \in J_1^{-1}(1)$, then 2 divides m_x (J_1), and this is necessary to get the right ramification of J_1 in Hurwitz's formula.

Let S_{01} and S_{03} be the two points of $\mathbb{P}^1$ whose J_1 -value is 0, let S_{12} and t_{12} be the two points of $\mathbb{P}^1$ whose J_1 -value is 1, and let $S_{\infty 1}$ and $S_{\infty 3}$ be the two points of $\mathbb{P}^1$ whose J_1 -value is ∞, where the second subcript is used to indicate the multiplicity of J1 at these points.

Let C be a genus 1 curve and let f: C $\rightarrow \mathbb{P}^1$ be a triple of cover of $\mathbb{P}^1$, which is totally ramified over S_{01}, $S_{\infty 1}$ and $S_{\infty 3}$ (change coordinates in $\mathbb{P}^1$ if necessary). Hence by Hurwitz's formula there is no other ramification of f, and locally f is given by f(z) = z_3 (i.e., f is just a base change of order 3).

Let E be the pull-back of the surface X_{431} via f, i.e.,

$$\pi : E = X_{431} \times_{\mathbb{P}^1} C \rightarrow C.$$

then clearly E is a minimal elliptic surface with a section and exactly the pair (I_3, I_9) of singular fibers. Moreover the J-map associated to this fibration is given by J = $J_1$of, and it is easy to check that J: C $\rightarrow \mathbb{P}^1$ is a degree 12 map ramified as given in Theorem 2.3. Hence this J-map is the desired unique J-map, and the surface π: E $\rightarrow$ C is the desired surface.

Q.E.D.

Next we give a final remark on this paper.

Remark 2.8: Another way to get our surface is to pull-back (via the J-map) the rational elliptic surface which has a Weierstrass equation

$$Y^2 = X^3 - 3t (t - 1)^3 X + 2 t (t - 1)^5.$$

This surface has J = t, and it has exactly three singular fibers: a fiber of type II over t = 0, a fiber of type III^* over t = 1, and a fiber of type I_1 over ∞ (see [6], page 203).

The resulting surface will be a minimal elliptic surface with the pair (I_3, I_9) of singular fibers and another 10 I_0^* -fibers (see [7], Table 7.1). Now using the process of deflating two *'s five times (see [6], page 203) we get the desired surface.

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