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A generalized differential transform method for linear partial differential equations of fractional order

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Abstract

In this letter we develop a new generalization of the two-dimensional differential transform method that will extend the application of the method to linear partial differential equations with space- and time-fractional derivatives. The new generalization is based on the two-dimensional differential transform method, generalized Taylor's formula and Caputo fractional derivative. Several illustrative examples are given to demonstrate the effectiveness of the present method. The results reveal that the technique introduced here is very effective and convenient for solving linear partial differential equations of fractional order.

Keywords: Differential transform method; Generalized Taylor formula; Caputo fractional derivative; Fractional differential equations

1. Introduction

Mathematical modeling of many physical systems leads to linear and nonlinear fractional differential equations in various fields of physics and engineering. The numerical and analytical approximations of such systems have been intensively studied since the work of Padovan [1]. Recently, several mathematical methods including the Adomian decomposition method [2–8], variational iteration method [6–9], homotopy perturbation method [10,11] and fractional difference method [12] have been developed to obtain exact and approximate analytic solutions. Some of these methods use transformation in order to reduce equations into simpler equations or systems of equations and some other methods give the solution in a series form which converges to the exact solution.

In this work, we develop a semi-numerical method based on the two-dimensional differential transform method [13–15], generalized Taylor's formula [16] and Caputo fractional derivative [17]. This new generalization of the two-dimensional differential transform method will extend the application of the method to linear partial differential equations of fractional order. To the authors' knowledge, this work represents the first application of the generalized differential transform method to solve partial differential equations with space- and time-fractional derivatives.

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There are several definitions of a fractional derivative of order $\alpha > 0$ [12,17]. The Caputo fractional derivative is defined as

$$D_a^{\alpha} f(x) = J_a^{m-\alpha} D^m f(x), \tag{1.1}$$

where $m-1 < \alpha \le m$. Here D^m is the usual integer differential operator of order m and J_a^{μ} is the Riemann–Liouville integral operator of order $\mu > 0$, defined by

$$J_a^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\mathrm{d}t, \quad x > 0. \tag{1.2}$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem [12]. For more information on the mathematical properties of fractional derivatives and integrals one can consult the aforementioned references.

2. Generalized two-dimensional differential transform method

The differential transform method was first introduced by Zhou [13] who solved linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method computationally takes a long time for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of ordinary or partial differential equations. The method is well addressed in [13–15].

In this section we shall derive the generalized two-dimensional differential transform method that we have developed for the numerical solution of linear partial differential equations with space- and time-fractional derivatives. The proposed method is based on a generalized Taylor's formula (for details, see [16]).

Consider a function of two variables $u(x, y)$, and suppose that it can be represented as a product of two single-variable functions, i.e. $u(x, y) = f(x)g(y)$. On the basis of the properties of generalized two-dimensional differential transform [14,15], the function $u(x, y)$ can be represented as

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{\infty} F_{\alpha}(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_{\beta}(h)(y-y_0)^{h\beta} \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}, \end{aligned} \tag{2.1}$$

where $0 < \alpha, \beta \le 1$, $U_{\alpha,\beta}(k, h) = F_{\alpha}(k)G_{\beta}(h)$ is called the spectrum of $u(x, y)$. The generalized two-dimensional differential transform of the function $u(x, y)$ is as follows:

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \left[(D_{x_0}^{\alpha})^k (D_{y_0}^{\beta})^h u(x,y)\right]_{(x_0,y_0)}, \tag{2.2}$$

where $(D_{x_0}^{\alpha})^k = D_{x_0}^{\alpha} D_{x_0}^{\alpha} \cdots D_{x_0}^{\alpha}$, k-times. In this work, the lower case $u(x, y)$ represents the original function while the upper case $U_{\alpha,\beta}(k, h)$ stands for the transformed function. On the basis of the definitions (2.1) and (2.2), we have the following results:

Theorem 2.1. *Suppose that $U_{\alpha,\beta}(k, h)$, $V_{\alpha,\beta}(k, h)$ and $W_{\alpha,\beta}(k, h)$ are the differential transformations of the functions $u(x, y)$, $v(x, y)$ and $w(x, y)$, respectively;*

(a) *if $u(x, y) = v(x, y) \pm w(x, y)$, then $U_{\alpha,\beta}(k, h) = V_{\alpha,\beta}(k, h) \pm W_{\alpha,\beta}(k, h)$,*
(b) *if $u(x, y) = av(x, y)$, $a \in \mathbf{R}$, then $U_{\alpha,\beta}(k, h) = aV_{\alpha,\beta}(k, h)$,*
(c) *if $u(x, y) = v(x, y)w(x, y)$, then $U_{\alpha,\beta}(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} V_{\alpha,\beta}(r, h-s)W_{\alpha,\beta}(k-r, s)$,*
(d) *if $u(x, y) = (x-x_0)^{n\alpha}(y-y_0)^{m\beta}$, then $U_{\alpha,\beta}(k, h) = \delta(k-n)\delta(h-m)$,*
(e) *if $u(x, y) = D_{x_0}^{\alpha} v(x, y)$, $0 < \alpha \le 1$, then $U_{\alpha,\beta}(k, h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} U_{\alpha,\beta}(k+1, h)$.*

Theorem 2.2. *If $u(x, y) = f(x)g(y)$ and the function $f(x) = x^{\lambda}h(x)$, where $\lambda > -1$, $h(x)$ has the generalized Taylor series expansion $h(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{\alpha k}$, and*

(a) $\beta < \lambda + 1$ *and* α *is arbitrary, or*
(b) $\beta \geq \lambda + 1$, α *is arbitrary and* $a_n = 0$ *for* $n = 0, 1, \dots m - 1$, *where* $m - 1 < \beta \leq m$.

Then the generalized differential transform (2.2) becomes

$$U_{\alpha,\beta}(k, h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} \left[D_{x_0}^{\alpha k} (D_{y_0}^{\beta})^{h} u(x, y) \right]_{(x_0, y_0)}. \tag{2.3}$$

Proof. The proof follows immediately from the fact that $D_{x_0}^{\gamma_1} D_{x_0}^{\gamma_2} f(x) = D_{x_0}^{\gamma_1+\gamma_2} f(x)$, under the conditions given in Theorem 2.2. □

Theorem 2.3. *If $v(x, y) = f(x)g(y)$, the function $f(x)$ satisfies the conditions given in Theorem 2.2, and $u(x, y) = D_{x_0}^{\gamma} v(x, y)$, then*

$$U_{\alpha,\beta}(k, h) = \frac{\Gamma(\alpha(k + 1) + \gamma)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}(k + \gamma/\alpha, h). \tag{2.4}$$

3. Applications and results

In this section we consider a few examples that demonstrate the performance and efficiency of the generalized differential transform method for solving linear partial differential equations with time- or space-fractional derivatives.

Example 3.1. Consider the following linear inhomogeneous time-fractional equation:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 2t^{\alpha} + 2x^2 + 2, \quad t > 0, \tag{3.1}$$

where $0 < \alpha \leq 1$, subject to the initial condition

$$u(x, 0) = x^2. \tag{3.2}$$

Suppose that the solution $u(x, t)$ can be represented as a product of single-valued functions. Selecting $\beta = 1$ and applying the generalized two-dimensional differential transform to both sides of Eq. (3.1), the linear inhomogeneous time-fractional equation (3.1) transforms to

$$U_{\alpha,1}(k, h + 1) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha(h + 1) + 1)} \Bigg[-\sum_{r=0}^{k} \sum_{s=0}^{h} \delta(r - 1)\delta(h - s)(k - r + 1)U_{\alpha,1}(k - r + 1, s) - (k + 1)(k + 2)U_{\alpha,1}(k + 2, h) + 2\delta(k)\delta(h - 1) + 2\delta(k - 2)\delta(h) + 2\delta(k)\delta(h) \Bigg]. \tag{3.3}$$

The generalized two-dimensional differential transform of the initial condition (3.2) is

$$U_{\alpha,1}(k, 0) = \delta(k - 2). \tag{3.4}$$

Utilizing the recurrence relation (3.3) and the transformed initial condition (3.4), we get $U_{\alpha,1}(2, 0) = 1$, $U_{\alpha,1}(0, 2) = 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}$ and $U_{\alpha,1}(k, h) = 0$ for $k \neq 2, h \neq 2$. Therefore, according to (2.1), the solution of Eq. (3.1) is given by

$$u(x, t) = x^2 + 2\frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} t^{2\alpha}, \tag{3.5}$$

which is the exact solution of the linear inhomogeneous time-fractional equation (3.1).

Table 1
The first components of $U_{1,1/2}(k,h)$ for Eq. (3.6)

	$U_{1,1/2}(0,h)$	$U_{1,1/2}(1,h)$	$U_{1,1/2}(2,h)$	$U_{1,1/2}(3,h)$	$U_{1,1/2}(4,h)$	$U_{1,1/2}(5,h)$
$U_{1,1/2}(k,0)$	1	0	1	$\frac{1}{\Gamma(5/2)}$	0	$\frac{1}{\Gamma(7/2)}$
$U_{1,1/2}(k,1)$	-1	0	-1	$-\frac{1}{\Gamma(5/2)}$	0	$-\frac{1}{\Gamma(7/2)}$
$U_{1,1/2}(k,2)$	$\frac{1}{2!}$	0	$\frac{1}{2!}$	$\frac{1}{2!\Gamma(5/2)}$	0	$\frac{1}{2!\Gamma(7/2)}$
$U_{1,1/2}(k,3)$	$-\frac{1}{3!}$	0	$-\frac{1}{3!}$	$-\frac{1}{3!\Gamma(5/2)}$	0	$-\frac{1}{3!\Gamma(7/2)}$
$U_{1,1/2}(k,4)$	$\frac{1}{4!}$	0	$\frac{1}{4!}$	$\frac{1}{4!\Gamma(5/2)}$	0	$\frac{1}{4!\Gamma(7/2)}$
$U_{1,1/2}(k,5)$	$-\frac{1}{5!}$	0	$-\frac{1}{5!}$	$-\frac{1}{5!\Gamma(5/2)}$	0	$-\frac{1}{5!\Gamma(7/2)}$

Example 3.2. Consider the following linear space-fractional telegraph equation:

$$\frac{\partial^{1.5} u}{\partial x^{1.5}} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \quad x > 0, \tag{3.6}$$

subject to the initial conditions

$$u(0,t) = \exp(-t), \qquad u_x(0,t) = \exp(-t). \tag{3.7}$$

Suppose that the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t) = v(x)w(t)$ where the function $v(x)$ satisfies the conditions given in Theorem 2.2. Selecting $\alpha = 1$, $\beta = 0.5$ and applying the generalized two-dimensional differential transform to both sides of Eq. (3.6), the linear space-fractional telegraph equation (3.6) transforms to

$$\begin{aligned} U_{1,1/2}(k+3,h) = {} & \frac{\Gamma(k/2+1)}{\Gamma(k/2+5/2)} \\ & \times \left[(h+1)(h+2)U_{1,1/2}(k,h+2)(h+1)U_{1,1/2}(k,h+1) + U_{1,1/2}(k,h)\right]. \end{aligned} \tag{3.8}$$

The generalized two-dimensional differential transforms of the initial conditions (3.7) are given by

$$\begin{aligned} &U_{1,1/2}(0,h) = (-1)^h/h!, \\ &U_{1,1/2}(1,h) = 0, \\ &U_{1,1/2}(2,h) = (-1)^h/h!. \end{aligned}$$

Utilizing the recurrence relation (3.8) and the transformed initial conditions, the first few components of $U_{1,1/2}(k,h)$ are calculated and given in Table 1.

Therefore, from (2.1), the approximate solution of the linear space-fractional Telegraph equation (3.6) can be derived as

$$\begin{aligned} u(x,t) = {} & \left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \frac{1}{5!}t^5\right) + \left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \frac{1}{5!}t^5\right)x \\ & + \left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \frac{1}{5!}t^5\right)\frac{x^{1.5}}{\Gamma(5/2)} \\ & + \left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \frac{1}{5!}t^5\right)\frac{x^{2.5}}{\Gamma(7/2)}, + \cdots \end{aligned}$$

that is

$$u(x,t) = \exp(-t)\left(1 + x + \frac{x^{1.5}}{\Gamma(5/2)} + \frac{x^{2.5}}{\Gamma(7/2)} + \frac{x^3}{\Gamma(4)} + \frac{x^4}{\Gamma(5)} + \frac{x^{4.5}}{\Gamma(11/2)} + \cdots\right), \tag{3.9}$$

which is the same solution as was obtained in [18] using the Adomian decomposition method.

Table 2
The first components of $U_{1/2,1/4}(k,h)$ for Eq. (3.10)

	$U_{1/2,1/4}(0,h)$	$U_{1/2,1/4}(1,h)$	$U_{1/2,1/4}(2,h)$	$U_{1/2,1/4}(3,h)$	$U_{1/2,1/2/4}(4,h)$
$U_{1/2,1/4}(k,0)$	a_0	a_1	a_2	a_3	a_4
$U_{1/2,1/4}(k,1)$	0	0	0	0	0
$U_{1/2,1/4}(k,2)$	b_0	b_1	b_2	b_3	b_4
$U_{1/2,1/4}(k,3)$	$\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3$	$\frac{\Gamma(8/4)}{\Gamma(5/2)\Gamma(3/4)}a_4$	$\frac{\Gamma(9/4)}{\Gamma(5/2)\Gamma(4/4)}a_5$	$\frac{\Gamma(10/4)}{\Gamma(5/2)\Gamma(5/4)}a_6$	$\frac{\Gamma(11/4)}{\Gamma(5/2)\Gamma(6/4)}a_7$
$U_{1/2,1/4}(k,4)$	0	0	0	0	0
$U_{1/2,1/4}(k,5)$	$\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3$	$\frac{\Gamma(8/4)}{\Gamma(7/2)\Gamma(3/4)}b_4$	$\frac{\Gamma(9/4)}{\Gamma(7/2)\Gamma(4/4)}b_5$	$\frac{\Gamma(10/4)}{\Gamma(7/2)\Gamma(5/4)}b_6$	$\frac{\Gamma(11/4)}{\Gamma(7/2)\Gamma(6/4)}b_7$

Example 3.3. Consider the following linear space–time-fractional wave equation:

$$\frac{\partial^{1.5}u}{\partial t^{1.5}} = \frac{1}{2}x^2\frac{\partial^{1.25}u}{\partial x^{1.25}} \quad x > 0,\ t > 0, \tag{3.10}$$

subject to the initial conditions

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad u_t(x,0) = g(x) = \sum_{n=0}^{\infty} b_n x^n. \tag{3.11}$$

Suppose that the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t) = v(x)w(t)$ where the functions $v(x)$ and $w(t)$ satisfy the conditions given in Theorem 2.2. Selecting $\alpha = 0.5$, $\beta = 0.25$ and applying the generalized two-dimensional differential transform to both sides of Eq. (3.10), the linear space–time-fractional wave equation (3.10) transforms to

$$U_{1/2,1/4}(k,h+3) = \begin{cases} \frac{1}{2}\frac{\Gamma(h/2+1)\Gamma(k/4+7/4)}{\Gamma(h/2+5/2)\Gamma(k/4+2/4)}U_{1/2,1/4}(k+3,h), & k \geq 2 \\ 0, & k < 2. \end{cases} \tag{3.12}$$

The generalized two-dimensional differential transforms of the initial conditions (3.11) are given by

$$U_{1/2,1/4}(k,0) = a_k,$$
$$U_{1/2,1/4}(k,1) = 0,$$
$$U_{1/2,1/4}(k,2) = b_k.$$

Utilizing the recurrence relation (3.12) and the transformed initial conditions, the first few components of $U_{1/2,1/4}(k,h)$ are calculated and given in Table 2.

Therefore, from (2.1), the approximate solution of the linear space–time-fractional wave equation (3.10) can be derived as

$$\begin{aligned} u(x,t) = {} & \left(a_0 + b_0 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3 t^{5/2}\right) \\ & + \left(a_1 + b_1 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4 t^{5/2}\right)x^{1/4} \\ & + \left(a_2 + b_2 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5 t^{5/2}\right)x^{2/4} \\ & + \left(a_3 + b_3 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6 t^{5/2}\right)x^{3/4}, \\ & + \left(a_4 + b_4 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7 t^{5/2}\right)x + \cdots. \end{aligned}$$

4. Conclusions

A new generalization of the two-dimensional differential transform method has been developed for linear partial differential equations with space- and time-fractional derivatives. The new generalization is based on the two-dimensional differential transform method, generalized Taylor's formula and Caputo fractional derivative. It may be concluded that this technique is very powerful and efficient in finding the analytical solutions for a large class of linear partial differential equations of fractional order. This technique provides more realistic series solutions as compared with the Adomian decomposition technique.

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