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INDECOMPOSABLE REPRESENTATIONS OF ORDER OF $\tilde{E}_6$

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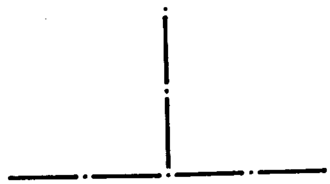
ABSTRACT

The extended Dynkin diagram

$\tilde{E}_6$:



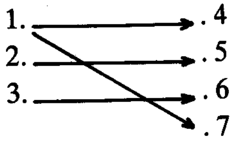
is a valued graph. We are going to construct a Baechstrom order A associateed to E_6. We prove, by constructions, that the order A of infinite lattice-type but can be listed (tame-type), i.e., we put all indecomposable A - lattices in finite number of general forms. Finally we give a method to obtain easily and directly the lattices from its associated representations.

1. Baechstrom order of $\tilde{E}_6$

Ringel and Roggenkamp have introduced for each basic Bachstrom order a valued graph (4).

In this section we construct an R-order A for $\tilde{E}_6$, where R is a complete valuation ring.

The orientation and the numerical of the vertices of the diagram $\tilde{E}_6$ are given as follows:



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Let its modulation M be given as follows,

${}_iS_j = F$ and $F_i = F_j = F$ ($F = R/\pi$ where π is the maximal ideal of R), $1 \leqslant i \leqslant 3, 4 \leqslant j \leqslant 7$.

We construct an R-order Γ, satisfying the conditions:

(i) M is hereditary and (ii) $\Gamma/\text{rad}\,\Gamma = \prod_{j=4}^{7} (F_j)_n$ as follows

$$\Gamma = \begin{bmatrix} R & R & R & R & R & R \\ \pi & R & R & R & R & R \\ \pi & \pi & R & R & R & R \\ \pi & \pi & \pi & R & R & R \\ \pi & \pi & \pi & R & R & R \\ \pi & \pi & \pi & R & R & R \end{bmatrix}$$

Then

$$\text{rad}\ \Gamma = \begin{bmatrix} \pi & R & R & R & R & R \\ \pi & \pi & R & R & R & R \\ \pi & \pi & \pi & R & R & R \\ \pi & \pi & \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi & \pi & \pi \end{bmatrix}$$

$$\text{and } \Gamma/\text{rad}\ \Gamma = \begin{bmatrix} F & O & O & O & O & O \\ O & F & O & O & O & O \\ O & O & F & O & O & O \\ O & O & O & F & F & F \\ O & O & O & F & F & F \\ O & O & O & F & F & F \end{bmatrix}$$

so the simple $\Gamma/\text{rad}\ \Gamma$-modules are:

$$S_4 = \begin{bmatrix} F \\ O \\ O \\ O \\ O \\ O \end{bmatrix}, S_5 = \begin{bmatrix} O \\ F \\ O \\ O \\ O \\ O \end{bmatrix}, S_6 \begin{bmatrix} O \\ O \\ F \\ O \\ O \\ O \end{bmatrix}, \text{and } S_7 = \begin{bmatrix} O \\ O \\ O \\ O \\ O \\ F \end{bmatrix}$$

Now we construct a Bächstrom order A of $\tilde{E}_6$, satisfying the conditions:

(i) $A \subset \Gamma$

(ii) $\Lambda/\text{rad}\,\Lambda = \prod_{i=1}^{3} F_i \quad , F_i = F$

(iii) $\text{rad}\,\Lambda = \text{rad}\,\Gamma$

(iv) ${}_iS_j = F_i \underset{\Lambda}{\otimes} S_j = F,\ 1 \leqslant i \leqslant 3,\ 4 \leqslant j \leqslant 7,$

as follows:

$$\Lambda = \begin{bmatrix} \alpha & R & R & R & R & R \\ \pi & \beta & R & R & R & R \\ \pi & \pi & \gamma & R & R & R \\ \pi & \pi & \pi & \alpha' & \pi & \pi \\ \pi & \pi & \pi & \pi & \beta' & \pi \\ \pi & \pi & \pi & \pi & \pi & \gamma' \end{bmatrix}$$

where $\alpha = \alpha' \pmod{\pi}$, $\beta = \beta' \pmod{\pi}$, and $\gamma = \gamma' \pmod{\pi}$.

2. The positive roots of $\tilde{E}_6$.

Let (G,d) be an extended Dynkin diagram, and let c be a Coxeter transformation of the vector space Q^G of all vectors $x = (x_i)_{i \in G}$ over the rational field Q. Then all positive roots of negative, positive and zero defect with respect to c are the vectors (see [1]):

(1) $x = c^{-r} P_{k_t}$, $O \leqslant r$, $1 \leqslant t \leqslant n$

(2) $x = c^{r} q_{kt}$, $O \leqslant r$, $1 \leqslant t \leqslant n$ and

(3) $x = x_o + r g\bar{n}$, $O \leqslant r$, $X_O \leqslant g\bar{n}$, $\partial_c x_o = o$,

where $\bar{n}$ is the canonic vector respectively.

In the case of $\tilde{E}_6$ we have

$c = s_1 s_2 \ldots s_7$ the Coxeter transformation,

$$\left.\begin{array}{l} C^+ = s^+_1 S^+_2 \ldots S^+_7 \\ C^- = S^-_7 S^-_6 \ldots S^-_1 \end{array}\right\} \text{The Coxeter functors,}$$

and

$$\left.\begin{array}{l} q_t = s_1 s_2 \ldots s_{t-1}\, T \\ \\ P_t = s_7 s_6 \ldots s_{t-1} T \end{array}\right\},\ 1 \leqslant t \leqslant 7\ ,$$

where T is the vector in Q^G defined by:

$T_t = 1$ and $T_i = o$ for all $i \neq t$.

The defect of $\tilde{E}_6$ with the given orientation has the following components:

$$d_c = 3 \begin{array}{l} \swarrow -2 \longrightarrow 1 \\ \leftarrow -2 \longrightarrow 1 \\ \nwarrow -2 \longrightarrow 1 \end{array}$$

2.1 The positive roots with negative defect:

These roots are $C^{+r}q_t$, $o \leqslant r$, $1 \leqslant t \leqslant 7$, we deduce the general forms as follows ($n \geqslant o$):

t = 1: there are three general forms:

$$3n \begin{array}{l} \diagup 2n+1 - n \\ - 2n - n \\ \diagdown 2n - n \end{array},\quad 3n+1 \begin{array}{l} \diagup 2n+1 - n+1 \\ - 2n+1 - n+1 \\ \diagdown 2n+1 - n+1 \end{array},\quad 3n+2 \begin{array}{l} \diagup 2n+1 - n \\ - 2n+1 - n+1 \\ \diagdown 2n+1 - n+1 \end{array}$$

t = 2: We obtain the roots by interchanging the edges (1-4) and (2-5) in the case (t = 1)

t = 3: Similarly by interchanging the edges (1 - 4) and (3-6) in the case (t = 1)

t = 4: There are six general forms:

$$3n+1 \begin{array}{l} \diagup 2n-2 - n \\ - 2n+1 - n \\ \diagdown 2n+1 - n \end{array},\quad 3n+2 \begin{array}{l} \diagup 2n+2 - n+1 \\ - 2n+1 - n \\ \diagdown 2n+1 - n \end{array},\quad 3n+3 \begin{array}{l} \diagup 2n+1 - n \\ - 2n+2 - n+1 \\ \diagdown 2n+1 - n \end{array}$$

$$3n \begin{array}{l} \diagup 2n+2 - n+1 \\ - 2n - n \\ \diagdown 2n+1 - n \end{array},\quad 3n+1 \begin{array}{l} \diagup 2n+1 - n \\ - 2n+1 - n+1 \\ \diagdown 2n+1 - n+1 \end{array},\quad 3n+2 \begin{array}{l} \diagup 2n+1 - n+1 \\ - 2n+2 - n+1 \\ \diagdown 2n+2 - n+1 \end{array}$$

t = 5: We obtain the roots by interchanging the edges (1-4) and (2-5) in the case (t = 4)

t = 6: Similarly by interchanging the edges (1-4) and (3-6) in the case (t = 4)

t = 7: There are two general forms:

$$3n+1 \left\langle \begin{array}{l} 2n+1 — n \\ 2n+1 — n \\ 2n+ \;\; — n \end{array}\right. , \quad 3n+2 \left\langle \begin{array}{l} 2n+2 — n+1 \\ 2n+2 — n+1 \\ 2n+2 — n+1 \end{array}\right.$$

2.2 The positive roots with positive defect:

These roots are $C^{-r}_{p_t}$, $o \leqslant r$, $1 \leqslant t \leqslant 7$.

We deduce the general forms as follows ($n \leqslant o$):

t = 1: There are three general forms

$$3n+1 \left\langle \begin{array}{l} 2n+1 — n+1 \\ 2n \;\; — n \\ 2n \;\; — n \end{array}\right. , \quad 3n+2 \left\langle \begin{array}{l} 2n+1 — n \\ 2n+1 — n+1, \\ 2n+1 — n+1 \end{array}\right. \quad 3n+3 \left\langle \begin{array}{l} 2n+1 — n+1 \\ 2n+2 — n+1 \\ 2n+2 — n+1 \end{array}\right.$$

t = 2: We obtain the roots by interchanging the edges (1-4) and (2-5) in the case (t = 1)

t = 3: Similarly by interchanging the edges (1-4) and (3-6) in the roots of the case (t = 1)

t = 4: There are six general forms:

$$3n \left\langle \begin{array}{l} 2n — n+1 \\ 2n — n. \\ 2n — n \end{array}\right. \quad 3n+1 \left\langle \begin{array}{l} 2n+1 — n \\ 2n \;\; — n, \\ 2n \;\; — n \end{array}\right. \quad 3n+1 \left\langle \begin{array}{l} 2n \;\; — n \\ 2n+1 — n+1 \\ 2n+1 — n+1 \end{array}\right.$$

$$3n+2 \left\langle \begin{array}{l} 2n+1 — n+1 \\ 2n+1 — n+1, \\ 2n+1 — n+1 \end{array}\right. \quad 3n+2 \left\langle \begin{array}{l} 2n+2 — n+1 \\ 2n+1 — n+1, \\ 2n+1 — n+1 \end{array}\right. \quad 3n+3 \left\langle \begin{array}{l} 2n+1 — n \\ 2n+2 — n+1 \\ 2n+2 — n+1 \end{array}\right.$$

t = 5: We obtain the roots by interchanging the edges (1-4) and (2-5) in the case (t = 4)

t = 6: Similarly by interchanging the edges (1-4) and (3-6) in the case (t = 4).

t = 7: There are two general forms

$$3n+1 \left\langle \begin{array}{l} 2n — n \\ 2n — n \\ 2n — n \end{array}\right. , \quad 3n+2 \left\langle \begin{array}{l} 2n+1 — n+1 \\ 2n+1 — n+1 \\ 2n+1 — n+1 \end{array}\right.$$

3. Construction of all indecomposable representations with non-zero defect of $\tilde{E}_6$.

These representations correspond the roots calculated in the previous sections, we use the following notations:

(i) FFF ... instead of the vector space F+F+F+ ...., for any number of F, where F = R/π. Also the vector of the representations is denoted by its dimensions, e.g. FFF : = 3.

(ii) The linear mappings of the representations are:

(a) 1: F → F , 11: F F → F F ... , ...
f → f (f$_1$, f$_2$) → (f$_1$, f$_2$)

(b) o: F → o or o → F , oo : FF → o or o → FF , ...

(c) 1 = 1:F → FF , 1 = 1 = 1: F → FFF , ...
f → (f,f) f → (f,f,f)

(d) +: F F → F , ++: F F F F → F F , ...
(f$_1$,f$_2$) → (f$_1$+f$_2$) (f$_1$,f$_2$,f$_3$,f$_4$) → (f$_1$+f$_2$, f$_3$+f$_4$)

Moreover, we may also combine the above notations, for example:

10: F F → F or F → F F
(f$_1$, f$_2$) → f$_1$ f$_1$ → (f$_1$, 0)

1+: F F F → F F , 101: F → F F F , and
(f$_1$,f$_2$,f$_3$) → (f$_1$,f$_2$+f$_3$) f → (f,o,f)

$(10)^n$: 10101010 ... 10 (10 is repeated n times), similalry

$(+)^n$ and the other $(...)^n$.

Since we have a one-to-one correspondence between all positive roots of non-zero defect and all indecomposable representations of non-zero defect, it is enough to give only the linear mappings

$j^{\Phi}i$, i = 1, 2, 3, j = 4, 5, 6, 7 of the general forms.

3.1 The indecomposable representations $C^{+}Q_t$ of $\tilde{E}_6$.

The general forms of these representations are:

t = 1: There are three general forms:

(a) $4^{\Phi}1 = o(10)^n$, $5^{\Phi}2 = (01)^n$, $6^{\Phi}3 = (+)^n$,

$$7\Phi 1 = \begin{cases} o & \text{for } n = o \\ 111 & \text{for } n = 1 \\ 111(011)^{n-1} & \text{for } n \geqslant 2 \end{cases}$$

$$7\Phi 2 = \begin{cases} o & \text{for } n = o \\ (-f_1, f_1+f_2, f_2 & \text{for } n = 1 \\ (-f_1, f_1+f_2, f_2, g_1, g_2, \ldots, g_i, \ldots, g_{n-1}) & \text{for } n \geqslant 2 \end{cases}$$

(note that we have defined the linear mapping with its value of $(f_1, \ldots, f_n)$ where $g_i = f_{2i} - f_{2i+1}$, f_{2i+1} , f_{2i+2} , f_{2i+2}, $i = 1, 2, \ldots, n-1$),

and

$$7\Phi 3 = \begin{cases} 0 & \text{for } n = 0 \\ 1\ 1 = 1 & \text{for } n = 1 \\ f_1, f_2, f_2, g'_1, g'_2, \ldots, g'_i, \ldots, g'_{n-1} & \text{for } n \geqslant 2 \end{cases}$$

where

$g'_i = f_{2i} + f_{2i+1}$, f_{2i+2} , f_{2i+2} , $i = 1, 2, \ldots, n-1$

(b) $4\Phi 1 = 1(10)^n$, $5\Phi 2 = 0(1)^n$, $6\Phi 3 = (+)^n O$,
$7\Phi 1 = 1(1 = 1\ \ 1)^n$, $7\Phi 2 = 1(110)^n$

$$7\Phi 3 = \begin{cases} 1 & \text{for } n = O \\ f_1, g''_1, g''_2, \ldots, g''_i, \ldots g''_n & \text{for } n \geqslant 1, \end{cases}$$

where $g''_i = f_{2i}$, $-f_{2i-1}$, $f_{2i-2} + f_{2i} + f_{2i-1}$, $i = 1, 2, \ldots, n$

(c) $4\Phi 1 = O(+)^n$, $5\Phi 2 = (+)^{n-1}$, $6\Phi 3 = (01)^{n+1}$

$$7\Phi 1 = \begin{cases} 01 & \text{for } n = 0 \\ 0, f_1, g'''_1, g'''_2, \ldots, g''_n & \text{for } n \geqslant 1 \end{cases}$$

where $g'''_i = o$, $f_{2i-2} + f_{2i}$, f_{2i+2} $i = 1, 2, \ldots, n$

$7\Phi 2 = 11(1 = 11)^n$ and

$$7\Phi 3 = \begin{cases} 11 & \text{for } n = 0 \\ f_1, f_2, g_1, \ldots, g_i, \ldots, g_n . & \text{for } n \geqslant 1 \end{cases}$$

where $g_i = f_{2i+1}$, f_{2i+1} , f_{2i+2} , $i = 1, 2, \ldots, n$.

$t = 2$, $t = 3$: by the same interchanging as in the roots.

t = 4 There are six general forms:

(a) $4\Phi 1 = (+1)^n$, $5\Phi 2 = O(+)^n$

$$6\Phi 3 = \begin{cases} O & \text{for } n = O \\ f_1+f_3 & \text{for } n = 1 \\ f_1+f_3,\ h_2,\ h_3,\ \ldots,\ h_i,\ldots,h_n & \text{for } n \geqslant 2 \end{cases}$$

where $h_i = f_{2i-2}+f_{2i+1}$, i = 2, 3, ... , n ,
$7\Phi 1 = 0(101)^n$, $7\Phi 2 = 1(110)^n$ and $7\Phi 3 = 1(011)^n$

(b) $4\Phi 1 = (+)^{n+1}$, $5\Phi 2 = 6\Phi 3$ in case (a) , $6\Phi 3 = O(+)^n$,
$7\Phi 1 = 11(101)^n$, $7\Phi 2 = 10(110)^n$ and $7\Phi 3 = 10(011)^n$

(c) $4\Phi 1 = O(+)^n o$, $5\Phi 2 = (+)n+^1$, $6\Phi 3 = (+)^{n+1}$,
$7\Phi 1 = (101)^{n+1}$, $7\Phi 2 = (110)^{n+1}$ and $7\Phi 3 = (011)^{n+1}$

(d) $4\Phi 1 = 1(01)^n$, $5\Phi 2 = (10)^n$, $6\Phi 3 = (01)^n$,
$7\Phi 1 = 111 = 111 = \ldots = 111$ (111 repeated n once),
$7\Phi 2 = (\ 1 = 11)^n$ and $7\Phi 3 = (11 = 1)^n$

(e) $4\Phi 1 = 0(10)^n$, $5\Phi 2 = 1(10)^n$, $6\Phi 3 = 1(10)^n$,
$7\Phi 1 = 1(\overline{111})^n$, $7\Phi 2 = 1(1 = 11)^n$, and
$$7\Phi 3 = \overline{1(\underbrace{111)(111}_{=})\ldots(\underbrace{111)(111}_{=})}_{n}$$

(f) $4\Phi 1 = 1(10)^n$, $5\Phi 2 = (10)^{n+1}$, $6\ \ 3 = (01)^{n+1}$,

$7\Phi 1 = 1 = 1(\overline{111})^n$, $7\Phi 2 = 11\underbrace{(\overline{111)(111})\ldots(\overline{111)(111})}_{n}$,

$7\Phi 3 = 11(11 = 1)^n$

t = 5, t = 6: by the same interchanging as in the roots.

t = 7: We have two general forms:

(1) $4\Phi 1 = (+)^{no}$,

$$5\Phi 2 = \begin{cases} o & \text{for } n = o \\ f_1+f_3 & \text{for } n = 1 \\ f_1+f_3,1_2,\ldots,1_i,\ldots,1_n & \text{for } n \geqslant 2 \end{cases}$$

where $l_i = f_{2i-2}+f_{2i-1}$, $i = 2, 3, \ldots, n$;
$6 \Phi 3 = 5\ 2$, $7 \Phi 1 = 1(101)^n$, $7 \Phi 2 = 1(110)^n$, and $7 \Phi 3 = 1(011)^n$

(b) $4 \Phi 1 = (10)^{n+1}$, $5 \Phi 2 = 6 \Phi 3 = (01)^{n+1}$, $7 \Phi 1 = \underbrace{11\ 111\ 111 \ldots 111}_{n}$,

$7 \Phi 2 = \underbrace{11\ 111\ 111 \ldots 111\ 111}_{n}$, $7 \Phi 3 = 1\ 1{+}1\ 111\ 111 \ldots 111$,

3.2 The indecomposable representations $\bar{C}Q_t$ of $\tilde{E}_6$.

The general forms of these representations are:

$t = 1$ We have the following three general forms:

(a) $4 \Phi 1 = 1(01)^n$, $5 \Phi 2 = (10)(01)^{n-1}$, $6 \Phi 3 = (10)^n$,
$7 \Phi 1 = 1101\ (1\ 1 = 1)^{n-1}$, $(n \neq o)$

$$7 \Phi 2 = \begin{cases} O & \text{for } n = o \\ f_1,o,f_2,f_1 & \text{for } n = 1 \\ f_1+f_4,f_4,f_2,f_1,o,f_3,f_4 & \text{for } n = 2 \\ f_1,f_4,f_4,f_2,f_1,m_1,\ldots,m_1\ldots,m_{n-2},o,f_{2n-1},f_{2n} & \text{for } n \geq 3 \end{cases}$$

where $m_i = -f_{2i+4}\ \ f_{2i+1}$, f_{2i+2}, $i = 1, 2, \ldots , n-2$,

$$7 \Phi 3 = \begin{cases} o & \text{for } n = o \\ 1 = 1\ 1 = 1 & \text{for } n = 1 \\ f_1,f_1,f_2+f_3,f_2,f_3,o,f_4 & \text{for } n = 2 \\ f_1,f_1,f_2+f_3+f_6,f_2,f_3,f_5,f_4,f_5,o,f_6 & \text{for } n = 3 \\ f_1,f_1,f_2+f_3+f_6,f_2,m_2,\ldots,m_i,\ldots,m_{n-2}, \\ f_{2n-3}, f_{2n-1}, f_{2n-2}, f_{2n-1}, o , f_{2n} & \text{for } n \geq 4 \end{cases}$$

where $m^1_i = f_{2i-1}$, $f_{2i+1}+f_{2i+4}$, f_{2i} , $i = 2, \ldots , n-2$

(b) $4 \Phi 1 = o(+)^n$,

$$5 \Phi 2 = \begin{cases} 1 & \text{for } n = o \\ f_1+f_3,f_2 & \text{for } n = 1 \\ f_1+f_3-f_5 , m_i'',\ldots,m''_{i-1}, f_{2n} & \text{for } n \geq 2 \end{cases}$$

,where $m''_i = -f_{2i} + f_{2i+3}$, 1, 2,...,n−1

$$6^{\Phi}3 = \begin{cases} 1 & \text{for } n = o,\ 7^{\Phi}1 = 01(011)^n, \\ +1\ (01)^{n-1} & \text{for } n \geq 1 \end{cases}$$

$$7^{\Phi}2 = \begin{cases} 10 & \text{for } n = o \\ 10101 & \text{for } n = 1 \\ f_1,o,f_2,f_4,f_3,f_4,o,f_5 & \text{for } n = 2 \\ f_1,o,f_2,f_4,f_3,f_2,m_2^*,\dots,m_1^*,\dots,m_{n-}^*1,\ f_{2n},o,f_{2n+1} & \text{for } n \geq 3 \end{cases}$$

where $m_i^* = f_{2i}+f_{2i}+_{2i+3}$, f_{2i-2}, f_{2i-1} , $i = 3, \dots , n-1$
and $7^{\Phi}3 = (1 = 1\ (110)^n)$

(c)

$$4^{\Phi}1 = 1(10)^n\ ,\ 5^{\Phi}2 = \begin{cases} 01 & \text{for } n = o \\ 010 & \text{for } n = 1 \\ 010(+)^{n-1}1 & \text{for } n \geq 2 \end{cases}$$

$$6^{\Phi}3 = (+)^{n-1}\ ,\ 7^{\Phi}1 = (1 = 1\ (101)^n) \quad ,$$

$$7^{\Phi}2 = \begin{cases} 011 & \text{for } n = o \\ o,f_1,f_2+f_3,\underline{m}_1,\dots,\underline{m}_1,\dots,\underline{m}_n & \text{for } n \geq 1 \end{cases}$$

where $\underline{m}_i = f_{2i-1}$, f_{2i-2} , o , $i = 1, 2,\dots, n$

and

$$7^{\Phi}3 = \begin{cases} 101 & \text{for } n = o \\ f_1,f_4,f_2,o,f_3,f_4 & \text{for } n = 1 \\ f_1,f_4,f_2,f_5+f_6,f_3,f_4,o,f_5,f_6 & \text{for } n = 2 \\ f_1,f_4,f_2,f_5+f_6,f_3,f_4,n_3,\dots,n_i,\dots,n_n,\ b,\ f_{2n+1},\ f_{2n+2} & \text{for } n \geq 3 \end{cases}$$

where $n_i = -(f_{2i+1}+f_{2i+2}),\ f_{2i-1}, f_{2i}$, $i = 3,\dots,n$.

t = 2, t = 3: by the same interchanging as in the roots.

t = 4: we have the following six general forms:

(a)

$$4^{\Phi}1 = \begin{cases} o & \text{for } n = o \\ 11 & \text{for } n = 1 \\ 11(+)^{n-1} & \text{for } n \geq 2 \end{cases}\ ,\ 5^{\Phi}2 = (+)^n$$

$$6^{\Phi}3 = \begin{cases} o & \text{for } n = o \\ + & \text{for } n = 1 \\ o_1,\dots,o_i,\dots,o_{n-1},f_1,f_{2n} & \text{for } n \geq 2 \end{cases}$$

where $o_i = f_{2i}+f_{2i+1}$, $i = 1,2,\ldots,n-1$,
$7\Phi 1 = (110)^n$, $7\Phi 2 = (011)^n$ and $7\Phi 3 = (101)^n$

(b) $4\Phi 1 = 0(01)^n$, $5\Phi 2 = (10)^n$, $6\Phi 3 = (10)^n$, $7\Phi 1 = 1(11 = 1)^n$,

$$7\Phi 2 = \begin{cases} o & \text{for } n = o \\ \overline{10\overset{=}{1}1} & \text{for } n = 1, \\ \underbrace{\overline{1011 = 111 = 111 = \overset{=}{\ldots} = 111 = 111}} & \text{for } n \geqslant 2 \end{cases}$$

,and $7\Phi 3 = (1 = 11)^n$.

(c)

$$4\Phi 1 = (+)^n \ , \ 5\Phi 2 = \begin{cases} 1 & \text{for } n = o \\ f_2,o'_1,\ldots,o'_i,\ldots,o'_{n-1},\ f_1+f_{2n-1} & \text{for } n \geqslant 1 \end{cases}$$

where $o'_i = f_{2i-1}+f_{2i-2}$, $i = 1, 2, \ldots , n-1$,
$6\Phi 3 = 1(+)^n$, $7\Phi 1 = 0(011)^n$, $7\Phi 2 = 1(101)^n$ and $7\Phi 3 = 1(110)^n$

(d) $4\Phi 1 = 1(10)^n$, $5\Phi 2 = 0(10)^n$, $6\Phi 3 = 0(10)^n$,

$$7\Phi 1 = \begin{cases} 1 = 1 & \text{for } n = o \\ 11 = 110 & \text{for } n = 1 \\ 11 = 110(110)^{n-1} & \text{for } n \geqslant 2 \end{cases} , \quad 7\Phi 2 = 01(1 = 11)^n$$

$$7\Phi 3 = \begin{cases} 10 \overset{=}{} & \text{for } n = o \\ \overline{10\ 11 = 1+1} & \text{for } n = 1 \\ \overline{10\ 11 = 1-1 = 11 = 1+1} & \text{for } n = 2 \\ \overline{10\ 11 = 1-1 = 11 = 1-1 = \ldots = 11 = 1-1 = 11 = 1-1} & \text{for } n \geqslant 3 \end{cases}$$

(e)

$$4\Phi 1 = \begin{cases} + & \text{for } n = o \\ 0(+)^n 1 & \text{for } n \geqslant 1 \end{cases} , \quad 5\Phi 2 = 1(+)^n \ , \ 6\Phi 3 = 1(+)^n,$$

$$7\Phi 1 = \begin{cases} 11 & \text{for } n = o \\ 11(101)^{n-1}101+1 & \text{for } n \geqslant 1 \end{cases} , \quad 7\Phi 2 = 10(110)^n,$$

and $7\Phi 3 = 01(110)^n$.

(f) $4\Phi 1 = O(10)^n$, $5\Phi 2 = (01)^{n+1}$, $6\Phi 3 = (10)^{n+1}$,
$7\Phi 1 = 001(1 = 11)^n$, $7\Phi 2 = (11 = 1)^{n+1}$.

and

$$7\Phi 3 = \begin{cases} \overset{=}{\overbrace{111}} & \text{for } n = o \\ \overset{=}{\overbrace{\underbrace{111 = 111 = 111 = \dots = 111 = 111}_{n+1}}} & \text{for } n \geq 1 \end{cases}$$

t = 5 , t = 6: By the same interchanging as in the roots.

t = 7 We have following two general forms:

(a) $4\Phi 1 = (+)^n$, $5\Phi 2 = (01)^n$, $6\Phi 3 = (10)^n$,

$$7\Phi 1 \begin{cases} o & \text{for } n = o \\ 1 = 1 = 1+\overset{=}{\overline{1-}}(1 = 1+\overset{=}{\overline{1\ -}}\ 1)^{n-1} & \text{for } n \geq 1 \end{cases}$$

$$7\Phi 2 \begin{cases} o & \text{for } n = o \\ 1 = 110\ (110)^{n-1} & \text{for } n \geq 1 \end{cases}$$

and

$$7\Phi 3 = \begin{cases} o & ao \\ \underbrace{001\overset{=}{\overbrace{1\ (01\ 1+1)\ (011+1)\ \dots\ (o1\ 1}}+1)}_{n-1} & \text{for } n \geq 1 \end{cases}$$

(b)

$$4\Phi 1 = 1(+)^n \text{ , } 5\Phi 2 = \begin{cases} 1 & \text{for } n = o \\ 1+ & \text{for } n = 1 \text{ ,} \\ 1(01)^n & \text{for } n \geq 2 \end{cases} \qquad 6\Phi 3 = 1(01)^n,$$

$$7\Phi 1 = 1 = 1(\overset{=}{\overbrace{101}}+1)^n \text{ ,} 7\Phi 2 = \begin{cases} 10 & \text{for } n = o \\ 10-1\ -1\ 1 & \text{for } n = 1 \\ 10(1 = 1 = 1+1)^n & \text{for } n \geq 2 \end{cases}$$

and

$$7\Phi 3 = 0\overset{=}{\overbrace{1\ \underbrace{(011+1)(011+1)\ ..\ (011}_{n}}}+1)$$

4. The regular representations of $\tilde{E}_6$:

The regular representations of $\tilde{E}_6$ include the homogeneous and the nonhomogeneous regular representations. Therefore we give first the simple regular representations and then the indecomposable regular representations.

4.1: The simple regular representations of $\tilde{E}_6$:

For E_6 we have the following eight simple regular representations:

$$E_o = 1 \leftarrow \begin{array}{l} 1 \to 1 \\ 1 \to 1 \\ 1 \to 1 \end{array}, \quad E_1 = 2 \leftarrow \begin{array}{l} 1 \to o \\ 1 \to o \\ 1 \to o \end{array},$$

$$E'_o = 1 \leftarrow \begin{array}{l} o \to O \\ 1 \to 1 \\ 1 \to o \end{array}, \quad E'_1 = 1 \leftarrow \begin{array}{l} 1 \to o \\ O \to o \\ 1 \to 1 \end{array}, \quad E'_2 = 1 \leftarrow \begin{array}{l} 1 \to 1 \\ 1 \to O \\ o \to o \end{array},$$

$$E''_o = 1 \leftarrow \begin{array}{l} o \to o \\ 1 \to o \\ 1 \to 1 \end{array}, \quad E''_1 = 1 \leftarrow \begin{array}{l} 1 \to o, \\ 1 \to 1 \\ o \to o \end{array}, \quad E''_2 = 1 \leftarrow \begin{array}{l} 1 \to 1 \\ o \to o \\ o \to o \end{array}$$

4.2: The indecomposable regular homogeneous representations of $\tilde{E}_6$.

We construct these representations for $n \geq 2$, they can be summarized in the following two cases:

Case 1: n is odd

$$\Phi_1 = (F \oplus F \oplus \dots \oplus F) \otimes F \to F \oplus F \oplus F \oplus \dots \oplus F \oplus F$$
$$(f_1, f_2, \dots, f_n)\, xm_1 \to (f_1+f_2), f_3, f_4, \dots, f_n, f_1)\, m_1$$

$$\Phi_2 = (F \oplus F \oplus \dots \oplus F) \otimes F \to F \oplus F \oplus F \oplus \dots \oplus F \oplus F$$
$$(f_1, f_2, \dots, f_n)\, xm_2 \to (f_1, f_2, f_3, \dots, f_{n-1}, f_n) m_2$$

$$(C_1)_n = \{((f'_1+f'_2), f'_3, f'_4, \dots, f'_n, f'_1), (f'_1, f'_2, \dots, f'_n), (\overbrace{o,o,\dots,o}^{n}) \;|(f'_1, \dots, f'_n) \quad (F)^n\}$$

$$(C_2)_n = \{(f_1, f_2, \dots, f_n), (f_1, f_2, \dots, f_n), (f_1, f_2, \dots, f_n) \;|(f_1, f_2, \dots, f_3 n) \in F^n\} + (C_1)_n$$

$(C_i)_n = C_i$ $(i = 1, 2)$ in the case $\dim U = \dim V = n$

Case 2: n is even.

$$\Phi_1 : (F \oplus F \oplus \dots \oplus F) \otimes F \to F \oplus F \oplus F \oplus F \oplus F \oplus \dots \oplus F \oplus F$$
$$(f_1, f_2, \dots, f_n)\, xm_1 \to (f_1+f_2, o, f_3, +f_4, o, f_5+f_6, \dots, f_{n-1}, f_n, o) m_1$$

$$\Phi_2 : (F \oplus F \oplus \ldots \oplus F) \otimes F \rightarrow F \oplus F \oplus F \oplus F \oplus F \oplus \ldots F \oplus F \oplus F$$
$$(f_1, f_2, \ldots, f_n) \times n \rightarrow (o, f_1, +f_2, o, f_3+f_4, o, \ldots, f_{n-3}+f_{n-2}, o, (f_1+f_2+\ldots+f_n))m_2$$

$$C_1 = \{((f'_1+f'_2, o, f'_3+f'_4, o, \ldots, f'_{n-1}+f'_n, o), (f'_1, f'_2, \ldots, f'_n), \overbrace{(0, \ldots, 0)}^{n} \mid (f'_1, \ldots, f'_n) \in F^n\}$$

$$C_2 = \{(o, f_1+f_2, o, f_3+f_4, o, \ldots, f_{n-3}+f_{n-4}, o, f_1+f_2+\ldots+f_n), (f_1, f_2, \ldots, f_n), (f_1, f_2, \ldots, f_n) \mid (f_1, \ldots, f_n) \in F^n\} + C_1$$

5. A method of constructing the Λ - lattices:

One can construct at once the Λ - lattices, where A is the Baechstrom order of $\tilde{E}_6$. Using the following method:

Let $x = (x_1, x_2, \ldots, x_7, j \Phi i, i = 1, 2, 3, j = 4, 5, 6, 7)$ be a representations of $\tilde{E}_6$, and let

dim x = (dim x_i) = (n_i) , i = 1, 2, ..., 7.

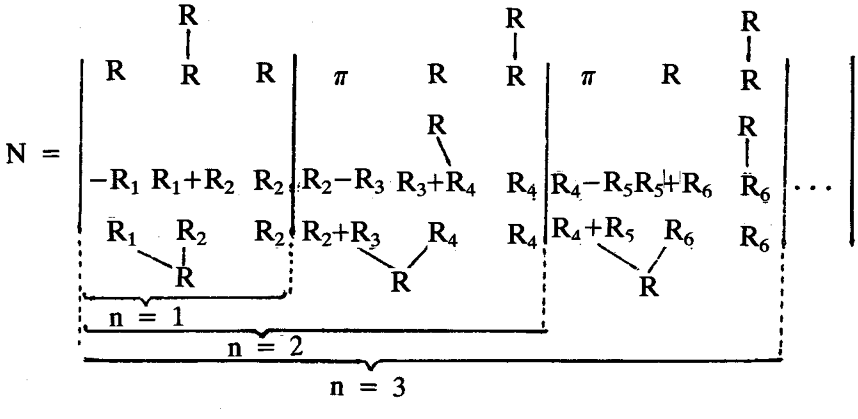
Then the A - lattice M, which corresponds to x has the following form:

$$M = \begin{bmatrix} R^{n4} & R^{n5} & R^{n6} & \overbrace{R \ldots R}^{n7} \\ \pi^{n4} & R^{n5} & R^{n6} & R \ldots R \\ \pi^{n4} & \pi^{n5} & R^{n6} & R \ldots R \\ \pi^{n4} & \pi^{n5} & \pi^{n6} & \\ \pi^{n4} & \pi^{n5} & \pi^{n6} & \\ \pi^{n4} & \pi^{n5} & \pi^{n6} & N \\ \pi^{n4} & \pi^{n5} & \pi^{n6} & \end{bmatrix}$$

where N is the $3 \times n_7$ - matrix (Im $7 \Phi 1$, Im $7 \Phi 2$, Im $7 \Phi 3)^T$. It is clear that R^{n4} and Im $7 \Phi 1$ are related by $4 \Phi 1$, R^{n5} and Im $7 \Phi 2$ are related by $5 \Phi 2$, and R^{n6} and Im $7 \Phi 3$ are related by $6 \Phi 3$.

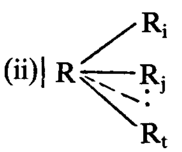
Some examples of Λ - lattices: It is enough to give for each Λ - lattice the block N and the relations indicated above.

(1) The following Λ - lattices are the lattices, which are correspond to the representations included in the general form (a) in 3.1 (t = 1), i.e. the representations $C^{+3}Q_1$, $C^{+6}Q_1$, ... See also the general form (a) of roots in 2.1 (t = 1).



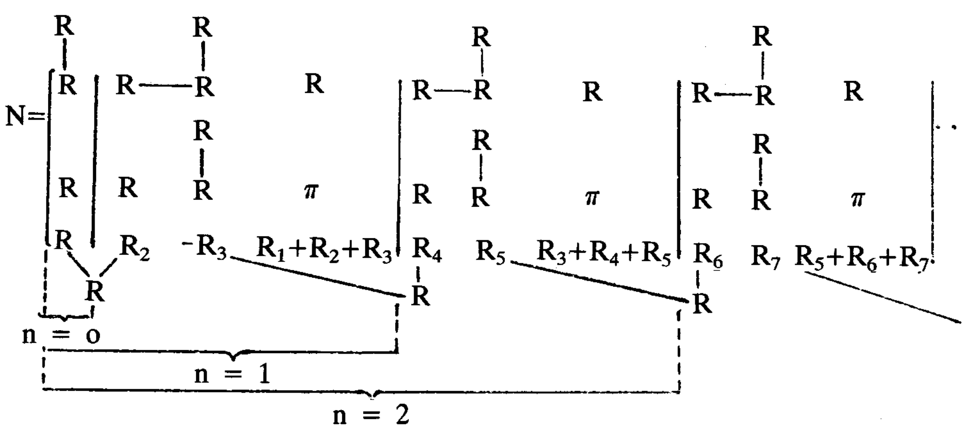
Note that we have used the following notations:

(i) R —— R_i means $r \times r_i$ (π) for all $r \in R$, $r_i \in R_i$

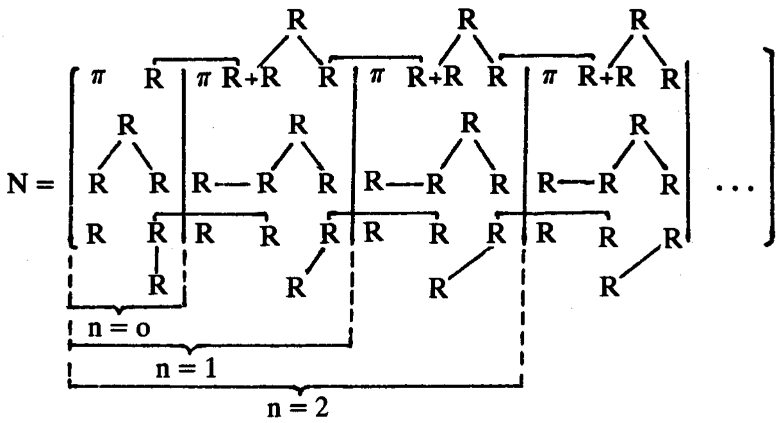
(ii)  means $r = (r_i r_j \ldots + r_t)$ (π) for all $r \in R$ and $r_s \in R_s$, $s = i, \ldots, t$

(iii) $R_i = R$ for all $r = 1, 2, \ldots$ and the R'^s with the same index means there exists the relation $=$ (π) between the elements in R'^s.

(2) The following Λ-lattices are the lattices, which correspond to the representations included in the general form (b) in 3.1 ($t = 1$), i.e. the representations $C^{+}Q_1$, $C^{+4}Q_1$, ... See also the general form (b) of roots in 2.1 ($t = 1$)

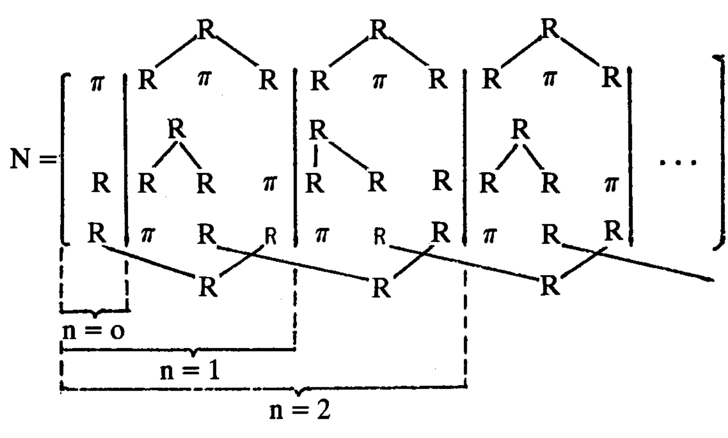


(3) $C^{+2}Q_1$, $C^{+5}Q_1$, ... are:

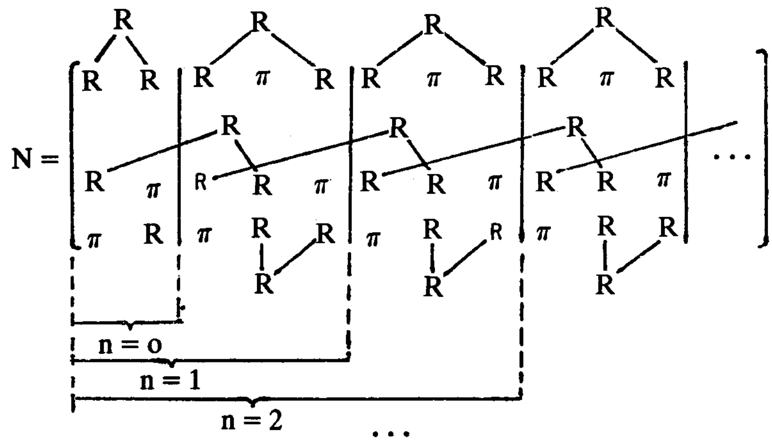


The following Λ - lattices are the lattices, which correspnd to the representations included in the general forms (a), (b) and (c) in 3.1 (t = 4) see also (a), (b), (c) in 2.1 (t = 4)), i.e. the representations $C^{+}Q_4$, $C^{+7}Q_4$, ..., $C^{+3}Q_4$, $C^{+4}Q_4$, ... and $C^{+5}Q_4$, $C^{+11}Q_4$, ...

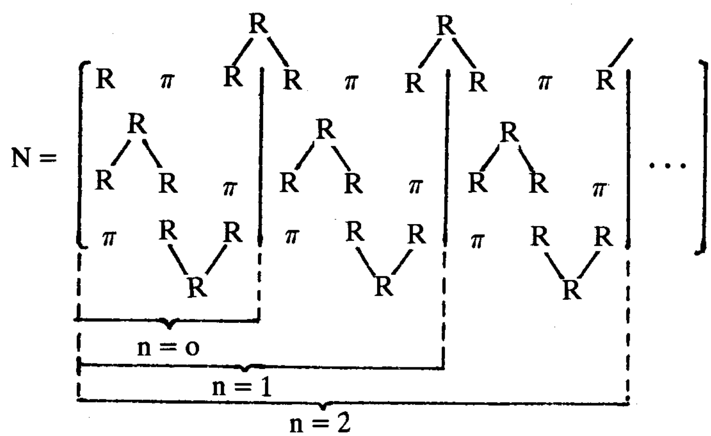
(4)



(5)



(6)



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