# *Translation-invariant Positive-definite Generalized Kernels of Infinite Number of Variables*

by

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# النواة اللامتغيرة بالازاحة والمعممة
# ذو متغيرات لا نهائية البعد

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وفي هذه الورقة تدرس نواة ذات متغيرات لا نهائية البعد في فراغات نووية وتحقق الخواص اللاتغير بالنسبة للازاحة وفي الوقت نفسه معممة وموجبة ويوجد الشرط الكافي للحصول على تمثيل تكاملي لها .

## Introduction

In the theory of spectral analysis of positive-definite kernels, there exist well-developed methods based on the ideas of Krein connected with the construction of a Hilbert space by means of a kernel [1,11]. In the main, our formulation deals with translation-invariant positive-definite generalized kernels.

Consider the rigged Hilbert spaces $H_- \supseteq H_o \supseteq H_+$ [1] with the involution $\omega \to \bar{\omega}$ defined in $H_-$ and also in $H_o$ and $H_+$. Let $K \in H_- \otimes H_-$ be a generalized kernel. If $(K, u \otimes \bar{u})_{H_o \otimes H_o} \geqslant 0$, then K is said to be positive-definite (p.d.).

Now, let $\phi$ be a topological space of functions on X, $\phi'$ be its adjoint, $K \epsilon \phi' \otimes \phi'$ a generalized kernel and G a commutative group in X.

The kernel $K \in \phi' \otimes \phi'$ is said to be G quasi-invariant if there exists a function $\rho(x, a)$, $x \epsilon X$, $a \epsilon G$ which for each $a \epsilon G$ is a multiplier in $\phi$ such that

$$(K, v(a+.)\,\rho(.,a) \otimes \overline{u(a+.)\,\rho(.,a))} = (K, v \otimes \bar{u}), \; u, v \epsilon \phi; \; a \epsilon G \tag{0.1}$$

Now, let $H_k$ be the Hilbert space constructed from the quasi-scalar product $\langle u,v\rangle_k = (K, v \otimes \bar{u})$ by means of completion and factorization. Let B be the continuous operator in $\phi$ which commutes with involution, and let $B^+$ be its adjoint in $\phi'$. We say that B is K symmetric if

$$(B^+ \otimes I)\,K = (I \otimes \overline{B^+})\,K \tag{0.2}$$

which is equivalent to the symmetry of B in $H_k$ : $\langle Bu, v\rangle_k = \langle u, Bv\rangle_k$.

The formula $(T_a u)(x) = u(a + x)\,\rho(x, a)$ makes sense for the representation of G in $\phi$, for which the generalized kernel K is translation-invariant, i.e. $(T_a^+ \otimes T_a^+)\,K = K$, and thus $T_a$ is unitary in $H_k$.

In what follows, we apply the proceding theory to obtain an integral representation of p.d. translation-invariant kernels on spaces of $R_o^\infty$ and of the type $S'_g \otimes S'_g$ and $\sigma'_g \otimes \sigma'_g$ ($R_o^\infty \subset R^\infty$ is the space of finite sequences), where

$$S_g(R^\infty) = \bigcap_{m = (m_i)_{i=1}^\infty} S_g^m(R^\infty), \text{ and}$$

$$S_g^m(R^\infty) = \left\{ u(t) = \sum_{k=0}^{\infty} u_k e^{ikt} \mid \|u\|_m^2 = \sum_{k=0}^{\infty} |u_k|^2 (1 + |k|^2)^m < \infty \right\}$$

Also,

$$\sigma_g^m(R') = \left\{ u(t) = \sum_{k=0}^{\infty} u_k e^{ikt} \mid \|u\|_m^2 = \sum_{k=0}^{\infty} |u_k|^2 m^{|k|} < \infty \right\},$$

and

$$\sigma_g(R^\infty) = \bigotimes_{i=1,e}^{\infty} \sigma_g(R'), \text{ where } \sigma_g(R') = \bigcap_{1}^{\infty} \sigma_g^m(R'),$$

(see [7,12]).

## 1. The case of $K \in S'_g \otimes S'_g$

Consider the kernel $K \in S'_g \otimes S'_g$ which satisfies the following conditions:

a) p.d., i.e. $(K, u \otimes u) \geqslant 0, \; u \in S_g(R^\infty)$.

b) $R_0^\infty$ quasi-invariant with density

$$\rho(x,a) = \exp\left\{-\sum_{i=1}^{n(a)} (2a_i x_i + a_i^2)\right\}, \quad x \in R^\infty, \; a\,(a_1, \ldots, a_n) \in R_0^\infty \tag{1.1}$$

As in [8], we can show that the density $\rho(x,a)$ which takes the form (1.1) is a multiplier in $S_g(R^\infty)$. In fact, consider the Fourier-Venar transform [10] of the function $u \in L_2(R^\infty, dg)$ in the form

$$\hat{u}(\lambda) = \lim_{n \to \infty} \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{R^n} e^{\sum_1^n \lambda_i^2/2} \, e^{i \sum_{i=1}^n \lambda_i x_i} u_n(x_1, \ldots, x_n) \,.$$

$$\cdot \; e^{-\sum_1^n x_i^2/2} \, dx_1 \ldots dx_n, \tag{1.2}$$

where $u_n(x_1, \ldots, x_n)$ is the corresponding cylindrical function generated by $u \in L_2(R^\infty, dg)$. In finite-dimensional cases, the Fourier-Venar transform $F_\omega$ is the unitary image of the Fourier transform F in transforming from $L_2(R^n, dx)$ to $L_2(R^n, dg)$, i.e.,

$$u_n : L_2(R^n, dx) \ni \phi \to \pi^{-n/4} e^{\sum_1^n x_i^2} \phi \in L_2(R^n, dg) \tag{1.3}$$

So, $F_\omega = u_n F u_n^{-1}$. And so, the Fourier-Venar transform exists as a unitary operator in $L_2(R^\infty, dg)$. In addition, by simply checking the relation $\hat{h}_\alpha(\lambda) = i^{|\alpha|} h_\alpha(\lambda)$ ; $|\alpha| = \alpha_1 + \ldots + \alpha_\upsilon$, this transformation is a unitary operator in each of the Hilbert spaces $S_g^{\pm m}$ and $\sigma_g^{\pm m}$. Thus, for an arbitrary $u \in S_g(R^\infty)$, we have

$$u(\,.\,)\rho(\,.\,,a) = \hat{u}(\lambda) e^{i\Sigma \lambda_k a_k}$$

But $e^{i\Sigma\lambda_k a_k}$ is a multiplier function in $S_g(R^\infty)$, so we have the required.

Using the preceding Fourier-Venar transform, we have the following theorem:

*Theorem 1*
Every translation-invariant p.d. kernel $K \epsilon S_g' \otimes S_g'$ admits the representation

$$(K, v \otimes \overline{u}) = \int_{R_k^\infty} (\hat{u}\,\hat{\bar{v}})(\lambda)\, d\rho(\lambda) \tag{1.4}$$

where $d\rho(\lambda) = c(\lambda)\, d\sigma(\lambda)$ is a finite measure on $R^\infty$ and $c(\lambda) \leqslant c \exp\left(1/2 \sum \frac{\lambda_j^2}{n(m_j)}\right)$ and $d\sigma(\lambda)$ is a finite measure defined on a $\sigma$-algebra of sets from $R^\infty$. Conversely, every measure $d\rho(\lambda)$ in the given form generates a translation-invariant p.d. generalized kernel.

*Proof:* First we construct the Hilbert space $H_k$ as the completion of the space $S_g(R^\infty)$ w.r.t. the quasi-scalar product $\langle u,v\rangle_k = (K, v \otimes \overline{u})$. By $B^+$ we denote the corresponding adjoint of the operator $B: S_g \to S_g$ in $S_g'$. The p.d. kernel K will be called B-translation-invariant if $(B^+ \otimes B^+)K = K$.

The formula $(T_a u)(x) = u(a+x)\,\rho(x,a)$ makes sense for the representation of the group G from $R^\infty$ in $S_g(R^\infty)$, for which the kernel K is translation-invariant and thus $T_a$ is unitary in $H_k$, (see [6]). Therefore, the corresponding infinitesimal operators of the representation will be K-symmetric, i.e., if B is an infinitesimal operator of the representation $T_a$, then, $(B^+ \otimes I)K = (I \otimes \overline{B}^+)K$. Moreover, these operators generate a commuting system of self-adjoint operators in $H_k$ (see [3]).

Henceforth, we shall consider $B_k$ as an infinitesimal operator of the representation $T_a$ with k variables, $k = 1, 2, \ldots$

Since $K \epsilon S_g' \otimes S_g'$, we can find $\ell = (\ell_i)_{i=1}^\infty$ such that $K \epsilon S_g^{-\ell} \otimes S_g^{-\ell}$. It is clear that $H_k \supseteq S_g^\ell$, moreover, the inclusion is continuous. Therefore, from the nuclearity of $S_g(R)$ we find $m = (m_i)_{i=1}^\infty$ such that $H_k \supsetneqq S_g^m(R^\infty)$ and the inclusion is quasi-nuclear (Hilbert-Schmidt operator).

Thus, we have the chain

$$H_{-m,k} \supsetneqq H_k \supsetneqq S_g^m(R^\infty) \supseteq S_g(R^\infty) \tag{1.5}$$

in which $H_{-m,k}$ is the dual space of $S_g^m$ w.r.t. $H_k$ and $S_g(R^\infty)$ is the extension of the equipment. The operators $(B_k)_{k=1}^\infty$ form a system of commuting self-adjoint operators in $H_k$ and define a differential expression $B_k$ in the form

$$B_k = i\, p^{-1}(x_k) \frac{\partial}{\partial x_k} (p(x_k)) \tag{1.6}$$

where $p(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$ is the density of a Gaussian measure.

Moreover, for every k, $D(B_k) \supset S_g(R^\infty)$ where $D(B_k)$ is the domain of definition of the operators $B_k$.

Now, applying theorem '2' from [2] to the system of operators $(B_k)_{k=1}^\infty$ we have the Parseval equality in the form

$$1 = \int_{R^\infty} p(\lambda)\, d\sigma(\lambda) \tag{1.7}$$

Here, $d\sigma(\lambda)$ is a non-negative finite measure defined on a $\sigma$-algebra of cylindrical sets from $R^\infty$, $p(\lambda)$ defines $d\sigma(\lambda)$ almost everywhere for each $\lambda$ and the operator valued function of which gives a non-negative quasi-nuclear operator, operates from $s_g^m$ to $H_{-m,k}$ for which the Hilbert norm $|p(\lambda)| \leqslant 1$. The integral (1.7) converges in the sense of the Hilbert norm. So, there exists a set $A \subset R^\infty$ with total measure $d\sigma(\lambda)$ such that $R(p(\lambda))$, for $\lambda \epsilon \Lambda$, consists of the generalized eigenvectors of the operators $B_k$ with eigenvalues $\lambda_k$:

$$<p(\lambda)u, B_k v> = \lambda_k <p(\lambda)u,v> \quad (u \epsilon S_g^m(R^\infty), v \epsilon S_g(R^\infty)) \tag{1.8}$$

Now, we consider the chain

$$S_g^{-m} \times S_g^{-m} \supset L_2(R^\infty \times R^\infty, dg(x) \times dg(y)) \supset S_g^m \times S_g^m \tag{1.9}$$

With the help of the procedure in [4, 5], the set of operators $p(\lambda)$ defines the family of elementary kernels

$$\Omega(\lambda) \epsilon S_g^{-m} \times S_g^{-m} \text{ where } \|\Omega(\lambda)\|_{S_g^{-m} \times S_g^{-m}} \leqslant c < \infty .$$

The connection between $p(\lambda)$ and $\Omega(\lambda)$ is given by the equality

$$<p(\lambda)u,v> = (\Omega(\lambda), v \otimes \bar{u}) \quad (u,v \epsilon S_g^m(R^\infty)) \tag{1.10}$$

From the positive-definiteness of $p(\lambda)$ and with the help of (1.10) and the inclusion $S_g^m \supseteq S_g$, it follows that $\Omega(\lambda)$ is p.d.,

$$(\Omega(\lambda), u \otimes \bar{u}) \geqslant 0 \quad (u \epsilon S_g(R^\infty)) \tag{1.11}$$

Hence, from (1.8) and the form of $B_k$ it follows that $\Omega(\lambda)$ satisfies the relations

$$(\Omega(\lambda), ip^{-1}(x_k) \frac{\partial}{\partial x_k} (p(x_k)v)(x)\overline{u(y)} - \lambda_k v(x)\overline{u(y)} = 0 \tag{1.12}$$

$$(\Omega(\lambda), \overline{v(x) ip^{-1}(y_k) \frac{\partial}{\partial y_k} (p(y_k)u)(y)} - \lambda_k v(x)\overline{u(y)}) = 0 \tag{1.13}$$

From (1.7) and (1.10), K has the integral representation

$$K = \int_{R^\infty} \Omega(\lambda)\, d\sigma(\lambda) \tag{1.14}$$

which converges in the sense of the space $S_g^{-m} \otimes S_g^{-m}$.

Now, we seek the solution of the system of equations (1.12) and (1.13) in the sense or generalized functions. Namely, $\Omega(\lambda) = \lim_{n\to\infty} \Omega_n(\lambda)$ where $\Omega_n(\lambda)$ is the corresponding cylindrical kernel which is obtained from the representation of the kernel $\Omega(\lambda)$ in the form of a series excluding the terms which contain variables with large number n and its convergence takes place in $S_g^{-m} \otimes S_g^{-m}$.

If $u_n(y_1, \ldots, y_n)$ and $v_n(x_1, \ldots, x_n)$ are cylindrical functions from $S_g(R^\infty)$ then for $u, v \in S_g(R^\infty)$ we have

$$(\Omega_n(\lambda), v \otimes \bar{u}) = (\Omega_n(\lambda), v_n \otimes \overline{u_n})$$

Therefore, we have the following system of equations:

$$(\Omega_n(\lambda), ip^{-1}(x_k)\frac{\partial}{\partial x_k}(p(x_k)\, v(x_k)) \otimes \overline{u(y)} - \lambda_k v(x)\overline{u(y)}) = 0$$
$$k = 1, 2, \ldots.$$

and an analogous equation for u.

Now, if

$$\hat{\Omega}_n(\lambda) = (U_{n,x} \otimes U_{n,y})\Omega_n(\lambda),\ \Psi_n = U_n v_n\,;\phi_n = U_n u_n \qquad (1.15)$$

where $U_n = p(x_1) \ldots p(x_k)$, we have

$$(\hat{\Omega}_n(\lambda).\ (i\frac{\partial}{\partial x_k}\Psi_n) \times \bar{\phi}_n - \lambda_k \Psi_n \otimes \phi_n)_{L_2(R^n x R^n, dg(x) \times dg(y))}$$
$$= 0,\ k = 1, 2, \ldots \qquad (1.16)$$

and a similar equation is obtained for $\phi_n$. Applying to (1.16) the generalized solution for a system of differential equations, we find that $\Omega_n(\lambda)$ is an ordinary function with 2n variables and has the form

$$\Omega_n(\lambda, x_1, \ldots, x_n; y_1, \ldots y_n)$$
$$= [\pi^n e^{\sum_1^n x_k^2} e^{\sum_1^n y^2_k}]\, e^{i\sum_1^n \lambda_k (x_k, y_k)} \hat{\Omega}_n(\lambda, 0, \ldots, 0).$$

$$\text{i.e. } \Omega_n(\lambda) = \pi^n e^{\sum_1^n x^2_k} e^{\sum_1^n y_k^2} \hat{\Omega}_n(\lambda, 0, \ldots, 0) e^{i\sum_1^n \lambda_k (x_k - y_k)}$$

or

$$\Omega_n(\lambda, x, y) = e^{\sum_1^n x_k^2} e^{\sum_1^n y_k^2} \hat{\Omega}_n(\lambda, 0, \ldots, 0)\, e^{i\sum_1^n \lambda_k (x_k - y_k)} \qquad (1.17)$$

Namely, $(\Omega_n(\lambda), 1 \otimes 1) = (\Omega(\lambda), 1 \otimes 1)$

$$= \left(\int_{R^n} e^{\sum_1^n x_k^2} e^{i\sum_1^n \lambda_k x_k} \left(\frac{2\pi}{3}\right)^{-n/2} e^{-3/2 \sum_1^n x_k^2} dx_1 \dots dx_n\right)$$

$$\left(\int_{R^n} e^{\sum_1^n y_k^2} e^{-i\sum_1^n \lambda_k y_k} \left(\frac{2\pi}{3}\right)^{-n/2} e^{-3/2 \sum_1^n y_k^2} dy_1 \dots dy_n\right)\Omega_n(\lambda, 0, \dots, 0)$$

$$= \frac{(2\pi)^n}{(2\pi/3)^n} \exp\left\{- \sum_1^n \lambda_k^2\right\} \Omega_n(\lambda, 0, \dots, 0).$$

But $(\Omega(\lambda), 1 \otimes 1) = c(\lambda) \geqslant 0$. Then evidently,

$$\Omega_n(\lambda, 0, \dots, 0) = c(\lambda) \frac{(2\pi/3)^n}{(2\pi)^n} e^{\sum_1^n x_k^2}$$

So, we have obtained the general form for $\Omega_n(\lambda)$;

$$\Omega_n(\lambda) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\sum_1^n x_k^2} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\sum_1^n y_k^2} e^{\sum_1^n \lambda^2{}_k} \left(\frac{2\pi}{3}\right)^n c(\lambda) e^{i\sum_1^n \lambda_k (x_k - y_k)} \tag{1.18}$$

Considering (1.18) we can write (1.14) in the form

$$K = \lim_{n \to \infty} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\sum_1^n x_k^2} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\sum_1^n y_k^2} \int_{R^n} e^{i\sum_1^n \lambda_k (x_k - y_k)}$$

$$e^{\sum_1^n \lambda_k^2} \left(\sqrt{\frac{2\pi}{3}}\right)^{2n} c(\lambda) d\sigma(\lambda) \tag{1.19}$$

Now, using the Fournier-Venar transform and from (1.19) we have

$$(K,\upsilon \otimes \overline{u}) = \lim_{n\to\infty} \int_{R^n} \hat{u}_n(\lambda)\hat{\overline{v}}_n(\lambda)c(\lambda)d\sigma(\lambda)$$

$$= \int_{R^\infty} \hat{u}(\lambda)\hat{\overline{v}}(\lambda)d\rho(\lambda) \tag{1.20}$$

Finally, we have to obtain the form of the measure $d\rho(\lambda)$. For this purpose we use $\| \Omega(\lambda) \|_{S_g^{-m} \otimes S_g^{-m}} \leqslant c < \infty$.

According to (1.18) it is sufficient to see the norm in the elements of $S_g^{-m}$ with

$$e^{i \sum_1^n \lambda_k x_k} e^{\sum_1^n x_k^2} = \omega_n(\lambda).$$

Then,

$$\| \Omega(\lambda) \|_{S_g^{-m} \otimes S_g^{-m}} = \lim_{n\to\infty} \left(\frac{2\pi}{3}\right)^n \frac{1}{(2\pi)^n} c(\lambda) e^{\sum_1^n \lambda_k^2} \|\omega_n\|^2_{S_g^{-m}}$$

According to [9], we can expand $\omega_n(\lambda)$ in the form of a series by $\{h_\alpha(x)\}_\alpha$

$$\omega_n(\lambda) = \sum_\alpha c_\alpha^{(n)} h_\alpha(x) \tag{1.21}$$

where

$$c_\alpha^{(n)} = \int_{R^n} \omega_n(\lambda) h_\alpha(x) dg(x)$$

$$= \prod_{k=1}^n \int_{R'} e^{i\lambda_k x_k} h_{\alpha_k}(x_k) e^{x_k^2} \frac{1}{\sqrt{\pi}} e^{-x_k^2} dx_k$$

$$= \prod_{k=1}^n \int_{R'} e^{i\lambda_k x_k} h_{\alpha_k}(x_k) \frac{1}{\sqrt{2\pi}} e^{-x_k^2/2} dx_k$$

$$= \left(\frac{\sqrt{2\pi}}{\sqrt{2\pi}}\right)^n i^{|\alpha|} h_\alpha(\lambda^{(n)}).$$

and $\lambda^{(n)} = (\lambda_1, \ldots, \lambda_n, 0, 0, \ldots)$. Therefore, by using the definition of the space $S_g^{-m}$ we obtain

$$\|\Omega(\lambda)\|_{S_g^{-m} \otimes S_g^{-m}} = (\sum_\alpha \frac{h_\alpha^2(x)}{(\alpha_i)^{m_i}} \frac{1}{(3\pi)^n} c(\lambda) \tag{1.22}$$

Evidently, $c(\lambda) \leqslant c_1 \|\delta_\lambda\|^{-2}_{S_g^{-m}}$, where $\delta_\lambda = \sum_\alpha h_\alpha(\lambda) h_\alpha(\,.\,)$ is a $\delta$-function from $S_g'(R^\infty)$.

One can show that for $\|\delta_\lambda\|_{S_g^{-m}}$ we have the inequality

$$a\,(m) \exp \sum_1^\infty \frac{\lambda_k^2/2}{m_k} \leqslant \|\delta_\lambda\|^2_{S_g^{-m}} \leqslant \|\delta_0\|^2_{S_g^{-m}} \exp \sum_1^\infty \frac{\lambda_k^2/2}{n(m_k)}\ , \tag{1.23}$$

where $n(m_k) \to \infty$ as $m_k \to \infty$. Now, let $m = (m_k)_{k=1}^\infty$ be chosen so large that $\|\delta_\omega\|^2_{S_g^{-m}} < \infty$ and this is usually possible by preserving the construction of (1.23) and hence

$$c(\lambda) \leqslant c \exp \sum_1^\infty \frac{\lambda_k^2/2}{n(m_k)} \tag{1.24}$$

Formula (1.24) gives the expression of the measure $d\rho(\lambda)$, corresponding to translation-invariant p.d. generalized kernel, given by the set

$$\hat{R}_k^\infty = \left\{ \lambda \in R^\infty \mid \frac{\lambda_k^2}{n(m_k)} < \infty \right\}$$

Then,

$$(K, v \otimes \overline{u}) = \int_{\hat{R}_k^\infty} (\hat{u}\,\overline{\hat{v}})(\lambda)\, d\rho\,(\lambda) \tag{1.25}$$

Conversely, every kernel in the (1.25) is a translation-invariant p.d. kernel on $S_g' \otimes S_g'$ (see [1]). It can be shown that the measure $d\rho(\lambda) = c(\lambda) d\sigma(\lambda)$ generates a kernel $K \epsilon S_g' \otimes S_g'$.

## 2. The Case of $K \epsilon\, \sigma_g' \otimes \sigma_g'$

Let us consider the kernel $K \epsilon\, \sigma_g' \otimes \sigma_g'$ and then following the preceding subsection we obtain

$$\|\Omega(\lambda)\|_{\sigma_g^{-m} \otimes \sigma_g^{-m}} = c(\lambda) e^{1/2 \sum_1^n \lambda^2{}_k/m_k}$$

and therefore,

$$c(\lambda) \leq c\, e^{-\sum_{1}^{\infty} \frac{\lambda_k^2/2}{m_k}} \tag{2.1}$$

and hence we can prove the following theorem:

*Theorem 2*

Every translation-invariant p.d. kernel $K \epsilon \sigma'_g \otimes \sigma'_g$ admits the representation

$$(K, \upsilon \otimes \overline{u}) = \int_{R_k^\infty} \hat{u}(\lambda) \overline{\hat{v}}(\lambda)\, d\rho(\lambda) \tag{2.2}$$

where $d\rho(\lambda) = c(\lambda)\, d\sigma(\lambda)$, $c(\lambda)$ satisfies (2.1)

From Theorem 1 and Theorem 2 we have:

*Theorem 3*

Every Translation-invariant p.d. kernel $k \epsilon \sigma'_g \otimes \sigma'_g$ is contained in $S'_g \otimes S'_g$

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