مكتبة البنين
قسم الدوريات

University of Qatar

QATAR UNIVERSITY SCIENCE BULLETIN

(Qatar Univ. Sci. Bull.)

VOL. 10 **1990**

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Published by the Faculty of Science
University of Qatar

Qatar Univ. Sci. Bull. (1990) 10: 25–39

COLLINEATIONS OF POLAR SPACES

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ABSTRACT

In this paper we consider general embeddable polar spaces with Witt index at least three.

If $\pi : S(\xi) \to S(\xi')$ be a thick collineation between polar spaces with $i(\xi) = i(\xi') \geqslant 3$. Then there exists a place $\varnothing : k \to k' \cup \infty$ with valuation ring $A = \varnothing^{-1}(k')$, an A– module M in V with $Mk = V$ and a $\varnothing$-semilinear mapping

$\beta : M \to V'$ such that

$\pi(X) = \beta(M \cap X) K'$ for all points X in $S(\xi)$.

INTRODUCTION

The fundamental theorem of projective geometry describes the bijective collineations between two projective spaces PV and PV' of finite dimension (greater than one) over division rings k and k' in terms of an isomorphism $\varnothing : k \to k'$ and a $\varnothing$-semilinear bijective mapping between the underlying vector spaces V and V'. Tits [9, Theorem 8. 6II] has given an extensive generalization of this theorem to embeddable polar spaces induced by polarities coming from either (σ, ϵ)-hermitian forms of from (σ, ϵ)-quadratic forms with Witt indices at least two. In another direction, Klingenberg [7] and later André [1] and Rado [8], have generalized the fundamental theorem by considering non-injective collineations. Now the isomorphism $\varnothing$ must be replaced by a place $\varnothing: k \to k' \cup \infty$ and an integral structure over the valuation ring $A = \varnothing^{-1}(k')$ is induced into the projective space PV. In [6,XXII] and [10, p.366], Weisfeiler asks for analogues of this to other Tits buildings. Recently, Faulkner and Ferrar [3] gave this for Moufang planes. In [5], where generalization of work of Chow [2] is given, we were also led to this type of theorem for polar spaces defined over symmetric and alternating forms with maximal Witt index. In the present work we will consider general embeddable polar spaces with Witt index at least three.

Let $S(\xi)$ be a polar space on the underlying finite dimensional k-vector spece V with polarity ξ of trace type coming from either a non-degenerate trace-valued (α, ϵ)-hermination form on a non-degenerate (α, ϵ)-quadratic form q associated with a(α, ϵ)-hermitian form f, with Witt index $i(\xi) \geqslant 3$. Similarly let $S(\xi')$ be a polar space on the k'-vector space V' with polarity ξ' ' of trace type having Witt index $i(\xi') \geqslant 3$.

Theorem. Let π: $S(\xi) \rightarrow S(\xi')$ be a thick collineation between polar spaces with $i(\xi) = i(\xi') \geqq 3$. Then there exists a place $\varnothing : k \rightarrow K' \cup \infty$ with valuation ring $A = \varnothing^{-1}(k')$, an A-module M in V with Mk = V and a $\varnothing$-semilinear mapping $ß:M \rightarrow V'$ such that

$\pi X = ß(M \cap X)k'$ for all points X in $S(\xi)$.

Moreover, the hermitian forms f and f' associated with the polarities ξ and ξ' be chosen so tha M has an orthogonal splitting $M = M(1) \perp M(2)$ where M(1) is unimodular and free with rank equal to the dimension of ß(M) over $\varnothing(A)$, while $ß(M(2)) = 0$, and

$\varnothing(f(x,y)) = f'(ß(x), ß(y))$ for all $x,y \in M$.

The definition of a thick collineation is given in the following section. In the above theorem M(2) need not be free. Also, $f(M(2), M(2)) \subseteq m$, the unique maximal two-sided ideal of A. If both the polar spaces $S(\xi)$ and $S(\xi)$ are associated with pseudoquadratic forms then, with the appropriate interpretation, $q' \circ ß = \varnothing \circ q$. Conversely, it is fairly easily seen that any $\varnothing$-semilinear mapping ß : M ---- V', as in the theorem, induces a collineation. In the sympletic situation where σ is the identity and $\epsilon = -1$, the above theorem has been completely proved in [5, theorem 2.1]. When π is bijective, the result reduces to Theorem 8.6II in [9]; however, $\Delta k \cong A \simeq k'$, $V = M = M(1) \cong V'$ and the forms f and f' are isometric. See also note added in proof.

If the collineation π is surjective, the $\varnothing(A) = k'$ and $ß(M) = V'$.

2. Pseudo-quadratic forms and polar spaces. We give now the definition of (σ,ϵ)-hermitian forms and pseudo-quadratic forms and the connection with polar spaces; further details can be found in [9, Section 8]. Let k be a division ring, V a finite dimensional right k-vector spacc and $\sigma : k \rightarrow k$ an antiautomorphism, that is, and additive automorphism of k such that

$$(ab) = b^{\sigma} a^{\sigma} \text{ for all } a,b \in k.$$

A function $f: V X V \rightarrow k$ is called a σ-sequilnear form if it is biadditive and if

$$f(xa,yb) = a^{\sigma} f(x,y)b \text{ for all } x,z, \in V \text{ and } a, b, \in k.$$

The form f is reflexive if the relation $f(x,y) = 0$ is symmetric for $x,y \in V$. This condition is equivalent to the existence of a nonzero $\epsilon \in k$ such that

$$f(y,x) = f(x,y)^{\sigma} \epsilon \qquad \text{for all } x,y \in V .$$

Necessarily

$$\epsilon = \epsilon^{-1} \quad \text{and } t^{\sigma^2} = \epsilon\, t\, \epsilon^{-1} \text{ for } t \in k .$$

A form f satisfying these conditions is said to be (σ, ϵ)-hermitian.

Now assume $\epsilon \neq -1$ when σ is the identiy and the characteristic of k is not two. Set

$$k_{\sigma,\epsilon} = \{t - t^{\sigma} / t \in k\},$$

an additive subgroup of k, and denote by $k(\sigma, \epsilon)$ the quotient group $k/k_{\sigma,\epsilon}$.

A Function $q: V \to k(\sigma, \epsilon)$ is called a (σ, ϵ)-quadratic form or a psedo -quadratic form relative to σ and ϵ, if there exists a σ-sequilinear form $g: V \times V \to k$ such that

$$q(x) = g(x,x) + k_{\sigma,\epsilon}, \text{ for all } x \in V. \text{ Then,}$$

$$q(x\, a) = a^{\sigma} q(x)\, a \text{ for } a \in k \text{ and } x \in V.$$

Also,

$$q(x+y) = q(x) + q(y) + (f(x,y) + k_{\sigma,\epsilon}) \text{ for all } x,y \in V, \text{ where}$$

$f: V \times V \to k$ is the trace-valued (,)-hermitian form defined by

$$f(x,y) = g(x,y) + g(x,y)^{\sigma} \epsilon .$$

The form f is uniquely determined by q. The pseudo-quadratic form q is determined by the associated form f and the values taken by q on the elements of a basis of V. A pseudo-quadratic form is called nondegenerate when the associated hermitian form f is non-degenerate, that is, $f(x,V) = 0$ only when $x = 0$.

$$q(x+y) = g(x+y\, ,\, x+y) + k_{\sigma,\epsilon},$$

$$= g(x) + g(y) + g(x,y) + g(y,x)$$

A subspace U of V is called totally singular with respect to the pseudo-quadratic form q if q vanishes on U. If U is totally singular for q, then U is also totally isotropic with respect to the associated hermitian form f, that is, $f(U,U) = 0$. All amximal totally singular (respectively, totaly isotropic) subspaces of V have the same dimension called the Witt index of q (respectively, f).

If the characteristic of k is not two, all totally isotropic subspaces of V are also totally singular.

The projective space PV of V is the set of all one-dimensional subspaces of V. Let f be a non-degenerate trace-valued (σ, ϵ)-hermitian form on V with Witt index $i(f) \geq 2$. Then f determines a polarity ξ of trace type for the space PV. Denote by $S(\xi)$ the set of all isotropic points X in PV. Thus $f(X,X) = 0$.

The $S(\xi)$ is the polar space relative to the polarity ξ (or form f). More strictly, $S(\xi)$ should be defined relative to an equivalence class of proportional forms, rather than to a representative of the class, as we have done. Let q be a non-degenerate (σ, ϵ)-quadratic form on V with Witt index $i(q) \geqslant 2$ and f as its associated hermitian form. Again, this determines a polarity ξ of trace type. Denote by $S(\xi)$ the set of all singular points X in PV. Thus, $q(X) = 0$ on $S(\xi)$. Then $S(\xi)$ is the polar space relative to the proportionality class of the pseudo-quadratic form q. The linear subspaces of $S(\xi)$ are the subspaces of V which are totally isotropic, respectively totally singular, with respect to the hermitian form f, respectively pseudo-quadratic form q, associated with ξ. In particular, a line of $S(\xi)$ is totally isotropic, respectively totally singular, in two dimensional subspace of V. If X and Y are points in a polar space with $f(X,Y) = 0$, the line joining X and Y is denoted by $X + Y$.

Now let k' be a second division ring and $S(\xi')$ a polar space with polarity ξ' aasociated with either a non-degenerate trace-valued (σ', ϵ')-hermitian form $f': V' \longrightarrow k'$ on the finite dimensional k'-vector space V', or with a non-degenerate (σ', ϵ')-quadratic form

$$q' : V' \rightarrow k'(\sigma', \epsilon')$$

with associated (σ', ϵ')-hermitian form f'. Assume $i(\xi') \geqslant 2$.
A collineation between the polar spaces $S(\xi)$ and $S(\xi')$ is a mapping

$$\pi : S(\xi) \rightarrow S(\xi')$$

with the following properties. Let $X,Y \in S(\xi)$ with $X + Y$ a line (so $f(X,Y) = 0$). Then,

$$f'(\pi X, \pi Y) = 0.$$

Moreover, if $\pi X \neq \pi Y$, then for any point Z on the line $X + Y$ of $S(\xi)$, the point πZ is on the line $\pi X + \pi Y$ of $S(\xi')$. In particular, it follows that any line $S(\xi)$ is carried by π into a line of $S(\xi')$ (usually not surjectively). It is possible for π to carry all the points of a line of $S(\xi)$ into a single point in $S(\xi')$.

Let $S(\xi)$ be a polar space with Witt index $i(\xi) = n \geqslant 2$. A polar frame for $S(\xi)$ a set of points $F = \{X_1, Y_1, \ldots, X_n, Y_n\}$ in $S(\xi)$ with

$$f(X_i, X_j) = f(Y_i, Y_j) = 0 \text{ for } 1 \leqslant i, j \leqslant n, \text{ and } i \neq j$$

but

$f(X_i,Y_i) \neq 0$ or $1 \leqslant i \leqslant n$.

since $i(\xi) = n$ and f is o- trace-valued and non-degenerate, it follows that $S(\xi)$a polar frame. Let span F be the set of points in $S(\xi)$ that are also in the subspace of PV spanned by the points in F.

A collineation $\pi : S(\xi) \rightarrow S(\xi')$ is called thick if there exists a polar frame F of $S(\xi')$ such that π F is a polar frame of $S(\xi')$ (so necessarily $i(\xi) = i(\xi'))$ and, moreover, for each line L of $S(\xi)$ in span F the cardiality of the set $\{\pi X \mid X$ a point on L$\}$ is at least three. Thus, in particular, each line L' of span π F coming (via π) from a line $L \subset$ span F contains at least three points coming from points on L (in general, L' will also contain many points not coming from $S(\xi)$).

As a consequence of our theorem, the image $\pi S(\xi)$ of a thick colineation π is a polar space defined over the subring $\emptyset(A)$ of k'. if the mapping π is surjective, then $\emptyset(A) = k'$. However, in general, the image $\pi S(\xi)$ will be properly inside a polar space defined over the larger division ring k'.

3. Theorem on Thick collineations. In this section we prove the theorem in the special case where $S(\xi)$ is spanned by any of its polar frames, that is, when dim V=2n where $n = i(\xi) \geqslant 3$, by generalizing the ideas of Theorem 2.1 in [5] to our present situation. The polar spae $S(\xi)$ is associated with either a non-degenerate trace-valued (σ, ϵ)-hermitian form f, or with a (σ, ϵ)-quadratic form q, with non-degenerate trace-valued (σ, ϵ)-hermitian form f. In the first case the linear subspaces of $S(\xi)$ are totally isotropic and in the second case they are totally singular. Likewise for $S(\xi')$.

Let π: $S(\xi) \rightarrow S(\xi')$ be a thick collineation. It is possible for $S(\xi)$ to be assocciated with a pseudo-quadratic form while $S(\xi')$ is associated with an hermitian forn, for example, let π be the identity mapping but in the image space forget the pseudo-quadratic form and consider the larger space determined by the totally isotropic points (both k and k' will have characteristic two). It is also possible for $S(\xi')$ to be associated with an hermitian form and $S(\xi')$ with a pseudo-quadratic form: for example, f a symmetric form over the 2-adic number field Q_2 and q' a quadratic form over the finite field F_2.

Now assume dim $V = 2i(\xi) \geqslant 6$. Since π is thick collineation, there exists a polar frame $\{X_1,Y_1,\ldots,X_n,Y_n\}$ for $S(\xi)$ with $\{\pi X_1, \pi Y_1,\ldots,\pi X_n, \pi Y_n\}$ a polar frame for $S(\xi')$. Let S be the totally isotropic or totally singular subspace of V spanned by $X_1,\ldots, X_n$ and T the totally isotropic or totally singular subspace spanned by $Y_1,\ldots, Y_n$. Then

$$S \cap T = 0 \quad \text{and} \quad V = S \oplus T.$$

Likewise, if $\pi X_1,\dots, \pi X_n$ spans S' in V' and $\pi Y_1,\dots, \pi Y_n$ spans T'.,

then

$$S' \cap T' = 0 \text{ and } V' = (S' \oplus T') \perp W'$$

with W' a subspace of V'. Since $n \geqslant 3$, the restriction

$$\pi\ PS \rightarrow PS'$$

satisfies the conditions of Teorem 3.1 in [4] (the proof remains valid over division rings). Hence there exists a place

$$\varnothing_S : k \rightarrow k' \cup \infty$$

with valuation ring

$$A_S = \varnothing^{-1}_S (k'),$$

a free A_S - module $M_S = u_1A_S + \dots + u_nA_S$ in S with rank n and a $\varnothing_S$-semilinear mapping $ß_S : M_S \rightarrow S'$ defined by

$$ß_S(\Sigma\, u_i\, a_i) = \Sigma\, u'_i\, \varnothing_S\, (a_i)$$

such that

$$\pi X = ß_S(M_S \cap X)\, k'$$

for all points X in PS. Here u_i is a nonzero element from the one-dimensional subspace $X_i$of V, and u'_i is a non-zero element from πX_i, $1 \leqslant i \leqslant n$. The module M_S and the mapping $ß_S$ are not uniquely determined but can be changed by multiplication by scalar. Likewise, considering the restriction

$$\pi : PT \rightarrow PT',$$

there exist corresponding $\varnothing_T$, A_T, $M_T = A_T + \dots +$

$$A_T \text{ in T with } v_i \in Y \text{ and } ß_T : M_T \rightarrow T' \text{ where}$$
$$ß_T(\Sigma\, v'_i\, b_i) = \Sigma\, v'_i\, \varnothing_T\, (b_i) \text{ and}$$
$$\pi Y = ß_T(M_T \cap Y)\, k'$$

for points Y in PT.

Since π is a thick collineation, there exists a point $(u_l + v_n b)\, k$ on the line $u_l k + v_n k$ with image

$$(u'_l + v'_n\, b')\, k' \text{ where } b' \neq 0.$$

Replacing M_T by $M_T b$ and adjusting $ß_T$ we may assume b = l. Likewise, by changing the choice of the v'_i, we may assume b' = l. Let $0 \neq a \in A_S$. Since π is a collineation, the image of $(u_l a + u_2 + v_n a)k$ must be

$$(u'_l\, \varnothing_S(a) + u'_2 + v'_n\, \varnothing_S\, (a)\, k'.$$

Hence the image of $(u_2 + v_n a)\, k$ must be

$$(u'_2 + v'_n \varnothing_S\, (a)k'.$$

Assume first that $a^{-1} \in A_T$. Then, by a similar argument, the image of $(u_2 + v_n a)k$ is also

$$(u'_2 \varnothing_T(a^{-1}) + v'_n)K'.$$

Hence $\varnothing_S(a) = \varnothing_T(a)$. On the other hand, if $a^{-1} \notin A_T$, then $\varnothing_T(a) = 0$. Now the image of $(u_2 + v_l + v_n a)k$ is in $u'_2 k' + v'_l k'$ and hence the image of $(u_2 + v_n a)k$ is $u'_2 k'$. Thus $\varnothing_S(a) = 0$ also. Hence

$$\varnothing_S(a) = \varnothing_T(a) \text{ for all } a \in A_S.$$

By symmetry $A_S = A_T = A$ and $\varnothing_S = \varnothing_T = \varnothing$, say. Now $M = M_S + M_T$ is a free A-module of rank 2 n.

From the definition of a polar frame,

$$f(u_i, v_j) = \delta_{ij} c_i$$

where $0 \neq c_i \in k$. Assume there exist c_i, c_j with

$$\varnothing(c^{-1}_j c_i) = 0.$$

Since $u_i + u_j$ and $v_i - v_j c^{-1}_j c_i$ are orthogonal, it follows from the properties of π that $u'_i + u'_j$ and $v'_i - v_j \varnothing(c^{-1}_j c_i)$ are orthogonal. This is a contradiction of

$$\varnothing(c^{-1}_j c_i) = 0.$$

Hence $c^{-1}_j c_i$ is a unit in A and M is a modular A-module. Changing the choice of each $u_i \in X_i$ by a unit, we may assume

$$f(u_i, v_i) = c \neq 0, \ 1 \leqslant i \leqslant n.$$

Replacing f by the proportional form $c^{-1} f$ we may assume M is unimodular and c = 1. Morover, now

$$f'(u_i, v_i) = c' \neq 0, \ 1 \leqslant i \leqslant n.$$

For any $a \in A$ the elements $u_i + u_2 a$ and $v_1 a^{\sigma} - v_2$ are orthogonal. Hence, $u_1' + u_2' \varnothing(a)$ and $v_1' \varnothing(a^{\sigma}) - v_2'$ are orthogonal so that

$$\varnothing(a^{\sigma}) = C'^{-1} \varnothing(a)^{\sigma'} c'.$$

Again, since $u_1 + v_2$ is orthogonal to $v_1 - u_2$, it follows that

$$c' \varnothing(\epsilon) = c'^{\sigma'} \epsilon'.$$

If we now replace f' by the proportional (σ'', ϵ'')-hermitian form $c'^{-1} f'$ where

$$\epsilon'' = c'^{-1} c'^{\sigma'} \epsilon' \text{ and } b^{\sigma''} = c'^{-1} b^{\sigma'} c' \text{ for } b \in k', \text{ then}$$

$$\varnothing(\epsilon) = \epsilon'' \quad \text{and} \quad \varnothing(a^{\sigma}) = \varnothing(a)^{\sigma''} \quad \text{for all } a \in A$$

(If ϵ' is associated with a pseudo-quadratic form q', this will also be changed to a proportional form). Now we may assume

$$f'(u'_i, v'_i) = 1, \ 1 \leqslant i \leqslant n,$$

$\varnothing(\epsilon) = \epsilon'$ and

$\varnothing(a^{\sigma}) = \varnothing(a)^{\sigma'}$ for $a \in A$.

Define $\beta : M \to V$ by

$\beta(x+y) = \beta_S(x) + \beta_T(y)$ for $x \in M_S$ and $y \in M_T$.

We must prove

$\pi X = \beta(M \cap X)\, k'$

for all points X in $S(\xi)$. This has already been done for X in PS or PT, and for

$X = (u_i + v_j a)k$ with $i \neq j$ and $a \in A$.

Now consider $X = xk$ where

$x = \sum (u_i a_i + v_i b_i)$ with $a_i, b_i \in A$.

Without loss in generality, we may asssume $a_1 = 1$. Let $\pi X = x'k'$ where

$x' = \sum (u'_i a'_i + v'_i b'_i)$.

Since, for $i \geqslant 2$, x is orthogonal to $v_1 a^{\sigma}_i - v_i$, it follows that x' is orthogonal to $v'_i \varnothing(a^{\sigma}_i) - v'_i$ and hence that

$a'_i = \varnothing(a_i)\, a'_1$.

Similarly,

$b'_i = \varnothing(b_i)\, a'_1$ for $i \geqslant 2$.

Now assume that a_2, say, is a unit in A. Then x is orthogonal to $u_1 - v_2(b_1^{-1}a_2)^{\sigma}\epsilon$,

from which it follows that

$b'_1 = \varnothing(b_1)a'_1$.

Thus, in this situation,

$X' = \beta(x)\, a'_1$

and the proof is complete. It remains to consider the case where $a_2, b_2, \ldots, a_n, b_n$ are all nonunits in A. Then

$x' \in u'_1 k' + v'_1 k'$.

We can now find $y \in u_2 k + u_3 k$ such that

$f(x,y) = 0$, $\pi(yk) = \beta(y)k'$ and

$\pi(x+y)k = \beta(x+y)k'$.

It follows from collinearity that

$\pi X = \beta(x)k'$.

Also, it is clear that

$\varnothing(f(x,y)) = f'(\beta(x), \beta(y))$ for all $x,y \in M$.

This completes the proof when dim $V = 2n$. In particular, in the symplectic situation where σ is the identity mapping and $\epsilon = -1$, the theorem has been established.

4. The general case. We now complete the proof of the theorem in the general situation. Since $\pi : S(\xi) \to S(\xi')$ is a thick collineation, by the results of the previuos section, there exists a place $\emptyset : k \to k' \cup \infty$ and a free unimodular A-module

$$M_o = \perp_{i=1}^{n} (u_i A + v_i)$$

where

$$n = i(\xi) = i(\xi') \geq 3 \text{ and } f(u_i, v_j) = \delta_{ij}, \; i \leq i, j \leq n$$

(after normalizing the form f). Moreover, there is a $\emptyset$-semilinear mapping $\beta_o : M_o \to V'$ such that

$$\pi X = \beta_o(M_o \cap X)k' \text{ for all points X in } S(\xi) \cap P(M_o k).$$

We must extend M_o to an A-module M in V with $Mk = V$, and β_o to a $\emptyset$-semilinear mapping $\beta : M \to V'$ which induces π.

Let $\beta_o(u'_i) = u'_i$ and $\beta_o(v_i) = v'_i$, $1 \leq i \leq n$, and $V = M_o k \perp W$ where W is a subspace of V with Witt index $i(\xi \mid W) = 0$. The Polar space $S(\xi')$ may be associated with a (σ', ϵ')–hermitian form f', or with a (σ', ϵ')-quadratic form q' with hermitian form f'. In either case, f' can be normalized by changing to a proportional form so that

$$f'(u'_i, v'_j) = \delta_{ij}, \; 1 \leq i, j \leq n.$$

$$\emptyset(\epsilon) = \epsilon' \text{ and}$$

$$\emptyset(a^{\sigma}) = \emptyset(a)^{\sigma'} \quad \text{for all } a \in A.$$

Let

$$V'_o = \perp_{i=1}^{n} (u'_i k' + v'_i k') \quad \text{and} \quad V' = V'_o + W'$$

where W' is a subspace of V' with Witt index $i(\xi' \mid W') = 0$. Then

$$\beta_o(M_o)k' = 'V_o.$$

Let M_W be the set of all $w \in W$ for which the exists $c \in A$ with

$$f(w,w) = c + c^{\sigma}\epsilon \quad \text{and } q(w) = c + k_{\sigma,\epsilon}$$

when $S(\xi)$ is associated with a pseudo-quadratic form q. We will prove that M_W is an A–module in W. clearly $w\,a \in M_W$ for all $w \in M_W$ and $a \in A$, so it suffices to prove additivity. Let

$w \in M_w$ so there exists $c \in A$ with

$$f(w,w) = c + c^{\sigma}\epsilon$$

(and $q(w) = c + k_{\sigma,\epsilon}$). Put

$$x = u_1 - v_1c + w.$$

Then $X = xk$ is a point in $S(\xi)$ and by the properties of π, its image

$$\pi X = (u'_1 d' - v'_1 c' + w')k' \text{ for some } c', d' \in k' \text{ and } w' \in W'.$$

Let

$$r = u_1 + v_1c^{\sigma}\epsilon + u_2 - v_2c^{\sigma}\epsilon$$

then

$$f(x,r) = f(r,r) = 0,$$

so that $X + rk$ is a line in $S(\xi)$. Hence

$$f'(\beta_0(r), \pi X) = 0$$

and consequently $c' = \varnothing(c)\, d'$. If $d' = 0$, then $c' = 0$ and $w' = 0$ since

$$i(\xi' \mid W') = 0$$

As this is impossible, $d' \neq 0$ and we may asume after changning w' by a scalar, that $d' = 1$ and $c' = \varnothing(c)$. Now let z be a second element in M_w.

Then

$$f(z,z) = a + a^{\sigma}\epsilon$$

(and $q(z) = k_{\sigma,\epsilon}$) for some $a \in A$. Let

$$b = f(w,z) \in k.$$

If we show $b \in A$ then it follows that $w + z \in M_W$ since

$f(w+z, w+z) = e + e^{\sigma} \epsilon$ with $e = a + b + c \in A$ (and $q(w+z) = e + k_{\sigma,\epsilon}$ Let Y be the point $(v_2b + u_2 - v_2 a - z)\, k$ in $S(\xi)$.

Then $X + Y$ is a line in $S(\xi)$. Analogously to the argument with X above.

$$\pi Y = (v'_1 b' + u'_2 - v'_2 \varnothing(a) - z')\, k'$$

for some $b' \in k'$ and $z' \in w'$; it is impossible for $\pi Y = v'_1 k'$ since

$$f'(\pi X, \pi Y) = 0.$$

Thus

$$b' = f'(w', z')$$

Also $Y + (u_1 - v_2 b^{\sigma} \epsilon)\, k$ is a line in $S(\xi)$. If $b \notin A$, then

$$\pi(u_1 - v_2 b^{\sigma} \epsilon)\, k = v'_2\, k'$$

forces

$$f'(\pi Y, v'_2) = 0,$$

which is a contradiction. Hence $b \in A$ and then $b' = \varnothing(b)$. Thus M_w is an A-module. We have also shown that

$$\emptyset(f(w,z,)) = f'(w',z'),$$

and when $S(\xi)$ and $S(\xi')$ are both associated with pseudo-quadratic forms that

$$q'(w') = \emptyset(q(w)).$$

The above argument also shows that if $w \in M_W$, so there exists $c \in A$ with $f(w,w) = c + c^{\sigma} \epsilon$, then the image of the point $X = xk$ where $x = u_1 - v_1 c + w$ is

$$\pi X = (u'_1 - v'_1 \emptyset(c) + w')\, k' \text{ for some } w' \in W'.$$

Define $\beta_w : M_w \to W'$ by $\beta_w(w) = w'$. We will prove that β_w is a $\emptyset$-semilinear mapping. If $c \in m$, the unique two sided ideal of A, then

$$f'(w',w') = 0$$

(and $q'(w') = 0$). Since $i(\xi' \mid W') = 0$, it then follows that $w' = 0$. In particular,

$\beta_W(wa) = 0 = B_W(w)\, \emptyset(a)$ for all $w \in M_w$ and $a \in m$. For $w \in M_w$, let $t = v + w$ where v is a primitive element of M_o(so $v \notin M_o m$) with $tk \in S(\xi)$. Then. by an argument analogous to one used before,

$$\pi tk = (\beta_o(v) + w'')k' \text{ for some } w'' \in W'.$$

(In fact, after changing the basis of M_o, we may assume $v = \bar{u}_1 - \bar{v}_1 a$ with $a \in A$ in some new basis $\bar{u}_1, v_1, \ldots, \bar{u}_n, v_n$ associated with a polar frame, and the result follows from the $\emptyset$-semilinearity of β_o on M_o). We next show $w'' = \beta_w(w)$. for those $v \in M_o$ for which $f(x,t) = 0$, this is clear since $(x-t)k$ lies in $S(\xi) \cap P(M_o k)$, so that $\pi(x-t)k$ must be collinear with πX and πtk. In the remaining case, it follows after constructing a new point of this same type orthogonal o both X and tk. This construction is possible since $n \geqslant 3$. Thus

$$\pi(v + w)\, k = (\beta_o(v) + \beta_W(w))\, k'.$$

It is now easily seen that β_W is an additive homomorphism and

$\beta_W(wa) = \beta_W(w)\, \emptyset(a)$ for all units $a \in A$. Thus β_W is a $\emptyset$-semilinear mapping.

Next we prove $M_W k = W$ so that $M = M_o + M_W$ is an A-module with $Mk = V$. Let $w \in W$. Since f is trace-valued there exists $c \in k$ such that

$$f(w,w) = c + c^{\sigma}\epsilon$$

(and $q(w) = c + k_{\sigma,\epsilon}$, when $S(\xi)$ is associated with a pseudo-quadratic form q). If $c \in A$, then $w \in M_w$. Otherwise, $c^{-1} \in A$ and $wc^{-1} \in M_w$

$$f(wc^{-1}, w c^{-1})\, e + e^{\sigma} \epsilon$$

where

$$e = (c^{-1}) \in A.$$

Hence

$$M_w k = W.$$

Define a $\varnothing$-semilinear mapping $ß : M \longrightarrow V'$ by

$$ß(v+w) = ß_o(v) + ß_w(w) \text{ for all } v \in M_o \text{ and } \in M_w.$$

Then $ß$ induces π. for let $X = xk \in A(\xi)$ where $x = v + w$ with $v \in M_o$ primitive and $w \in W$. Then

$$f(w,w) = -f(v,v)$$

(and $q(w) = -q(v)$. Since $v \in M_o$ there now exists $c \in A$ such that

$$f(w,w) = c + c^{\sigma\epsilon}$$

(and $q(w) = c + k_{\sigma,\epsilon,}$). Hence $w \in M_w$ and $x \in M$. Since $M \cap X = xA$ it follows that

$$\pi X = ß(x)k' = ß(m \cap x)k'.$$

This completes the proof of the main statement of the theorem. We have also shown that f and f' can be chosen such that

$$\varnothing(f(x,y)) = f'ß(x), ß(y)) \text{ for all } x,y \in M.$$

If $f(M_w, M_w) \subseteq m$, then (see note added in proof)

$$ß(M_w) = 0 \text{ and } ß(M)k' = V'_o.$$

In this case take $M(1) = M_0$ and $M(2) = M_w$. Then

$$M = M(1) \perp M(2)$$

with $M(1)$ a free unimodular A-module of rank 2n. It is possible to construct examples where $M(2)$; is not free.

Now asume there exists $w,z \in M_w$ with $f(w,z) = 1$. Let

$$ß(w) = w' \text{ and } ß(z) = z'.$$

Then

$$f'(w',z') = \varnothing(1) = 1$$

and hence $w' \neq 0$ and $z' \neq 0$. If $f(w,w)$ is a unit in A, then wA is a free rank one, orthogonal direct summand of M_w. Likewise if $f(z,z)$ is a unit. If neither $f(w,w)$ nor $f(z,z)$ are units, then $wA + zA$ is unimodular free, rank two, orthogonal direct summand of M_w. Since now

$$f'(w',w') = \varnothing(f(w,w)) = 0,$$

the polar space $S(\xi')$ must be associated with a pseudo-quadratic form q' with $q'(w') \neq 0$ (and the characteristic of k' must be two).

Proceeding in this fashion we obtain a splitting

$$M = M(1) \perp M(2)$$

where

$$M(1) = M_o \perp B_1 \perp \dots \perp B_m$$

with the B_i free unimodular A-modules of rank one or two, while M(2) is an A-module with

$$f(M(2), M(2)) \subseteq m.$$

Thus $\beta(M(2)) = 0$. The A-module M(1) is unimodular and free with rank equal to the dimension of $\beta(M)$ over $\varnothing$ (A). This completes the proof of the theorem.

Remarks. In the splitting $M = M(1) \perp M(2)$ the components M(1) and M(2) are not uniquely determined. If the valuation ring A is discrete, then M(2) will be free and m-modular and $M = M(1) \perp M(2)$ is just a splitting into Jordan components. In general, M(2) will not be free.

Since there is no assumption that the collineation

$$\pi : S(\xi) \perp S(\xi')$$

is surjective, in general

$$\varnothing(A) \neq k' \text{ and } \beta(M)k' \neq V'.$$

In fact, it is not necessary ture that $\beta(M)k'$ and V' have the same dimension over k'. However, if it is assumed that $\pi$is surjective, although not necessarily injective, it is easily seen that

$$\varnothing(A) = k', \quad \beta_O (M_O) = V'_O \text{ and}$$
$$\beta(M) = \beta(M(1)) = V'.$$

Note added in proof. There is an exceptional situation for part of the theorem we had not noticed before. Assume the characteristic of k' is two, S(ξ') is a polar space associated with a pseudo-quadratic form and π is not surjective . It is then possible for $\beta(M(2)) \neq 0$, so the statement about M(1) and M(2) in the thorem should be omitted for this situation. The main part f the theorem is not affected.

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تجمعات الفراغات القطبية

ليـــلى رشـــيد

في هذا البحث نتناول الفراغات القطبية العامة ذات دليل ويت Witt index الذي قيمته على الأقل ثلاثة بفرض أن $\pi: S(\xi) \to S(\xi')$ هي تجمع كثيف ـ (a thick collineation) بين فراغات قطبية ذات $i(\xi) = i(\xi') \geq 3$

فانه يوجد رتبة (a Place)

$$\varnothing : K \to K' \cup \infty$$

ذات حلقة تقديرية $A = \varnothing^{-1}(K')$

كما يوجد تشكيل خطي M في V (AN A - Module M in V)

حيث $MK = V$ ، وحيث

$\varnothing$ راسم شبه خطي و $\beta : M \to V'$

بحيث $\pi X = \beta (M \cap X) K'$

لجميع نقط X في $S(\xi)$.