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A 6th-order local approximation by a cubic curve

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تقريب محلي من الرتبة السادسة بإستعمال منحنى من الدرجة الثالثة

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نقدم في هذا البحث برنامج مكتوب بلغة رمزية لإيجاد منحنى بشكل حدي بارميتري من الدرجة الثالثة والذي يلامس المنحنى المعطي من الرتبة السادسة عند نقطة التقريب . هذه الطريقة تعطي تقريباً محلياً من الرتبة السادسة لمنحنى إقتران معطى . نبين كذلك كيفية إستخدام البرنامج بتوضيح بعض الأمثلة .

KEY WORDS **:** Bernstein polynomials, Computer Algebra, computer graphics, computer aided geometric design.

ABSTRACT

We derive and present a MAPLE program for the construction of a parametric cubic polynomial curve which has sixth-order contract with a given smooth curve at a given point, thus providing a local sixth-order approximation to the given curve. Such a program is bound to be useful. We give also examples to show the implementation of the method.

INTRODUCTION

In this article we are dealing with computer algebra systems to get good results on approximating curves with low degree and high accuracy. This approach provides a strong tool in order to get representations and approximations of complicated forms of curves and to give a parametric form of curves which are given in implicit form. The Bernstein polynomials of degree 3 on an interval $I = [0, t_o]$ are defined by :

$$B_i^3(t) = \binom{3}{i}(t_o - t)^i t^{3-i}, \quad i = 0,1,2,3$$

For more on Bernstein Polynomials see [1].

Two curves

$$C : \text{given by } x : I \to \Re^2$$

$$P : \text{given by } y : I \to \Re^2$$

have contact of order $k > 0$ at $t = t^o \in I$ iff there exists reparametrizations α and β, which satisfy :

$$\frac{d^i(x(\alpha(t_o)))}{dt^i} = \frac{d^i(y(\beta(t_o)))}{dt^i}, \quad i = 0,1, \dots, k.$$

In [2], [3], [4], [6], [7], [8], [9], these are related results.

2. CONSTRUCTION OF THE APPROXIMANT

In [10] a cubic approximant in Bernstein form is constructed. In this article we write a Maple-Program to construct a cubic Polynomial approximant in Bernstein form, which has a 6^{th} order of contact at the point of approximation. For a given curve C, we want to find a curve P, which has contact of order 6 with C. Geometric continuity is an appropriate way to fit together two curves at a common point, and provides also additional free parameters for geometric modeling. It does not depend on the parameterization of the curves. For a given regular smooth planar curve :

$$C : t \to \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \qquad t \in I$$

a cubic polynomial curve P is required :

$$P : t \to \begin{pmatrix} \zeta(t) \\ \eta(t) \end{pmatrix}, \qquad t \in I$$

which has a 6th order of contact at a common point in I, where $\zeta(t)$ and $\eta(t)$ are cubic polynomials in Bernstein form; i.e.

$$\zeta(t) := a_o B_o^3(t) + a_1 B_1^3(t) + a_2 B_2^3(t) + a_3 B_3^3(t)$$

$$\eta(t) := c_o B_o^3(t) + c_1 B_1^3(t) + c_2 B_2^3(t) + c_3 B_3^3(t)$$

It is always possible to assume that

$$(f(t_o), g(t_o)) := (0,0), \ (f'(t_o), g'(t_o)) := (\lambda, 0),$$

where $\lambda \neq 0$, so that for $t \in I$ we can parametrize C in the form

$$C : t \to \zeta(t) \to \left(\frac{\zeta(t)}{\phi(\phi(t))}\right), \quad t \in I$$

Thus P approximates C near t_o with order 6 iff P and C have 6^{th} order of contact at t_o i.e. iff

$$\left(\frac{d}{dt}\right)^j \{\phi(\zeta(t) - \eta(t)\}\Big|_{t = t_o}, j = 0, 1, 2, \dots, 5 \qquad (1)$$

So, to get the required order of approximation, we have to solve

$$\phi_1(\zeta(t_o))\zeta_1(t_o) - \eta_1(t_o) = 0,$$

$$\phi_2(\zeta(t_o))\zeta_1^2(t_o) + \phi_1(\zeta(t_o))\zeta_2(t_o) - \eta_2(t_o) = 0,$$

$$\phi_3(\zeta(t_o))\zeta_1^3(t_o) + 3\phi_2(\zeta(t_o))\zeta_1(t_o)\zeta_2(t_o) + \phi_1(\zeta(t_o))\zeta_3(t_o) - \eta_3(t_o) = 0,$$

$$\phi_4(\zeta(t_o))\zeta_1^4(t_o) + 6\phi_3(\zeta(t_o))\zeta_1^2(t_o)\zeta_2(t_o) + 3\phi_2(\zeta(t_o))\zeta_2^2(t_o) + 4\phi_2(\zeta(t_o))\zeta_1(t_o)\zeta_3(t_o) = 0,$$

$$\phi_5(\zeta(t_o))\zeta_1^5(t_o) + 10\phi_4(\zeta(t_o))\zeta_1^3(t_o)\zeta_2(t_o) + 15\phi_3(\zeta(t_o))\zeta_1(t_o)\zeta_2^2(t_o) + 10\phi_3(\zeta(t_o))\zeta_1^2(t_o)\zeta_3(t_o) + 10\phi_2(\zeta(t_o))\zeta_2(t_o)\zeta_3(t_o) = 0$$

$$\zeta(t_o) = 0$$

$$\eta(t_o) = 0$$

where $\phi_i(\zeta(t_o))$, $\zeta_i(t_o)$ and $\eta_i(t_o)$ are the i^{th} derivatives of ϕ, ζ and η respectively at $t = t_o$. Since, we have a free parameter we set $\zeta_1 = s$, for some $s \neq 0$ and thus obtain

$$\phi_1(0)\, s - \eta_1(t_o) = 0,$$

$\phi_2(0)\, s^2 + \phi_1(0)\, \zeta_2(t_o) - \eta_2(t_o) = 0,$

$\phi_3(0)\, s^3 + 3\phi_2(0)\, s\, \zeta_2(t_o) + \phi_1(0)\, \zeta_3(t_o) - \eta_3(t_o) = 0,$

$\phi_4(0)\, s^4 + 6\phi_3(0)\, s^2\, \zeta_2(t_o) + 3\phi_2(0)\, \zeta_2^2(t_o)$

$+ 4\phi_2(0)\, s\, \zeta_3(t_o) = 0,$

$\phi_5(0)\, s^5 + 10\phi_4(0)\, s^3\, \zeta_2(t_o) + 15\phi_3(0)\, s\, \zeta_2^2(t_o)$

$+ 10\phi_3(0)\, s\, \zeta_3(t_o) + 10\phi_2(0)\, \zeta_2(t_o)\, \zeta_3(t_o) = 0$

$\zeta(t_o) = 0$

$\eta(t_o) = 0$

To simply the notations, we use the abbreviation :

$$\phi_i = \phi_i(0), \quad i = 0, 1, ..., 5$$

3. MAPLE - Program

In this section we give a Maple-Program to solve the system in the last section.

```
# Cubic polynomial approximation in Bernstein form

# First the cubic Bernstein polynomials are defined

bern : = proc(n, i, t)

bionomial (n,i,)*(t0 - t)^i * t^(n-i)

end :

# Input t0 : node of approximation, and the value of the
# function at t0.
# F [i] : i = 1, 2, 3, 4, 5 : the associated derivatives of the
# function to be approximated at t0.
# The cubic polynomials X(t) and Y(t) are defined by X and
# Y respectively.
X : = a0 * bern(3,0,t) + al * bern(3,1,t) + a2 * bern(3,2,t)
+ a3 * bern(3,3,t) ;
Y : = b0 * bern(3,0,t) + bl * bern(3,1,t) + b2 * bern(3,2,t)
+ b3 * bern(3,3,t) ;
# The derivatives of X and Y.
der X : = proc(j, t)
diff(X, t$j)
end :
der Y : = proc(j, t)
diff(Y, t$j)
end :
# We simplify the notations of derivatives of X and Y and
# substitute t = t0.
x : = array (1..3);
y : = array (1..3);
F : = array (1..5);
for j from 1 to 3 do
x [j] : = subs (t = t0, derX (j, t));
y [j] : = subs (t = t0, derY (j, t));
od;
# Now we substitute in the system of equations
e1 : = subs (t = t0, X);
e2 : = subs (t = t0, Y);
e3 : = x[1] - s;
e4 : = F[1] * s - y[1];
e5 : = F[2] * s^2 + F[1] * x[2] - y[2];
e6 : = F[3] * s^3 + 3*F[2]*s*x[2] + F[1] * x[3] - y[3];
e7 : = F[4] * s^4 + 6* F[3]*s^2*x [2] +3*F [2]*x[2]^2
     + 4* F [2]*s*x [3] ;
e8 : = F [5]*s^5 + 10* F [4]*s^3x [2] + 15*F [3]*s*x [2]^2
     + 10*F [3]* s^2*x [3] + 10*F [2]*x [2]*x [3] ;
sol : = solve ({e1 = t0, e2 = y0, e3 = 0, e4 = 0, e5 = 0, e6 =
0, e7 = 0, e8 = 0} , {a0, a1, a2, a3, b0, b1, b2, b3}) ;
assign (sol) ;
```

4. Solution of the System

Applying the Maple-Program in the last section gives the following output :

$c_\circ = 0,$

$a_\circ = 0,$

$$a1 = \frac{-s}{3t_\circ^2}$$

$$c1 = \frac{-\phi_1 s}{3t_\circ^2}$$

$a2 = R,$

$$a3 = \frac{72\phi_2 s^2 + \phi_4 s^4 t_\circ^2 + 24\phi_3 s^3 t_\circ + 36\phi_3 s^2 R t_\circ^3 + 216\phi_2 s R t_\circ^2 + 108\phi_2 R^2 t_\circ^4}{24\phi_2 s t_\circ^2}$$

$$c2 = \frac{\phi_2 s^2 + 6\phi_1 R t_\circ}{6t_\circ},$$

$$c3 = \frac{1}{24\phi_2 s t_\circ^2}(-4\phi_3 s^4 t_\circ^2 \phi_2 - 36\phi_2^2 s^3 t_\circ - 72\phi_2^2 s R t_\circ^3 + \phi_1\phi_4 s^4 t_\circ^2$$

$$+ 24\phi_1\phi_3 s^3 t_\circ + 36\phi_1\phi_3 s^2 R t_\circ^3 + 72\phi_1\phi_2 s^2 + 216\phi_1\phi_2 s R t_\circ^2$$

$$+ 108\phi_1\phi_2 R^2 t_\circ^4),$$

where R is a real root of the following cubic equation :

$$3240\phi_2^2 t_\circ^6 Z^3 + (540\phi_3\phi_2 s^2 t_\circ^5 + 6480\phi_2^2 s t_\circ^4)\, Z^2$$

$$+ (720\phi_3\phi_2 s^3 t_\circ^3 - 90\phi_2\phi_4 s^4 t_\circ^4 + 180\phi_3^2 s^4 t_\circ^2 + 4320\phi_2^2 s^2 t_\circ^2)\, Z$$

$$+ (-2\phi_5\phi_2 s^6 t_\circ^3 - 60\phi_2\phi_4 s^5 t_\circ^2 + 5\phi_3\phi_4 s^6 t_\circ^3 + 240\phi_2\phi_3 s^4 t_\circ$$

$$+ 960\phi_2^2 s^3 + 120\phi_3^2 s^5 t_\circ^2) = 0$$

This equation is of odd degree and has always real roots. The derivation of the program provides a proof of the existence of such a parametric cubic interpolant.

By applying the Maple-Program to required data consisting of values and derivatives of a function at a given point it is possible to find a cubic polynomial approximant in Berntein form which approximates with order 6. The computer algebra program enables us to mention the following theorem :

Theorem : A planar curve in C^5 can be approximated locally by a parametrically defined cubic polynomial curve with order 6.

5. Examples

We find a cubic approximant of the circle at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ by applying the Maple-Program in section 5. One specializes the program by setting :

$$t0 := \frac{1}{\sqrt{2}}, \qquad s := 1, \qquad y0 := \frac{1}{\sqrt{2}}$$

and supplies the program with the first, second, third, fourth and fifth derivatives at t0 in F [i]. Executing the program gives three solutions

```
a21 : = .9106836007;   b21 : = 2.422649735;
a31 : = 3.464101596;   b11 : = 2.666666668;
b31 : = 2.000000008;   a01 : = 2.000000001;
b01 : = 2.000000001;   a11 : = 1.333333334;

X1 : a01*t^3 + 3*all* (t0 - t)* t^2 + 3* a21 * (t0 - t)^2* t
+ a31* (t0 - t)^3 :

Y1 : b01*t^3 + 3*bll* (t0 - t)* t^2 + 3* b21 * (t0 - t)^2* t
b31* (t0 - t)^3 :

a22 : = - .2440169385;    b22 : = 3.577350273;
a32 : = - 3.464101620;    b32 : = 1.999999990;
b12 : = 2.666666668;      a02 : = 2.000000001;
b02 : = 2.000000001;      a12 : = 1.333333334;

X2 = : a02 * t^3 + 3*a12* (t0 - t)* t^2 + 3*a22* (t0 - t)^2* t
     + a32* (t0 - t)^3 :

Y2 : = b02 * t^3 + 3*b12* (t0 - t)* t^2 + 3*b22* (t0 - t)^2* t
     + b32* (t0 - t)^3 :

a23 : = .3333333400;    b33 : = 2.749999999;
b13 : = 2.666666668;    a03 : = 2.000000001;
b03 : = 2.000000001;    a13 : = 1.333333334;
b23 : = 2.999999995;    a33 : = - .7499999590;

X3 : = a03 * t^3 + 3*a13* (t0 - t)* t^2 + 3*a23* (t0 - t)^2* t
     + a33* (t0 - t)^3 :

Y3 : = b03 * t^3 + 3*b13* (t0 - t)* t^2 + 3*b23* (t0 - t)^2* t
     + b33* (t0 - t)^3 :
```

which represent three curves t $\rightarrow$ (X1(t), Y1(t)),

t $\rightarrow$ (X2(t), Y2(t)), t $\rightarrow$ (X3(t), Y3(t))

By drawing these curves, we see that the curve

t $\rightarrow$ (X3(t), Y3(t))

coincides graphically with the circle in the first quadrant.

In the second example, the function f (x) = exp (x^2) is approximated at (1 , E) where E = e. Applying the program in section 5 gives two complex solutions and the following real solution :

R : = 0.1689708683; a0 : = 1; b0 : = E; a1 : = 2/3; b2 : = 2*E*R ;

b3 : = 25/18*E-8*E*R + 9*E* R^2 ; b1 : = 1/3*E ;

a3 : = -59/36 + 5*R + 9/2*R^2; a2 : = R ;

X1 : = a0* t^3 + 3*a1 * (t0 - t)* t^2 + 3* a2* $(t0 - t)^2$* t + a3* $(t0 - t)^3$;

Y1 : = b0* t^3 + 3*b1 * (t0 - t) * t^2 + 3* b2* $(t0 - t)^2$ * t + b3* $(t0 - t)^3$;

which represents an acceptable solution around the point of approximation (1, E).

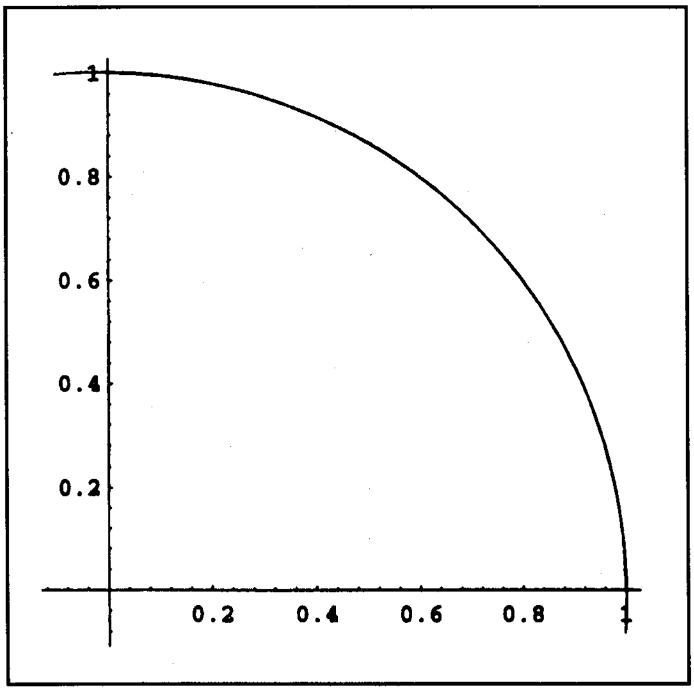

Fig 1 : Approximation of the circle

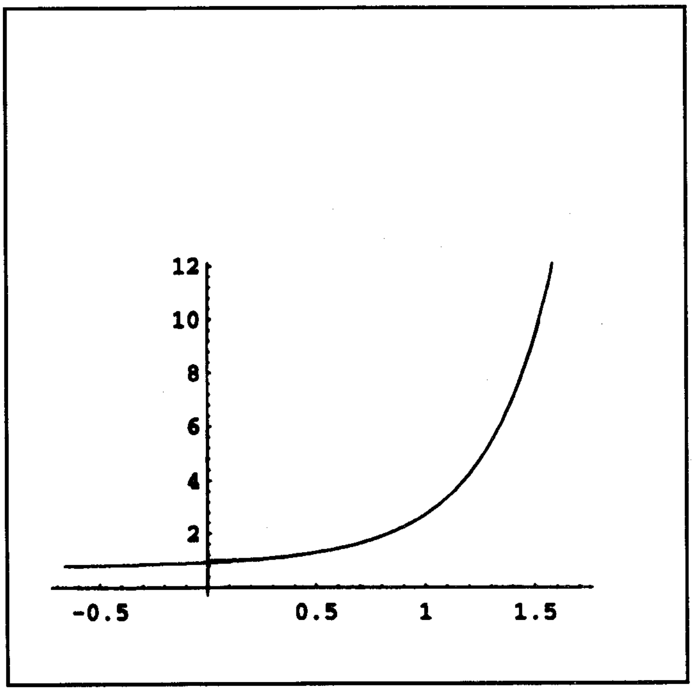

Fig 2 : Approximation of exp (x ^ 2)

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