Khan *Fixed Point Theory and Applications* (2018) 2018:8
https://doi.org/10.1186/s13663-018-0633-4

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Iterative approximation of common attractive points of further generalized hybrid mappings

Safeer Hussain Khan*

Dedicated to Professor H.K. Xu for his contributions towards Fixed Point Theory.

*Correspondence:
safeerhussain5@yahoo.com;
safeer@qu.edu.qa
Department of Mathematics,
Statistics and Physics, Qatar
University, Doha, State of Qatar

Abstract

Our purpose in this paper is (i) to introduce the concept of further generalized hybrid mappings, (ii) to introduce the concept of common attractive points (CAP), and (iii) to write and use Picard-Mann iterative process for two mappings. We approximate common attractive points of further generalized hybrid mappings by using iterative process due to Khan (Fixed Point Theory Appl. 2013:69, 2013, https://doi.org/10.1186/1687-1812-2013-69) generalized to the case of two mappings in Hilbert spaces without closedness assumption. Our results are generalizations and improvements of several results in the literature in different ways.

MSC: 47H09; 47H10; 47H99

Keywords: Common attractive points; Further generalized hybrid mappings; Weak convergence

1 Introduction and preliminaries

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and C be a nonempty subset of H. Let T be a mapping of C into H. Recall that the set of fixed points of T is denoted and defined by $F(T) = \{z \in C : Tz = z\}$. Fixed point problems are quite general, which covers many important problems, in particular, variational inequalities, equilibrium problems, and convex optimization problems, for example, see [2–4]. Takahashi and Takeuchi [5] introduced the concept of attractive points in Hilbert spaces. They defined and denoted the set of attractive points as follows:

$$A(T) = \left\{z \in H : \|Tx - z\| \leq \|x - z\|\right\} \quad \text{for all } x \in C.$$

From this definition, neither an attractive point is a fixed point nor conversely. However, for a relation between the two, see Lemmas 1 and 3. Basically this concept was introduced to get rid of the hypothesis of closedness and convexity as used in a well-celebrated Baillon's nonlinear ergodic theorem in Hilbert spaces [6]. They also proved an existence theorem for attractive points without convexity in Hilbert spaces. In these theorems, they used the so-called generalized hybrid mappings (to be defined in the sequel) whose class is larger than the class of nonexpansive mappings used in Baillon's theorem. Since we are interested in the existence theorem, we state it as follows.

Theorem 1 (Takahashi and Takeuchi [5]) *Let H be a Hilbert space and C be a nonempty subset of H. Let $T : C \to C$ be a generalized hybrid mapping. Then T has an attractive point if and only if $\exists z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded.*

Obviously, the hypothesis does not require any closedness or convexity. Takahashi and Takeuchi [5] also gave some properties of the attractive points as follows.

Lemma 1 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping. If $A(T) \neq \emptyset$, then $F(T) \neq \emptyset$.*

Lemma 2 *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $T : C \to H$ be a mapping. Then $A(T)$ is a closed and convex subset of H.*

Later, the following was noted by Takahashi et al. [7] for quasi-non-expansive mappings.

Lemma 3 *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $T : C \to H$ be a quasi-nonexpansive mapping (that is, $\|Tx - z\| \leq \|x - z\|$, $z \in F(T)$). Then $A(T) \cap C = F(T)$.*

Let l^∞ be the Banach space of bounded sequences with supremum norm and $(l^\infty)^*$ be its dual space (set of all continuous linear functionals on l^∞). It is well known that there exists $\mu \in (l^\infty)^*$ (that is, there exists a continuous linear functional on l^∞) such that $\|\mu\| = \mu(1) = 1$ and $\mu_n(x_{n+1}) = \mu_n(x_n)$ for each $x = (x_1, x_2, x_3, \dots) \in l^\infty$. Such μ is called a Banach limit. Sometimes $\mu_n(x_n)$ is denoted by $\mu(x)$. It is also known that for a Banach limit μ, $\liminf_{n\to\infty} x_n \leq \mu(x) \leq \limsup_{n\to\infty} x_n$ for each $x = (x_1, x_2, x_3, \dots) \in l^\infty$. As a special case, if $\lim_{n\to\infty} x_n$ exists and is a, then $\mu(x) = a$ too. This means the idea of a Banach limit is an extension of the idea of usual limits. It is also a well-known result that for a bounded sequence $\{x_n\}$ in a Hilbert space H, there exists unique $u_0 \in \overline{\mathrm{co}}\{x_n : n \in \mathbb{N}\}$ such that $\mu_n \langle x_n, v\rangle = \langle u_0, v\rangle$ for all $v \in H$.

Recall that for every closed convex subset C of a Hilbert space H, there exists a metric projection $P_C : H \to C$. That is, for each $x \in H$, there is a unique element $P_C x \in C$ such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. We also need the following lemma due to Takahashi and Toyoda [8].

Lemma 4 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $P_K : H \to K$ be the metric projection. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - k\| \leq \|x_n - k\|$ for any $k \in K$ and $n \in \mathbb{N}$, then $\{P_K x_n\}$ converges strongly to some $k_0 \in K$.*

Mathematicians started working on attractive points in various directions after the publication of Theorem 1, see, for example, [7, 9–16], and [17]. Let us start by recalling the definitions and possible comparisons of different types of mappings. In the sequel, we take the mapping $T : C \to H$ unless otherwise specified. T is called contractive if there exists a real number α with $0 < \alpha < 1$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$. T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. T is said to be quasi-nonexpansive if for $p \in F(T)$, $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$. T is called quasi-contractive (due to Berinde [18]) if there exist real numbers α with $0 < \alpha < 1$ and $L \geq 0$ such that $\|Tx - Ty\| \leq \alpha\|x - y\| + L\|x - Tx\|$ for all $x, y \in C$. Note that the class of quasi-contractive mappings already contains contractions, Kannan, Chatterji and Zamfirescu operators (for definitions, see [18]). Takahashi et al. [7] introduced a broader class of nonlinear mappings which contains the class of contractive mappings and the class of generalized hybrid mappings. T is called normally generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0 \tag{1.1}$$

for all $x, y \in C$. A normally generalized hybrid mapping with a fixed point is quasi-nonexpansive. Moreover, a normally generalized hybrid mapping with $\alpha = 1$, $\beta = \gamma = 0$, $-1 < \delta < 0$ is a contractive mapping. However, this class does not contain the class of quasi-contractive mappings due to Berinde [18]. Finally, we have also found another class of mappings in [11] which was originally introduced by Kawasaki and Takahashi [13] and called "widely more generalized hybrid" in a Hilbert space. T is called "widely more generalized hybrid" if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \varsigma, \eta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \varepsilon\|x - Tx\|^2 + \varsigma\|y - Ty\|^2 + \eta\big\|(x - Tx) - (y - Ty)\big\|^2 \leq 0 \end{aligned} \tag{1.2}$$

for all $x, y \in C$. They noted that the class of widely more generalized hybrid mappings contains the class of normally generalized hybrid mappings but not of quasi-nonexpansive mappings generally even with having a fixed point.

Our purpose in this paper is (i) to introduce the concept of further generalized hybrid mappings, (ii) to introduce the concept of common attractive points (CAP), and (iii) to write and use Picard-Mann iterative process for two mappings. We approximate common attractive points of further generalized hybrid mappings by using iterative process due to Khan [1] generalized to the case of two mappings in Hilbert spaces without closedness of C. First we introduce further generalized hybrid mappings as another generalization of normally generalized hybrid mappings. T is called a further generalized mapping if there exist $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 \\ &\quad + \delta\|x - y\|^2 + \varepsilon\|x - Tx\|^2 \leq 0 \end{aligned} \tag{1.3}$$

for all $x, y \in C$. Obviously, this is a generalization of (1.1) when $\varepsilon = 0$. It is noteworthy that it contains the class of quasi-nonexpansive mappings, quasi-contractive mappings due to Berinde [18] and, in turn, contractive mappings, Kannan mappings, Chatterjea mappings, Zamfirescu mappings. For definitions of these mappings, see, for example, [18]. To see that (1.3) actually contains quasi-contractive mappings due to Berinde [18], choose $\alpha = 1$, $\beta = \gamma = 0$, $\delta \in (-1, 0)$, $\varepsilon \in (-\infty, 0]$ and then use $a^2 + b^2 \leq (a + b)^2$ for all nonnegative real numbers a, b. Recall that quasi-contractive mappings due to Berinde [18] are not contained in (1.1). Apparently, this seems a special case of "widely more generalized hybrid" mappings (1.2) when $\varsigma = \eta = 0$. However, our class not only constitutes a simple generalization of (1.1) but also, as mentioned above, contains the class of quasi-nonexpansive mappings when it has a fixed point contrary to "widely more generalized hybrid" mappings (1.2). So the results obtained by our new mapping will not only be more general but also simpler.

Now, we introduce the concept of common attractive points for two mappings S and T denoted and defined as follows:

$$\mathrm{CAP}(S,T) = \{z \in H : \max(\|Sx - z\|, \|Tx - z\|) \leq \|x - z\|\}$$

for all $x \in C$. Obviously, $z \in \mathrm{CAP}(S,T)$ means that $z \in A(S)$ as well as $z \in A(T)$. Note also that $\mathrm{CAP}(S,T) = A(T)$ when $S = T$.

Recall that a Mann iterative process is as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \in \mathbb{N}. \end{cases} \tag{1.4}$$

Khan [1] introduced a new iterative process called Picard-Mann hybrid iterative process:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ is in $(0,1)$. It was proved to be independent but faster than all Picard, Mann, and Ishikawa processes. Finally, we generalize it to the case of two mappings S and T as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Sy_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is in $(0,1)$. This process reduces to Mann if $S = I$, the identity mapping and at the same time deals with common attractive points.

In short, we approximate common attractive points of (1.3) through (1.6) in Hilbert spaces without closedness of C. Our results are generalizations and improvements of several results in the literature as mentioned later in this paper.

2 Main results

Let us first give some useful properties of $\mathrm{CAP}(S,T)$ on the lines similar to Lemmas 1, 2, and 3. For the sake of simplicity, we take the same parameters $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{R}$ for the two further generalized hybrid mappings S, T as defined in (1.3).

Lemma 5 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $S, T : C \to C$ be two mappings. If $\mathrm{CAP}(S,T) \neq \emptyset$, then $F(S) \cap F(T) \neq \emptyset$. In particular, if $z \in \mathrm{CAP}(S,T)$, then $P_C z \in F(S) \cap F(T)$.*

Proof Let $z \in \mathrm{CAP}(S,T)$. Then $z \in A(S)$ and $z \in A(T)$ (and of course $z \in H$). Thus there is a unique element $u = P_C z \in C$ such that $\|u - z\| \leq \|y - z\|$ for all $y \in C$. Now $Tu \in C$

implies $\|u - z\| \leq \|Tu - z\|$. On the other hand, $z \in A(T)$, therefore $\|Ty - z\| \leq \|y - z\|$ for all $y \in C$ and, in particular, $\|Tu - z\| \leq \|u - z\|$. Thus $\|Tu - z\| \leq \|u - z\| \leq \|Tu - z\|$ and hence $u \in F(T)$. Similarly, $u \in F(S)$ and so $F(S) \cap F(T) \neq \emptyset$ and $u = P_C z \in F(S) \cap F(T)$. □

Lemma 6 *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $S, T : C \to C$ be two mappings. Then* $\mathrm{CAP}(S, T)$ *is a closed and convex subset of H.*

Proof Since the intersection of two closed sets is closed and that of two convex sets is convex, the proof follows on the lines similar to Lemma 2.3 of [5]. □

Lemma 7 *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $S, T : C \to H$ be two quasi-nonexpansive mappings. Then* $\mathrm{CAP}(S, T) \cap C = F(S) \cap F(T)$.

Proof Let $z \in \mathrm{CAP}(S, T) \cap C$. Then, by definition, $\max(\|Sx - z\|, \|Tx - z\| \leq \|x - z\|)$ for all $x \in C$. In particular, choosing $x = z \in C$, we obtain $\max(\|Sz - z\|, \|Tz - z\|) \leq 0$. That is, $z \in F(S) \cap F(T)$. Conversely, since $z \in F(S) \cap F(T)$ and $S, T : C \to H$ are quasi-nonexpansive mappings, we have $\|Sx - z\| \leq \|x - z\|, \|Tx - z\| \leq \|x - z\|$ for all $x \in C$. This implies that $\max(\|Sx - z\|, \|Tx - z\|) \leq \|x - z\|$ for all $x \in C$. Clearly, $z \in C$. Hence $z \in \mathrm{CAP}(S, T) \cap C$. This completes the proof. □

Our next result is an existence theorem on common attractive points of two further generalized hybrid mappings (1.3) without any use of closedness and convexity. This result is followed by some important remarks on comparing it with some results in the current literature.

Theorem 2 *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $S, T : C \to C$ be two further generalized hybrid mappings as defined in* (1.3) *which satisfy $\alpha + \beta + \gamma + \delta \geq 0$, $\varepsilon \geq 0$ and either $\alpha + \beta > 0$ or $\alpha + \gamma > 0$. Then* $\mathrm{CAP}(S, T) \neq \emptyset$ *if and only if there exists $z \in C$ such that both $\{S^n z, n = 0, 1, 2, \ldots\}$ and $\{T^n z, n = 0, 1, 2, \ldots\}$ are bounded.*

Proof Suppose that $\mathrm{CAP}(S, T) \neq \emptyset$ and $z \in \mathrm{CAP}(S, T)$. Then, by definition, $\max(\|Sx - z\|, \|Tx - z\|) \leq \|x - z\|$ for all $x \in C$. This means that $\|S^{n+1}x - z\| \leq \|S^n x - z\|$ and $\|T^{n+1}x - z\| \leq \|T^n x - z\|$ for all $x \in C$ and $n \in \mathbb{N}$. That is, both $\{S^n z, n = 0, 1, 2, \ldots\}$ and $\{T^n z, n = 0, 1, 2, \ldots\}$ are bounded.

Conversely, suppose that there exists $z \in C$ such that $\{S^n z, n = 0, 1, 2, \ldots\}$ as well as $\{T^n z, n = 0, 1, 2, \ldots\}$ is bounded. Suppose that $\max(\|Sx - z\|, \|Tx - z\|) = \|Tx - z\|$. After doing long calculations on the lines similar to Theorem 8 of [11], we find that there exists $p \in H$ such that $\|Tx - p\|^2 \leq \|x - p\|^2$. This means that $p \in A(T)$. However, by our supposition on maximum, we get $\|Sx - p\|^2 \leq \|x - p\|^2$. Thus $\mathrm{CAP}(S, T) \neq \emptyset$. In case, $\max(\|Sx - z\|, \|Tx - z\|) = \|Sx - z\|$, we can get the result by interchanging the roles of S and T. □

This theorem constitutes a generalization of Theorem 3.1 of [7] and the results generalized therein when $S = T$ and $\varepsilon = 0$. Clearly this theorem handles existence of common attractive points, so it is independent of Theorem 8 of [11]. But a special case of our result when $S = T$ can be obtained from Theorem 8 of [11] by choosing $\varsigma = \eta = 0$. However, even in this special case, it is more general in the sense that our class of mappings is simpler and always covers the class of quasi-nonexpansive mappings as opposed to Theorem 8 of [11]. The same holds for all the results of [11].

Let us now come to one of our main targets of proving a weak convergence theorem in Hilbert spaces without needing closedness of C.

Theorem 3 *Let H be a real Hilbert space, and let C be a nonempty convex subset of H. Let $S,T : C \to C$ be two further generalized hybrid mappings as defined in* (1.3) *which satisfy $\alpha+\beta+\gamma+\delta \geq 0$, $\varepsilon \geq 0$ and either $\alpha+\beta > 0$ or $\alpha+\gamma > 0$. Let $\mathrm{CAP}(S,T) \neq \emptyset$. If $\{x_n\}$ is defined by* (1.6), *where $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\liminf \alpha_n(1-\alpha_n) > 0$, then $\{x_n\}$ converges weakly to a point $q \in \mathrm{CAP}(S,T)$. Moreover, $q = \lim_{n\to\infty} Px_n$, where P is a projection of H onto* $\mathrm{CAP}(S,T)$.

Proof Let $z \in \mathrm{CAP}(S,T)$. Then, by (1.6),

$$\begin{aligned} \|y_n - z\|^2 &= \left\|(1-\alpha_n)x_n + \alpha_n Tx_n - z\right\|^2 \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|Tx_n - z\|^2 \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|x_n - z\|^2 \\ &= \|x_n - z\|^2, \end{aligned}$$

and so

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|Sy_n - z\|^2 \\ &\leq \|y_n - z\|^2 \\ &\leq \|x_n - z\|^2. \end{aligned}$$

Thus

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 \tag{2.1}$$

for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty}\|x_n - z\|^2$ exists and so $\{x_n\}$ must be bounded.

Since H is a Hilbert space, so

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|Sy_n - z\|^2 \\ &\leq \|y_n - z\|^2 \\ &= \left\|(1-\alpha_n)x_n + \alpha_n Tx_n - z\right\|^2 \\ &= (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|Tx_n - z\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|x_n - z\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ &= \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2. \end{aligned}$$

This implies that

$$\alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Now, using the condition $\liminf \alpha_n(1-\alpha_n) > 0$ and the above proved fact that $\lim_{n\to\infty}\|x_n - z\|^2$ exists, we have

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$$

We have also proved in the above lines that $\{x_n\}$ is a bounded sequence, therefore we have its subsequence $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup q \in C$. Since $T : C \to C$ is a further generalized mapping, therefore for any $y \in C$, we get

$$\begin{aligned} &\alpha\|Tx_{n_j} - Ty\|^2 + \beta\|x_{n_j} - Ty\|^2 + \gamma\|Tx_{n_j} - y\|^2 \\ &\quad + \delta\|x_{n_j} - y\|^2 + \varepsilon\|x_{n_j} - Tx_{n_j}\|^2 \le 0, \end{aligned}$$

and so

$$\begin{aligned} &\alpha\left(\|Tx_{n_j} - x_{n_j}\|^2 + \|x_{n_j} - Ty\|^2 + 2\langle Tx_{n_j} - x_{n_j}, x_{n_j} - Ty\rangle\right) \\ &\quad + \beta\|x_{n_j} - Ty\|^2 + \gamma\|Tx_{n_j} - y\|^2 \\ &\quad + \delta\|x_{n_j} - y\|^2 + \varepsilon\|x_{n_j} - Tx_{n_j}\|^2 \le 0. \end{aligned}$$

Making use of Banach limit μ, we get

$$(\alpha+\beta)\mu_n\|x_{n_j} - Ty\|^2 + (\gamma+\delta)\mu_n\|x_{n_j} - y\|^2 \le 0.$$

This yields that

$$\begin{aligned} &(\alpha+\beta)\mu_n\left[\|x_{n_j} - y\|^2 + \|y - Ty\|^2 + 2\langle x_{n_j} - y, y - Ty\rangle\right] \\ &\quad + (\gamma+\delta)\mu_n\|x_{n_j} - y\|^2 \le 0. \end{aligned}$$

Thus

$$\begin{aligned} &(\alpha+\beta+\gamma+\delta)\mu_n\|x_{n_j} - y\|^2 \\ &\quad + (\alpha+\beta)\|y - Ty\|^2 + 2(\alpha+\beta)\mu_n\langle x_{n_j} - y, y - Ty\rangle \le 0. \end{aligned}$$

But $\alpha+\beta+\gamma+\delta \ge 0$, therefore

$$(\alpha+\beta)\|y - Ty\|^2 + 2(\alpha+\beta)\mu_n\langle x_{n_j} - y, y - Ty\rangle \le 0.$$

Since $x_{n_j} \rightharpoonup q$, therefore

$$(\alpha+\beta)\|y - Ty\|^2 + 2(\alpha+\beta)\langle q - y, y - Ty\rangle \le 0.$$

Since H is a Hilbert space, so using

$$2\langle u-v,p-w\rangle = \|u-w\|^2 + \|v-p\|^2 - \|u-p\|^2 - \|v-w\|^2 \tag{2.2}$$

in the above inequality, we have

$$\begin{aligned}&(\alpha+\beta)\|y-Ty\|^2\\&\quad+(\alpha+\beta)\bigl[\|q-Ty\|^2+\|y-y\|^2-\|q-y\|^2-\|y-Ty\|^2\bigr]\leq 0.\end{aligned}$$

This implies that $(\alpha+\beta)[\|q-Ty\|^2-\|q-y\|^2]\leq 0$. Since $(\alpha+\beta)>0$, $\|q-Ty\|^2-\|q-y\|^2\leq 0$. Similarly, we get $\|q-Sy\|^2-\|q-y\|^2\leq 0$ and hence $q\in \mathrm{CAP}(S,T)$. Next we prove that $x_n \rightharpoonup q$ by proving that any two subsequences of $\{x_n\}$ converge weakly to the same limit q. Let $x_{n_j}\rightharpoonup q_1$ and $x_{n_k}\rightharpoonup q_2$. By what we have just proved, q_1 and q_2 belong to $\mathrm{CAP}(S,T)$, and from the initial steps of this proof we conclude that $\lim_{n\to\infty}(\|x_n-q_1\|^2-\|x_n-q_2\|^2)$ exists, call it ℓ. Now, using (2.2) again, $2\langle x_n,q_2-q_1\rangle = \|x_n-q_1\|^2+\|q_2\|^2-\|x_n-q_2\|^2-\|q_1\|^2$. This yields $\|x_n-q_1\|^2-\|x_n-q_2\|^2 = 2\langle x_n,q_2-q_1\rangle-\|q_2\|^2+\|q_1\|^2$. Thus

$$\begin{aligned}&\|x_{nj}-q_1\|^2-\|x_{n_j}-q_2\|^2=2\langle x_{n_j},q_2-q_1\rangle-\|q_2\|^2+\|q_1\|^2\quad\text{and}\\&\|x_{n_k}-q_1\|^2-\|x_{n_k}-q_2\|^2=2\langle x_{n_k},q_2-q_1\rangle-\|q_2\|^2+\|q_1\|^2.\end{aligned}$$

Now, taking weak limit on the above two equations and making use of $x_{n_j}\rightharpoonup q_1$ and $x_{n_k}\rightharpoonup q$, we get

$$\begin{aligned}&\ell=2\langle q_1,q_2-q_1\rangle-\|q_2\|^2+\|q_1\|^2,\\&\ell=2\langle q_2,q_2-q_1\rangle-\|q_2\|^2+\|q_1\|^2.\end{aligned}$$

Subtracting we get $2\langle q_1-q_2,q_2-q_1\rangle=0$ and hence $q_1=q_2$. In turn, $x_n\rightharpoonup q\in\mathrm{CAP}(S,T)$.

Finally, we show that $q=\lim_{n\to\infty}Px_n$, where P is the projection of H onto $\mathrm{CAP}(S,T)$. Now from (2.1) it follows that $\|x_{n+1}-z\|\leq\|x_n-z\|$ for all $z\in\mathrm{CAP}(S,T)$ and $n\in\mathbb{N}$. Since $\mathrm{CAP}(S,T)$ is closed and convex by Lemma 6, applying Lemma 4, $\lim_{n\to\infty}Px_n=p$ for some $p\in\mathrm{CAP}(S,T)$. It is well known for projections that $\langle x_n-Px_n,Px_n-z\rangle\geq 0$ for all $z\in\mathrm{CAP}(S,T)$ and $n\in\mathbb{N}$. Therefore, $\langle q-p,p-z\rangle\geq 0$ for all $z\in\mathrm{CAP}(S,T)$ and, in particular, $\langle q-p,p-q\rangle\geq 0$. Hence, $q=p=\lim_{n\to\infty}Px_n$. □

Although the following is a corollary to the above theorem, it is a new result in itself. As already mentioned, the iterative process (1.5) is independent but faster than several iterative processes, therefore this corollary has its own standing.

Corollary 1 *Let H, C, T and $\alpha,\beta,\gamma,\delta,\varepsilon$ be as in Theorem* 3. *Let $A(T)\neq\varnothing$. If $\{x_n\}$ is defined by the iterative process* (1.5), *where $\{\alpha_n\}$ is a sequence in* $(0,1)$ *with* $\liminf\alpha_n(1-\alpha_n)>0$, *then $\{x_n\}$ converges weakly to a point $q\in A(T)$. Moreover, $q=\lim_{n\to\infty}Px_n$, where P is the projection of H onto $A(T)$.*

Proof Choose $S=T$ in the above theorem. □

Corollary 2 *Let H, C, T and $\alpha, \beta, \gamma, \delta, \varepsilon$ be as in Theorem* 3. *Let $A(T) \neq \emptyset$. If $\{x_n\}$ is defined by Mann iterative process* (1.4), *where $\{\alpha_n\}$ is a sequence in* $(0,1)$ *with* $\liminf \alpha_n(1-\alpha_n) > 0$, *then $\{x_n\}$ converges weakly to a point $q \in A(T)$. Moreover, $q = \lim_{n\to\infty} Px_n$, where P is the projection of H onto $A(T)$.*

Proof Choose $S = I$ in the above theorem. □

Remarks In view of Lemma 2, in Theorem 3, instead of assuming $\mathrm{CAP}(S,T) \neq \emptyset$, we could have assumed that there exists $z \in C$ such that both $\{S^n z, n = 0,1,2,\ldots\}$ and $\{T^n z, n = 0,1,2,\ldots\}$ are bounded. Similar remark applies to Corollaries 1 and 2.

Now we give some remarks on how our above results are generalizations and improvements of the results in the existing literature.

Remarks

(1) Theorem 5.1 of [7] can now be obtained by choosing either $S = I$, $\varepsilon = 0$ in Theorem 3 or $\varepsilon = 0$ in Corollary 2.

(2) Corollary 2 can be viewed as an improvement and extension of Theorem 8 of [11] in the sense that (i) our class of mappings is simpler and (ii) it contains the class of quasi nonexpansive mappings as opposed to [11]. Corollary 1 not only keeps this sense but also gives faster convergence (see [1]).

(3) Corollary 1 (leave alone our Theorem 3) generalizes Corollary 4.3 of Zheng [17] in two ways: we do not need closedness of C and the class of our mappings is much more general than that of [17].

(4) Of course, all corresponding results generalized in [7] and [11] are part and parcel of the above remarks.

If, in addition, we use the closedness of C in Theorem 3, then we have the following:

Theorem 4 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $S, T : C \to C$ be two further generalized hybrid mappings, as defined in* (1.3), *which satisfy $\alpha + \beta + \gamma + \delta \geq 0$, $\varepsilon \geq 0$ and either $\alpha + \beta > 0$ or $\alpha + \gamma > 0$. Let* $\mathrm{CAP}(S,T) \neq \emptyset$. *If $\{x_n\}$ is defined by* (1.6), *where $\{\alpha_n\}$ is a sequence in* $(0,1)$ *with* $\liminf \alpha_n(1-\alpha_n) > 0$, *then $\{x_n\}$ converges weakly to a point $P_C q \in F(S) \cap F(T)$, where $q \in H$ and $P_C : H \to C$ is the metric projection.*

Proof By Theorem 3, $q \in \mathrm{CAP}(S,T)$. Now, using Lemma 5, $P_C q \in F(S) \cap F(T)$ as desired. □

3 Conclusions

In this paper, we have introduced the concepts of further generalized hybrid mappings and common attractive points (CAP). We have given some basic properties of common attractive points and have compared them with common fixed points. Further, we have shown that our newly introduced class of mappings contains many important classes and is better than some apparently looking more general mappings in the literature. We have given an existence theorem on common attractive points. Then, using a two-mapping variant of Picard-Mann iterative process, we have approximated the common attractive points of further generalized hybrid mappings in Hilbert spaces without closedness on its subsets. Our results also show a contrast of common attractive points with common fixed points. Our results can open the door for further research activity in the field for other mappings, other iterative processes, or other ambient spaces.

Acknowledgements
(1) The author remains grateful to his Ph.D. advisor Professor Wataru Takahashi for what he has learnt from him throughout his career. (2) The author thanks anonymous referees for their careful reading and suggestions.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The sole author contributed 100% to the article. The author read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 October 2017 Accepted: 5 February 2018 Published online: 05 March 2018

References

1. Khan, S.H.: A Picard–Mann hybrid iterative process. Fixed Point Theory Appl. **2013**, 69 (2013). https://doi.org/10.1186/1687-1812-2013-69
2. Qin, X., Yao, J.C.: Projection splitting algorithms for nonself operators. J. Nonlinear Convex Anal. **18**, 925–935 (2017)
3. Yildirim, I.: Approximating fixed points of two finite families of *I*-asymptotically quasi-nonexpansive mappings. J. Nonlinear Funct. Anal. **2016**, Article ID 42 (2016)
4. Takahashi, W.: Weak and strong convergence theorems for families of nonlinear and nonself mappings in Hilbert spaces. J. Nonlinear Var. Anal. **1**, 1–23 (2017)
5. Takahashi, W., Takeuchi, Y.: Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space. J. Nonlinear Convex Anal. **12**, 399–406 (2011)
6. Baillon, J.B.: Un theoreme de type ergodique pour les contractions nonlinears dans un espaces de Hilbert. C. R. Acad. Sci. Paris, Ser. A–B **280**, 1511–1541 (1975)
7. Takahashi, W., Wong, N.C., Yao, J.C.: Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces. J. Nonlinear Convex Anal. **13**(4), 745–757 (2012)
8. Takahashi, W., Toyoda, M.: Weak convergence theorems for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **118**(2), 417–428 (2003)
9. Atsushiba, S., Takahashi, W.: Nonlinear ergodic theorems without convexity for nonexpansive semigroups in Hilbert spaces. J. Nonlinear Convex Anal. **14**(2), 209–219 (2013)
10. Chen, L., Gao, L., Zhao, Y.: A new iterative scheme for finding attractive points of (α,β)-generalized hybrid set-valued mappings. J. Nonlinear Sci. Appl. **10**, 1228–1237 (2017)
11. Guu, S., Takahashi, W.: Existence and approximation of attractive points of the widely more generalized hybrid mappings in Hilbert spaces. Abstr. Appl. Anal. **2013**, Article ID 904164 (2013). https://doi.org/10.1155/2013/904164
12. Kaewkhao, A., Inthakon, W., Kunwai, K.: Attractive points and convergence theorems for normally generalized hybrid mappings in CAT(0) spaces. Fixed Point Theory Appl. **2015**, 96 (2015). https://doi.org/10.1186/s13663-015-0336-z
13. Kawasaki, T., Takahashi, W.: Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces. J. Nonlinear Convex Anal. **14**, 71–87 (2013)
14. Kocourek, P., Takahashi, W., Yao, J.C.: Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces. Taiwan. J. Math. **14**, 2497–2511 (2010)
15. Kohsaka, F., Takahashi, W.: Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces. Arch. Math. (Basel) **91**, 166–177 (2008)
16. Takahashi, W., Yao, J.C.: Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces. Taiwan. J. Math. **15**, 457–472 (2011)
17. Zheng, Y.: Attractive points and convergence theorems of generalized hybrid mapping. J. Nonlinear Sci. Appl. **8**, 354–362 (2015)
18. Berinde, V.: A convergence theorem for some mean value fixed point iterations procedures. Demonstr. Math. **38**(1), 177–184 (2005)