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MORPHISMS OF AFFINE SCHEMES AND EQUIVALENCE OF CATEGORIES

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تحولات الخطط وتناظر الطبقات

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في هذا البحث نكون العلاقة بين الطبقة R-filt (G(R)-gr) والطبقة $\underline{O}_X^{\mu}$-filt ($\underline{O}_X^{g}$ -Gr) النسخه الكميه من هذه النتيجة أمكن الحصول عليها بالتقييد إلى الأجزاء ذات الدرجة الصفرية . ونلخص الخواص الهندسية للخطه $Spec^g$ G(R) وفي النهاية باللغه الرسميه للخطط ، نحصل على العلاقة بين R-filt (G(R)-gr) والطبقه $\underline{O}_X^{\mu}$-filt ($\underline{O}_X^{g}$ -Gr) . وهذه النتائج لها معنى في طبقات المؤثرات التفاضلية .

Key words: Micro-Structure Sheaves, Zariski Filtered (rings) Sheaves, Quantum Sheaves, Affine Schemes and Morphisms

ABSTRACT:

In this work we construct the relation between the category R-filt (G(R)-gr) and the category $\underline{O}_X^{\mu} - \mathrm{Filt}(\underline{O}_X^{g} - \mathrm{Gr})$. The quantum (ungraded) version of this result can be obtained by restricting to parts of zero degree. Geometrically, we summarize geometrical properties for the affine scheme $spec^g$ G(R). Finally, in the language of formal affine schemes we obtain the relation between R-filt (G(R)-gr) and the category $\underline{O}_X^{\mu} - \mathrm{Filt}(\underline{O}_X^{g} - \mathrm{Gr})$. The results have a special meaning for sheaves of rings of differential operators.

INTRODUCTION

For a long time after its introduction by Jerary, sheaf theory was mainly applied successfully to the theory of functions of several complex variables and to algebraic geometry, until it became a basic tool for almost all mathematics, and cohomology a natural language for many people.

In recent years new topological methods, especially the theory of sheaves founded by Jerary, have been applied successfully to algebraic geometry. In the study of micro-structure sheaves $\underline{O}_X^{\mu}$ of filtered rings with Zariskian filtration, the author observed that, in order to find the regular and holonomic solutions of $\underline{O}_X^{\mu}$ (we come back to this later), it is necessary first to study the application in section 3. Using the filtered micro-structure sheaves $\underline{O}_X^{\mu}$ and its associated graded sheaves $G(\underline{O}_X^{\mu})$, [1], [9], it is possible to introduce the relation (commutative and non-commutative cases are considered) between the category of filtered (graded) affine schemes $(X, \underline{O}_X^{\mu})((X, G(\underline{O}_X^{\mu}))^g)$ with filtered geometric space morphism and the category of Zariski filtered rings with filtered ring morphisms. Geometrically, we summarize geometrical properties for $X = \mathrm{Spec}^g\, G(R)$ in section 2. Finally, in section 4, in the language of formal affine schemes we obtain the relation between R-filt (G(R)-gr) and the category $\underline{O}^{\mu}_X - \mathrm{Filt}(\underline{O}^{g}_X - \mathrm{Gr})$.

We can obtain the ungraded and quantum versions of our results, in sections 3 and 4, by restricting to the parts of zero degree. So far, this is the micro, formal and quantum versions of that application introduced in [5] as commutative case or that in [8] as non-commutative case.

I- PRELIMINARIES

We recall some basic notions here but for full details we have to refer to the references. Basic facts concerning schemes, affine schemes, coherent sheaves and formal completion may be found in [5]. Full detail on Zariskian filtration, filtrations and gradations on modules and micro-localizations of filtered modules may be found in [6], [7], [4]. For micro-structure sheaves $\underline{O}^{\mu}_X$ and quantum sections $F_0\underline{O}^{\mu}_X$ over projective (affine) schemes, coherent sheaves over $\underline{O}^{\mu}_X$ and formal schemes and quantum sections over projective (affine) schemes one may use [9], [1], [2]. Finally on Zariskian filtrations on sheaves we have to refer to [3].

In the sequel we assume that FR is a Zariski filtration such that the associated graded ring

$$G(R) = \bigoplus_{n\in Z}(F_nR / F_{n-1}R) \cong \tilde{R}/X\tilde{R}$$

is a commutative (Noetherian because of the Zariski hypothesis) strongly graded domain, this situation is general in the sense that it allows application of the results to most of the important examples; enveloping Algebras of Lie Algebras, Weyl Algebras, many rings of differential operators as well as the classical commutative Zariski rings.

We consider $X = \mathrm{Spec}^g\,(G(R))$, the graded prime spectrum of G(R). Write β for the basis of the Zariski topology on X consisting of the basic open sets $X(f)=\{p\in X,\ f\notin p\}$; $f\in G(R)$ homogeneous element. We may define structure sheaves on X. First we may associate to X(f), the graded commutative Noetherian ring $Q^g_f(G(R)) = G(R)[f^{-1}]$ and we obtain the graded Noetherian structure sheaf $\underline{O}^g_{G(R)}$ on X having as the stalk at $p\in X$ the Noetherian local graded ring $Q^g_f(G(R))$. Secondly we may associate to X(f) the Noetherian commutative ring $G(R)[f^{-1}]_0$ and we obtain the usual structure sheaf $\underline{O}_{G(R)}$ on X having as the stalk at $p\in X$ the Noetherian local ring $Q^g_p(G(R))_0$. Associating to $X(f)\in\beta$ the micro-localizations, $Q^{\mu}_{\tilde{f}}(\tilde{R})$, resp. $Q^{\mu}_f(R)$ we obtain sheaves $\tilde{\underline{O}}^{\mu}_X$, resp. $\underline{O}^{\mu}_X$, on X, having as the completed stalks at $p\varepsilon X$ the Noetherian rings $\tilde{Q}^{\mu}_p(\tilde{R})$, resp $Q^{\mu}_p(R)$, . The quantization of the micro-level for G(R) is obtained by looking at the parts of filtration degree zero $F_0\underline{O}^{\mu}_X$ This may also be viewed as parts of degree zero in the graded sense for the corresponding Rees sheaves $F_0\underline{O}^{\mu}_X \cong (\tilde{\underline{O}}^{\mu}_X)_0$.

1. 1 Theorem: With the above notations, [1], [3]

1 - $\underline{O}^{\mu}_X, F_0\underline{O}^{\mu}_X$ are Zariski coherently filtered sheaves on X.

2 - $\tilde{\underline{O}}^{\mu}_X$ is Noetherian graded sheaf on X.

II-AFFINE GRADED-SCHEMES FOR G (R)

In this section, we summarize properties for X $\mathrm{Spec}^g G(R)$. In general, obviously $X=\mathrm{Spec}^g(G(R))$ is not a scheme. However in case G(R) is positively graded, then we write P(X) = Proj(G(R)) for the Zariski open subset of X consisting of the graded prime ideals not containing $G(R)_+ = \bigoplus_{n>0}(G(R))_n$, and in this case the closed set $V(G(R)_+)$ in X is nothing but Spec $G(R)_0$. We may define the P(X) = Proj(G(R)) generally as the subset in X given by the graded prime ideals p of G(R) such that there is an $n \neq 0$, $G(R)_n \not\subset p$. Hence we have.

2.1 Theorem: $X = p(X) \Leftrightarrow G(R)$ is quasi-strongly graded.

So, there is no sensitivity in studying many other problems like coherence and theorems of formal completions, ... Only we have to assume the condition that G(R) is quasi-strongly graded as we will see in section 4.

Let R , R´ be non-commutative Zariski filtered rings and ϕ : $R \to R'$ be as strict epihomomorphism. Then ϕ induces a continuous map

$${}^{\alpha}G(\phi): (\mathrm{Spec}^g \circ G)(R') \to (\mathrm{Spec}^g \circ G)(R)$$

which is

$${}^{\alpha}G(\phi): (\mathrm{Spec}^g\,(G(R')) \to \mathrm{Spec}^g\,(G(R))$$

defined by: $q \to G(\phi)^{-1}(q)$, where $G(\phi)$: $G(R) \to G(R')$. Moreover, if R" is another non-commutative Zariski filtered ring and ϕ' : $R' \to R''$ is another strict epihomomorphisms, then

$${}^{\alpha}(G(\phi') \circ G(\phi)) = {}^{\alpha}G(\phi) \circ {}^{\alpha}(G(\phi')).$$

From this it follows that $\mathrm{Spec}^g \circ G$ is a contravariant graded functor from the category of non-commutative Zariski filtered rings and strict epihomomorphisms to the category of topological spaces and continuous maps.

Also, we can prove the following properties.

2.2. Theorem: With the same considerations as above,

$Spec^g G(R)$ is an affine graded-scheme and has a basis for the Zariski topology of affine Noetherian basic open sets.

2.3 Theorem: $X = Spec^g G(R)$ is compact and integral such that at each point $p \in X$, the coefficient for $\tilde{\underline{O}}^{\mu}_X$ is just the union of sections for $\tilde{\underline{O}}^{\mu}_{X'}$.

2.4 Theorem: with notations and considerations as above,

a-Since $Spec^g G(R)$ is path-connected then $Spec^g G(R)$ is not disconnected.

b- $Spe^g G(R)$ is irreducible and Noetherian.

2.5 *Remark:* On X, the graded sheaf $G(\underline{O}^{\mu}_X)$ is nothing but the graded structure sheaf $\underline{O}^g_X$, since, for each $X(f)\varepsilon\beta$ we have

$$G(\underline{O}^{\mu}_X)(X(f)) = G(\underline{O}^{\mu}_X(X(f))) = G(Q^{\mu}_{S_f}(R)) = Q^g_f(G(R)) = G(R)[f^{-1}]$$

and for each $p \in X$, we have

$$G(\underline{O}^{\mu}_X)_p = G(\underline{O}^{\mu}_{X,p}) = Q^{\mu}_p(G(R)).$$

Hence, as just terminology, we call $(Spec^g G(R), G(\underline{O}^{\mu}_X))$ as affine graded scheme of G(R). Now a graded-scheme needs not be a scheme, but this is just terminology of course.

III- MORPHISMS OF AFFINE SCHEMES

Let R′, R be non-commutative Zariski filtered rings and $\phi : R' \to R$ be strict filtered epihomomorphism.

Let $f \in h(G(R'))$, $p \in X = Spec^g G(R)$ such that ${}^{\alpha}G(\phi)(p) = q \in Y = Spec^g G(R')$. By the exactness of the localization functors $G(\phi)$ induces a graded ring epihomomorphism $G(R')_f \to G(R)_{G(\phi)(f)}$.

Thus, $G(\phi)$ induces a graded ring epihomomorphism $\underline{O}^g_{G(R')}(Y(f)) \to \underline{O}^g_{G(R)}(X(G(\phi)(f)))$ which already is compatible with the restriction graded homomorphism, hence we have a graded sheaf epimorphism $G(\phi)^{\sim} : \underline{O}^g_{G(R')} \to \underline{O}^g_{G(R)}$ as required.

Again, by the exactness of the localization functors associated to p, ${}^{\alpha}G(\phi)(p)=q$, $G(\phi)$ induces a graded local ring epihomomorphism of the stalks $G(\phi)^{\sim}_p : \underline{O}^g_{G(R'),q} \to \underline{O}^g_{G(R),p}$.

Therefore ϕ: $R' \to R$ induces a graded morphism

$$({}^{\alpha}G(\phi)G(\phi)^{\sim}) : (X, \underline{O}^g_{G(R)}) \to (Y, \underline{O}^g_{G(R')})$$

of graded affine schemes such that the graded homomorphisms induced on the stalks are graded local homomorphisms.

Conversely, let $(\psi,\theta)^g : (X, \underline{O}_X)^g \to (Y, \underline{O}_Y)^g$ be an epimorphism (where $X = Spec^g G(R)$, $Y = Spec^g G(R')$ and $\underline{O}_X$, $\underline{O}_Y$ are the graded structure sheaves $\underline{O}^g_X$, $\underline{O}^g_Y$) such that $\theta^g_p : G(R')_q \to G(R)_p$ is a graded local epihomomorphism for each $p \in X$ ($q=\psi(p)$). $(\psi,\theta)^g$ determines a graded ring epimorphism $G(\phi) : G(R') \to G(R)$ (and then one may define a filtered epimorphism $\phi_1 : R' \to R$ such that $G(\phi_1) = G(\phi)$. Since the induced embedding $\xi^g_p : Q_q \to Q_p$ of the residue fields is monomorphism, hence

$$\psi = {}^{\alpha}G(\phi) \text{ and } \theta^g_p : G(R')_q \to G(R)_p$$

is graded epimorphism induced by ϕ. Therefore $(\psi,\theta) = ({}^{\alpha}G(\phi), G(\phi)^{\sim})$. We have therefore proved:

3.1 Theorem: with conventions and notations as before, there is a map from the strict non-commutative Zariski filtered ring epihomomorphism ϕ: $R' \to R$ to the graded epimorphisms $(\psi,\theta)^g : (X, \underline{O}_X) \to (Y, \underline{O}_Y)$ such that $\underline{O}^g_p : \underline{O}_{Y,q} \to \underline{O}_{X,p}$, $p \in X$, $q = \psi(p)$, is local graded ring epihomomorphism.

3.2 Theorem: With conventions and notations as before, there is a one-to-one correspondence between the graded epimorphisms $G(\phi) : G(R') \to G(R)$ (where R′, R are non-commutative Zariski filtered rings) and the graded epimorphisms $(\psi,\theta)^g : (X, \underline{O}_X) \to (Y, \underline{O}_Y)$ such that θ^g_p is a local epimorphism for each $p \in X$.

Now let us return to filtered setting. Let $I \subset G(R')$ be a graded ideal in G(R′), Y(I) be an affine basic Zariski open set in $Y = Spec^g G(R')$. Since ${}^{\alpha}G(\phi)$ is a continuous map, hence ${}^{\alpha}G(\phi)^{-1}(Y(I)) = X(G(\phi)(I))$ is an affine basic Zariski open set in $X = Spec^g G(R)$. We write $\bar{K}_I(n)$ for the kernel functor, induced by $\bar{K}_I$, on $\tilde{R}'/X^n\tilde{R}'$ and $\tilde{K}(n)_{G(\phi)(I)}$ for the kernel functor, induced by $\bar{K}_{G(\phi)(I)'}$, on $\tilde{R}/X^n\tilde{R}$. By the saturation condition, $L(\tilde{K}_I(n))[L(\tilde{K}(n)_{G(\phi)(I)})]$ has a filter basis consisting of $\tilde{L}/X^n\tilde{L}$ of $\tilde{R}'/X^n\tilde{R}'$ with $\tilde{L} \in L(\bar{K}_I$ and $[\tilde{J}/X^n\tilde{J}$ of $\tilde{R}/X^n\tilde{R}$ with $\tilde{J} \in L(\bar{K}_{G(\phi)(I)})]$.

Write $Q^g_{n,I}[Q^g_{n,G(\phi)(I)}]$ for the localization functor associated to $\bar{K}_I(n)[\tilde{K}(n)_{G(\phi)(I)}]$. Since $\bar{K}_I(n)[\tilde{K}(n)_{G(\phi)(I)}]$ is perfect, hence ϕ induces the graded homomorphism.

$$Q^g_{n,I}(\tilde{R}'/X^n\tilde{R}') \to Q^g_{n,G(\phi)(I)}(\tilde{R}/X^n\tilde{R})$$

Taking the inverse limit in the graded sense, we have the graded ring homomorphism

$$Q^{\mu}_{\tilde{K}_I}(\tilde{R}') \overset{\tilde{\theta}^{\mu}_I}{\rightarrow} Q^{\mu}_{\tilde{K}_{G(\phi)(I)}}(\tilde{R})$$

Since $\tilde{\theta}^{\mu}_I(X_{Q^{\mu}_{\tilde{K}_I}(\tilde{R}')} Q^{\mu}_{\tilde{K}_I}(\tilde{R}')) \subset (X_{Q^{\mu}_{\tilde{K}_I}(\tilde{R})} Q^{\mu}_{\tilde{K}_I}(\tilde{R}))$

hence $\tilde{\theta}^{\mu}_I$ induces a filtered Zariski ring homomorphism $Q^{\mu}_{K_I}(R') \overset{\theta^{\mu}_I}{\rightarrow} Q^{\mu}_{K_{(\phi)(I)}}$.

Therefore, we have the filtered ring epihomomorphisms $\Gamma(Y(I), \underline{\theta}^{\mu}_Y) \overset{\theta^{\mu}_{Y(I)}}{\rightarrow} \Gamma(X(G(\phi)(I)), \underline{\theta}^{\mu}_Y)$, which are compatible with the restriction homomorphisms.

Now, $\theta^{\mu}_{Y(I)}; Y(I) \subset Y = \mathrm{Spec}^g G(R')$, define $\underline{\theta}^{\mu}_Y \overset{\theta^{\mu}}{\rightarrow} \underline{\theta}^{\mu}_X$ as a filtered sheaf morphism. For every prime ideal $p \in X$, $q = {}^{\alpha}G(\phi)(p) \in Y$ and since the local property exists in case of associated graded functor, hence the stalk sheaf morphism $\theta^{\mu}_p : \underline{\theta}^{\mu}_{Y,q} \rightarrow \underline{\theta}^{\mu}_{X,p}$ is filtered local homomorphism. All this leads to the following:

3.3 Theorem: There is a map from the strict non-commutative Zariski filtered ring epihomomorphisms $\phi: R' \rightarrow R$ to the filtered morphisms

$$({}^{\alpha}G(\phi), \theta)^{\mu} : (X, \underline{O}^{\mu}_X) \rightarrow (Y, \underline{O}^{\mu}_Y)$$

such that θ^{μ}_p is filtered local ring homomorphism for each $p \in X$.

3. 4 *Remark*

1 - Quantum affine schemes and quantum morphisms follow easily from this by restricting to parts of degree zero.

2 - The condition of commutativity of the associated graded rings is necessary for our study here in this note.

IV-MORPHISMS OF FORMAL SCHEMES

In this section, we will mainly be concerned with non-commutative formal schemes. On the formal level we study and prove the relation between R-filt (G(R)-gr) and $\underline{O}^{\mu}_{\hat{X}}$ - Filt ($\underline{O}^{\mu}_{\hat{X}}$ - Gr).

We may work over X = $\mathrm{Spec}^g G(R)$ or P(X). Consider a coherently filtered sheaf of ideals $\underline{I}_X$ in $\underline{O}^{\mu}_X$ (here of course X means X or P(X) unless we mention otherwise), then $G(\underline{I}_X)$ is a coherent subsheaf of graded ideals of $\underline{O}^g_X$. Let P(X)' be the closed subset of P(X) defined by $G(\underline{I}_{P(X)})$; $P(X)' = V(I) \subset P(X)$. Now we can defined two formal schemes. First, the commutative graded formal scheme $(P(\hat{X}), \underline{O}^g_{P(\hat{X})})$ where $P(\hat{X}) = P(X)'$ and

$$\underline{O}^g_{P(\hat{X})} \cong \underset{\underset{n}{\leftarrow}}{\mathrm{Lim}^g} (\underline{O}^g_{P(X)} / (G(\underline{I}_{P(X)}))^n) = \underset{\underset{n}{\leftarrow}}{\mathrm{Lim}^g} \underline{(G(R)/G(I)^n)}^g_{X'}$$

where Lim^g_n is calculated section wise as the graded limit in the category of graded modules. Second, the Noetherian formal scheme $(P(\hat{X}), \underline{O}_{P(\hat{X})})$ where (here in our case) $\underline{O}_{P(\bar{X})} \cong \underset{\underset{n}{\leftarrow}}{\mathrm{Lim}} \ \underline{O}_{P(X)} / (G(\underline{I}_{P(X)})_0)^n$.

For convenience, we write

$P(X) = Y$, $P(\hat{X}) = \hat{Y}$, $G(\underline{I}_Y)^g_{\hat{Y}} = \underline{J}^g_Y$ and $G(\underline{I}_Y)_O = \underline{J}_Y$.

Let $M \in G(R)\text{-gr}_f$ be any finitely generated graded G (R) - module. In view of the foregoing observations and conventions we obtain two formal schemes over $\hat{Y}$ of Y: graded formal coherent sheaf

$$\underline{M}^{g,\Delta}_{\hat{Y}} \cong \underset{\underset{n}{\leftarrow}}{\mathrm{Lim}^g} \ \underline{M}^g_Y / (\underline{J}^g_Y)^n \ \underline{M}^g_Y$$

and formal coherent sheaf

$$\underline{M}^{\Delta}_{\hat{Y}} \cong \underline{M}^{g,\Delta}_{\hat{Y}} \cong \underset{\underset{n}{\leftarrow}}{\mathrm{Lim}} \ (\underline{M}^g_Y)_o / \underline{J}^n_Y (\underline{M}_Y)_o.$$

4.1. *Note*

We can consider the defined sheaves as sheaves over Y, if it is necessary, this always means the extension by zero outside $\hat{Y}$. Also, since the open affine Noetherian subsets of Y form a base of the topology of Y, then the intersection of those with the closed subset $\hat{Y}$ forms a basis for the topology on $\hat{Y}$, see 2.2.

We aim to construct morphisms between $G(R) - \mathrm{gr}_f$ and $\underline{O}^g_{\hat{Y}} - \mathrm{Gr}_{Coh.}$. The zero-version of this result over $(Y, \underline{O}_{\hat{Y}})$ is in fact corollary 9.9 of (5), Hence it is true.

Hence we consider the following theorem with its proof under the above consideration, but with Y = $\mathrm{Spec}^g(G(R))$ and G(R) is I-adically complete, where I defines $\underline{J}^g_Y$.

4.2. Theorem (Commutative Level)

There is an equivalence of categories between $G(R)\text{-gr}_f$ and $\underline{O}^g_{\hat{Y}} - \mathrm{Gr}_{Coh.}$, where $G(R) - \mathrm{gr}_f$ is the category of finitely generated graded modules over G(R) and $\underline{O}^{g}_{\hat{Y}}-Gr_{Coh.}$ is the category of coherent graded $\underline{O}^{g}_{\hat{Y}}$ modules.

Proof:

The functor F_1: G(M) a $\underline{M}^{g,\Delta}_{\hat{Y}}$ from G(R)-gr_f to $\underline{O}^{g}_{\hat{Y}}-Gr_{Coh.}$ is a graded exact functor. Indeed, let $U\in\beta(Y)$; say U = $spe^{g}(B)$ and $G(M)\in G(R)\text{-}gr_f$. Hence $\underline{M}^{g}_{\hat{Y}}(U)=N\in B-gr_f$ is finitely generated graded B-module. Now $\underline{M}^{g,\Delta}_{\hat{Y}}(U)=\underset{\leftarrow n}{\mathrm{Lim}}^{g}\,\underline{M}^{g}_{Y}(U)/(\underline{I}^{g}_{Y}(U))^{n}\underline{M}^{g}_{Y}(U)$ which is $\underline{I}^{g}_{Y}(U)-$ adic graded completion of $\underline{M}^{g}_{Y}(U)$,i.e. $\underline{M}^{g,\Delta}_{\hat{Y}}(U)=\tilde{N}$. Since we have finitely generated graded modules, hence the completion functor is exact. Therefore G(M) a $\underline{M}^{g,\Delta}_{Y}(U)$ is graded exact functor.

The functor $F_2:\underline{\Im}^{g}_{\hat{Y}}\to\Gamma(\hat{Y},\underline{\Im}^{g}_{\hat{Y}})$ from $\underline{O}^{g}_{\hat{Y}}-Gr_{Coh}$ into G(R)-gr_f is exact and F_1, F_2 are inverse to each other. It is clear that $\Gamma(\hat{Y},\underline{M}^{g,\Delta}_{\hat{Y}})\cong G(M)$. Thus, the two composi-tions of our two functors are the identity. But, to show that the graded functor $\Gamma(\hat{Y},-)$ is exact on the category $\underline{O}^{g}_{\hat{Y}}-Gr_{coh}$, let

$$0\to{}^{1}\underline{\Im}^{g}_{\hat{Y}}\to{}^{2}\underline{\Im}^{g}_{\hat{Y}}\to{}^{3}\underline{\Im}^{g}_{\hat{Y}}\to 0 \qquad (1)$$

be an exact sequence in $\underline{O}^{g}_{\hat{Y}}-Gr_{coh.}$. Since $\Gamma(\hat{Y},{}^{i}\underline{\Im}^{g}_{\hat{Y}})=G(M_i)$, i = 1,2,3 are finitely generated graded modules, hence

$$0\to G(M_1)\to G(M_2)\to G(M_3)$$

is exact in G(R)-gr_f. Also

$$0\to G(M_1)\to G(M_2)\to G(M_3)\to G(M_4)\to 0 \qquad (2)$$

is exact in G(R)-gr_f, where $G(M_4)$ is in G(R)-gr_f too. Applying the exact functor $(\ )^{g,\Delta}$ on (2) we obtain the exact sequence

$$0\to{}^{1}\underline{M}^{g,\Delta}_{\hat{Y}}\to{}^{2}\underline{M}^{g,\Delta}_{\hat{Y}}\to{}^{3}\underline{M}^{g,\Delta}_{\hat{Y}}\to{}^{4}\underline{M}^{g,\Delta}_{\hat{Y}}\to 0 \qquad (3)$$

From (1) and (3) ${}^{4}\underline{M}^{g}_{\hat{Y}}=0$. This means that $G(M_4)=0$, hence

$$0\to G(M_1)\to G(M_2)\to G(M_3)\to 0$$

is exact. Therefore, $\Gamma(\hat{Y},-)$ is a graded exact functor.

4.3. Remark (Commutative real case)

Hartshorne's version, [5], of this over $(\hat{Y},\underline{O}_{\hat{Y}})$ is also true because, in this case, we have real schemes.

Let us consider the behaviour of the result 4.1. on the filtered level. Recall from [2] that for an ideal I or R equipped with the induced (hence good) filtration, we may define the filtered (non-commutative) Noetherian formal scheme over Y as follows. Put $G(I)=I_g$ and $V(I_g)=\hat{Y}$. On $\hat{\wp}$ we now introduce two structure coherently filtered sheaves $\underline{O}^{\mu}_{\hat{Y}}\cong\underset{\leftarrow n}{\mathrm{Lim}}^{f}(\underline{O}^{\mu}_{Y}/(\underline{I}^{\mu}_{Y})^{n})$ and $\underline{O}^{q}_{\hat{Y}}\cong F_0\underline{O}^{\mu}_{\hat{Y}}$ (the sheaf of quantum-sections over $\hat{\wp}$), where we write $\underset{\leftarrow n}{\mathrm{Lim}}^{f}$ to indicate that we have taken into account the filtration defined on $\underset{\leftarrow n}{\mathrm{Lim}}$ as in [2].

Similarly, for $M\in R\text{-filt}_{good}$; filtered module with good filtration, the coherently filtered formal sheaves of $\underline{M}^{\mu}_{Y}$ and $\underline{M}^{\mu,q}_{Y}$ may be defined as

$$\underline{M}^{\mu,\Delta}_{\hat{Y}}=\underset{\leftarrow n}{\mathrm{Lim}}^{f}\,\underline{M}^{\mu}_{Y}/(\underline{I}^{\mu}_{Y})\underline{M}^{\mu}_{Y}\ \text{and}\ \underline{M}^{\mu,q}_{\hat{Y}}=F_0\underline{M}^{\mu,\Delta}_{\hat{Y}}$$

respectively. Of course these define coherently filtered sheaves over $\underline{O}^{\mu}_{\hat{Y}}$ and $\underline{O}^{q}_{\hat{Y}}$ respectively, i.e. $\underline{M}^{\mu,\Delta}_{\hat{Y}}\in\underline{O}^{\mu}_{\hat{Y}}-\mathrm{Filt}_{coh.}$, $\underline{M}^{\mu,q}_{\hat{Y}}\in\underline{O}^{q}_{\hat{Y}}-\mathrm{Filt}_{coh.}$. The authors in [2] have studied these sheaves.

4.4 *Note*

If R is I-adic filtered complete, i.e. $R\cong\underset{\leftarrow n}{\mathrm{Lim}}^{f}\,R/I^{n}$, then

$$\underline{O}^{\mu}_{\hat{Y}}=\underset{\leftarrow n}{\mathrm{Lim}}^{f}\,\underline{O}^{\mu}_{Y}/(\underline{I}^{\mu}_{Y})^{n}\cong(\underset{\leftarrow n}{\mathrm{Lim}}^{f}\,R/I^{n})^{\mu}_{Y}\cong\underline{O}^{\mu}_{Y}..$$

Conversely, if $\underline{O}^{\mu}_{\hat{Y}}\cong\underline{O}^{\mu}_{Y}$, we can assume that $\hat{Y}=Y$. Then this gives rise to that the ideal I is contained in the prime radial of R. Therefore $I^{n}=0$ and this implies that $\hat{R}^{I}\cong R$. Similarly, we can say that $\underline{I}^{\mu}_{Y}$ is formal complete ⇔ I is I-adic filtered complete. It is clear that if M is I-adic complete, then $\underline{M}^{\mu}_{Y}$ will be formal complete, i.e. $\underline{M}^{\mu,\Delta}_{\hat{Y}} \cong \underline{M}^{\mu}_{Y}$.

Combined with section 3 in [2] and the conventions assumed in 4.2. this yields the non-commutative filtered level of the result in 4.2.

4.5. Theorem (non-commutative level)

There is an equivalence of categories between R-filt$_{good}$, and $\underline{O}^{\mu}_{\hat{Y}}-\text{Filt}_{coh.}$, where R-filt$_{good}$ is the category of good filtered I-adically complete R-modules and $\underline{O}^{\mu}_{\hat{Y}}-\text{Filt}_{coh.}$ is the category of coherent filtered $\underline{O}^{\mu}_{\hat{Y}}$ − modules.

Proof:

The functor $F_1 : M$ a $\underline{M}^{\mu,\Delta}_{\hat{Y}}$, from R-filt$_{good}$ into $\underline{O}^{\mu}_{\hat{Y}}-\text{Filt}_{coh.}$, is exact. Indeed, let U = Y(f)´ = $\text{spec}^{g}(G(Q^{\mu}_{f}(R)))$ for suitable f∈R with σ(f)∈G(R), for the definition of σ(f) see [7]. By the well known exactness property of $Q^{\mu}_{f}(-)$ on strict sequences we obtain that Γ(Y(f),-) from R-filt$_{good}$ to $Q^{\mu}_{f}(R)-\text{filt}_{good}$ is exact. Since $\Gamma(Y(f), M^{\mu,\Delta}_{\hat{Y}}) = Q^{\mu}_{f}(M)^{\wedge I}$ and the $Q^{\mu}_{f}(I)$-adic completion is exact on good filtration, it follows that M a $\underline{M}^{\mu,\Delta}_{\hat{M}}$ is filtered exact functor.

The functor $F_2 \underline{\mathfrak{I}}^{\mu}_{\hat{Y}}$ a $\Gamma(\hat{Y}, \underline{\mathfrak{I}}^{\mu}_{\hat{Y}})$ from $\underline{O}^{\mu}_{\hat{y}}-\text{Filt}_{coh.}$ to R-filt$_{good}$, is exact. Indeed, consider

$$0 \to {}^{1}\underline{\mathfrak{I}}^{\mu}_{\hat{Y}} \to {}^{2}\underline{\mathfrak{I}}^{\mu}_{\hat{Y}} \to {}^{3}\underline{\mathfrak{I}}^{\mu}_{\hat{Y}} \to 0 \qquad (4)$$

an exact sequence in $\underline{O}^{\mu}_{\hat{Y}}-\text{filt}_{coh.}$. Since ${}^{1}\underline{\mathfrak{I}}^{\mu}_{\hat{Y}} \cong \underline{M}^{\mu,\Delta}_{I_{\hat{Y}}}$ for some M_i; i=1,2,3 in R-filt$_{good}$ it follows that (4) becomes

$$0 \to \underline{M}^{\mu,\Delta}_{1_{\hat{Y}}} \to \underline{M}^{\mu,\Delta}_{2_{\hat{Y}}} \to \underline{M}^{\mu,\Delta}_{3_{\hat{Y}}} \to 0 \qquad (5)$$

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Now $\Gamma(\hat{Y},-)$ is left exact, then

$$0 \to M_1 \to M_2 \to M_3 \to M_4 \to 0 \qquad (6)$$

is an exact sequence in R-filt$_{good}$, where M_4 is coker(ϕ). From the first step, Δ is exact, hence we have an exact sequence

$$0 \to \underline{M}^{\mu,\Delta}_{1_{\hat{Y}}} \to \underline{M}^{\mu,\Delta}_{2_{\hat{Y}}} \to \underline{M}^{\mu,\Delta}_{3_{\hat{Y}}} \to \underline{M}^{\mu,\Delta}_{4_{\hat{Y}}} \to 0 \qquad (7)$$

Now it is sufficient to compare (5) and (6) to get $\underline{M}^{\mu,\Delta}_{4_{\hat{Y}}} = 0$.

i.e.$M_4 = 0$. Hence (6) with $M_4 = 0$ is exact, and this proves that $\underline{\mathfrak{I}}^{g}_{\hat{Y}}$ a $\Gamma(\hat{Y}, \underline{\mathfrak{I}}^{g}_{\hat{Y}})$ is filtered exact functor.

Finally, clearly $F_1F_2 = 1$ and $F_2F_1 = 1$.

From all of this, the theorem follows.

4.6. Final remark

1 - All arguments still hold if Δ is replaced by q in 4.5. This gives the quantum formal version of the result 4.5.

2 - The fact that the micro-or quantum-version of Serre's global sections results for coherent sheaves is also true. This follows from theorem 4.5.

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